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Finite Part Integrals and Hypersingular Kernels

Youn-Sha Chan[1], Albert C. Fannjiang[2], Glaucio H. Paulino[3], and Bao-Feng Feng[4]

[1]Department of Computer and Mathematical Sciences, University of Houston–Downtown
One Main Street, Houston, Texas 77002, U.S.A., Email: chany@uhd.edu

[2]Department of Mathematics, University of California, Davis, CA 95616, U.S.A.

[3]Department of Civil and Environmental Engineering, University of Illinois
2209 Newmark Laboratory, 205 North Mathews Avenue, Urbana, IL 61801, U.S.A.

[4]Department of Mathematics, The University of Texas–Pan American,
1201 W. University Drive, Edinburg, Texas 78541, U.S.A.

Abstract. Singular integral equation method is one of the most effective numerical methods solving a plane crack problem in fracture mechanics. Depending on the choice of the density function, very often a higher order of sigularity appears in the equation, and we need to give a proper meaning of the integration. In this article we address the Hadamard finite part integral and how it is used to solve the plane crack problems. Properties of the Hadamard finite part integral will be summarized and compared with other type of integrals. Some numerical results for crack problems by using Hadamard finite part integral will be provided.
Keywords. singular integral equations; Hadamard finite part integrals; fracture mechanics; integral transform method.
AMS (MOS) subject classification: 35, 44, 45, and 73.

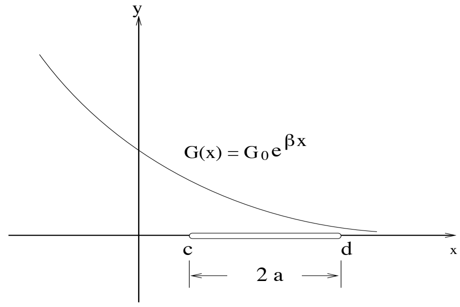

Figure 1: Antiplane shear problem for a nonhomogeneous material. Shear modulus $G(x) = G_0 e^{\beta x}$; c and d represent the left and right crack tip, respectively; a is the half crack length.

1 Introduction

In general, the solution to the crack problems in the linear elastic fracture mechanics (LEFM) often leads to a system of Cauchy type singular integral equations [4, 5, 16]

$$\frac{a_i}{\pi}\,-\!\!\!\!\!\!\int_c^d \frac{\phi_i(t)}{t-x}\,dt+\sum_{j=1}^{J}\int_c^d k_{ij}(x,t)\phi_i(t)\,dt+b_i\phi_i(x)=p_i(x)\,,$$

where $c<x<d$, a_i, b_i $(i=1,2,\cdots,I)$ are real constants, and the kernels $k_{ij}(x,t)$ are bounded in the closed domain $(x,t)\in[c,\,d]\times[c,\,d]$. Each function $p_i(x)$ is known and given by the boundary condition(s). Functions $\phi_i(x)$ are the unknowns of the problems, also called by the density functions which often are the derivatives of the displacements. However, if the unknown density function is chosen to be the displacement, say $w_i(x)$, then the order of singularity increases. Thus, a formulation of (a system of) hypersingular integral equations is made.

The choice of different unknown density function in the formulation leads to different order of singularity for the integral equation. For instance, consider a mode III crack problem in a nonhomogeneous elastic medium with the shear modulus variation $G(x) = G_0 e^{\beta x}$ (illustrated in Figure 1), then the governing a partial differential equation (PDE) in terms of the z component of the displacement vector $w(x,y)$ is

$$\nabla^2 w(x,y)+\beta\frac{\partial w(x,y)}{\partial x}=0 \qquad (1)$$

with the mixed boundary conditions

$$\begin{array}{ll} w(x,0)=0\ , & x\notin[c,d]\ ,\\ \sigma_{yz}(x,0^+)=p(x)\ , & x\in(c,d)\ , \end{array} \qquad (2)$$

where $p(x)$ is the traction function along the crack surfaces (c,d). By a process of Fourier integral transform PDE (1) can be reduced to a hypersingular integral equation [2]:

$$\frac{G_0e^{\beta x}}{\pi}\,=\!\!\!\!\!\!\int_c^d\left[\frac{1}{(t-x)^2}+\frac{\beta}{2(t-x)}+N(x,t)\right]w(t,0^+)\,dt=p(x)\,, \qquad (3)$$

where $c<x<d$, and $N(x,t)$ takes an integral form

$$N(x,t)=\int_0^\infty\frac{-\beta^2\sqrt{\xi}\cos[(t-x)\xi]}{\left(2\sqrt{\xi}+\sqrt{2\sqrt{A(\xi,\beta)}+\xi}\right)\left(\xi+\sqrt{A(\xi,\beta)}\right)}d\xi$$

$$-\int_0^\infty\frac{\beta^3\sqrt{\sqrt{B(\xi,\beta)}-\xi^2}\ \sin[(t-x)\xi]/2}{\left(\sqrt{\sqrt{B(\xi,\beta)}+\xi^2}+\sqrt{2}|\xi|\right)\left(2\xi^2+\beta^2+2\sqrt{B(\xi,\beta)}\right)}d\xi$$

with $A(\xi,\beta) = \xi^2 + \beta^2$ and $B(\xi,\beta) = \xi^4 + \beta^2\xi^2$. Unless a proper meaning of integration is given , the first integral in equations (3) is meaningless; the integral is regularized by "Hadamard finite part integral", and we have used "double-bar integral" to denote it. Hadamard finite part (HFP) integral was first introduced by Jacques Hadamard [8] to solve some linear PDE, which can be considered as a generalization of the Cauchy principal value (CPV) integral [6].

2 HFP and CPV Integrals

HFP integral is a generalization of CPV integral, thus let us look at CPV integral first.

2.1 CPV Integral

Equations that involve integrals of the type

$$\fint_c^d \frac{\phi(t)}{t-x}dt\,, \qquad |x|<1 \tag{4}$$

in not integrable in the ordinary (Riemann or Lebesgue integral) sense because of the kernel $1/(t-x)$ is not integrable over any interval that includes the point $t = x$. Thus, it is regularized by CPV integral [10, 13]:

$$\fint_c^d \frac{\phi(t)}{t-x}dt := \lim_{\epsilon\to 0}\left\{\int_c^{x-\epsilon}\frac{\phi(t)}{t-x}dt + \int_{x+\epsilon}^d \frac{\phi(t)}{t-x}dt\right\}\,, \tag{5}$$

where $c < x < d$. Notice that the ϵ-neighborhood about the singular point $x = t$ must be symmetric, and it is how CPV integral works out for canceling off the singularity.

For the existence of the CPV integral, the function $\phi(x)$ in (5) needs to be at least Hölder continuous on (c, d), that is, $\phi(x) \in C^{0,\alpha}(c, d)$. This requirement of regularity can be easily checked by following manipulation:

$$\begin{aligned}
&\fint_c^d \frac{\phi(t)}{t-x}dt \\
=\;& \lim_{\epsilon\to 0}\left\{\int_{|t-x|\geq\epsilon}\frac{\phi(t)-\phi(x)}{t-x}dt + \phi(x)\int_{|t-x|\geq\epsilon}\frac{dt}{t-x}\right\} \\
=\;& \int_c^d \frac{\phi(t)-\phi(x)}{t-x}dt + \phi(x)\fint_c^d \frac{dt}{t-x}.
\end{aligned} \tag{6}$$

Thus, for any $\phi \in C^{0,\alpha}, \alpha > 0$, the first integral on the right side of (6) is an ordinary Riemann integral and the second integral is

$$\fint_c^d \frac{dt}{t-x} = \log\frac{d-x}{x-c}\,, \quad c<x<d.$$

Although Cauchy principal value integral is defined for an interior point in (c, d) above, it can be evaluated separately on both sides of the end points:

$$\fint_c^x \frac{\phi(t)}{t-x}dt := \lim_{\epsilon\to 0}\left\{\int_c^{x-\epsilon}\frac{\phi(t)}{t-x}dt - \phi(x)\ln\epsilon\right\}\,,$$

where $x > c$, and

$$\fint_x^d \frac{\phi(t)}{t-x}dt := \lim_{\epsilon\to 0}\left\{\int_{x+\epsilon}^d \frac{\phi(t)}{t-x}dt + \phi(x)\ln\epsilon\right\}\,,$$

where $x < d$.

2.2 HFP Integral

CPV integral does not work for a higher singularity. For instance, consider $\phi(t) = 1$ and $x = 0$ in

$$\int_c^d \frac{\phi(t)}{(t-x)^2}\,dt\,, \qquad c<x<d\,, \tag{7}$$

that is,

$$\int_c^d \frac{dt}{t^2}\,, \qquad c<0<d\,.$$

The integral is not convergent, neither does the principal value exist, since

$$\int_{[c,\,d]\setminus(-\epsilon,\,\epsilon)}\frac{dt}{t^2} = \lim_{\epsilon\to 0}\left(\frac{1}{c}-\frac{1}{d}+\frac{2}{\epsilon}\right)$$

is not finite. Hadamard finite part integral is defined by disregarding the infinite part, $2/\epsilon$, and keeping the finite part, *i.e.*

$$\fint_c^d \frac{dt}{t^2} = \frac{1}{c}-\frac{1}{d}\,. \tag{8}$$

Definition 1 (Hadamard finite part integral) *Let $\epsilon > 0$, and denote*

$$F(\epsilon,\,x) = \int_{[c,\,d]\setminus(x-\epsilon,\,x+\epsilon)} f(t,\,x)\,dt\,, \quad c<x<d\,,$$

where the singularity appears at the point $t = x$. If $F(\epsilon, x)$ is decomposed into

$$F(\epsilon,\,x) = F_0(\epsilon,\,x) + F_1(\epsilon,\,x)\,,$$

and

$$\lim_{\epsilon\to 0}F_0(\epsilon,\,x) < \infty\,, \quad F_1(\epsilon,\,x) \overset{\epsilon\to 0}{\to} \infty\,,$$

then the finite part integral is defined by keeping the "finite part", i.e.

$$\fint_c^d f(t,\,x)\,dt = \lim_{\epsilon\to 0}F_0(\epsilon,\,x)\,.$$

Notice that HFP integral can be considered as a generalization of the CPV integral in the sense that if the principal value integral exists, then they give the same result [6]. We shall define the HFP integral for integrals with quadratic singularity as in (7). Denote by $C^{m,\alpha}(c, d)$ the space of functions whose m-th derivatives are Hölder continuous on (c, d) with index $0 < \alpha \leq 1$.

Definition 2 *If $\phi(x) \in C^{1,\alpha}(c, d)$, then*

$$\fint_c^d \frac{\phi(t)}{(t-x)^2}\,dt := \lim_{\epsilon\to 0}\left[\int_c^{x-\epsilon}\frac{\phi(t)}{(t-x)^2}dt + \int_{x+\epsilon}^d \frac{\phi(t)}{(t-x)^2}dt - \frac{2\phi(x)}{\epsilon}\right]\,. \tag{9}$$

The condition $\phi(x) \in C^{1,\alpha}(c,d)$ is required for the existence of the defined HFP integral [12].

Following observation may help to understand Definition 2 for HFP. By a step of integration by-parts, the first integral under the limit $\epsilon \to 0$ in (9) can be written as

$$\int_c^{x-\epsilon} \frac{\phi(t)}{(t-x)^2} dt = \frac{\phi(x-\epsilon)}{\epsilon} - \frac{c}{c-x} + \int_c^{x-\epsilon} \frac{\phi'(t)}{t-x)} dt\,.$$

Similarly,

$$\int_{x+\epsilon}^{d} \frac{\phi(t)}{(t-x)^2} dt = \frac{\phi(x+\epsilon)}{\epsilon} - \frac{d}{d-x} + \int_{x+\epsilon}^{d} \frac{\phi'(t)}{t-x)} dt\,.$$

Thus, the term $-2\phi(x)/\epsilon$ in (9) will kill the singularity $[\phi(x-\epsilon)+\phi(x+\epsilon)]/\epsilon$, and under the assumption that $\phi(x) \in C^{1,\alpha}(c,d)$ Definition 2 indeed takes the finite part of the integral according to Definition 1.

Another direction of viewing Definition 2 is by taking direct differentiation d/dx to (5) with Leibnitz's rule, *i.e.*

$$\begin{aligned}\frac{d}{dx}\int_c^d \frac{\phi(t)}{t-x} dt &= \lim_{\epsilon\to 0}\frac{d}{dx}\left[\int_c^{x-\epsilon} \frac{\phi(t)}{t-x}dt + \int_{x+\epsilon}^d \frac{\phi(t)}{t-x}dt\right] \\ &= \lim_{\epsilon\to 0}\left[\int_c^{x-\epsilon} \frac{\phi(t)}{(t-x)^2}dt + \int_{x+\epsilon}^d \frac{\phi(t)}{(t-x)^2}dt - \frac{\phi(x-\epsilon)+\phi(x+\epsilon)}{\epsilon}\right] \qquad (10)\end{aligned}$$

Comparing (10) with (9), one can conclude

Proposition 1 *If $\phi(x) \in C^{1,\alpha}(c,d)$, then*

$$\neq\!\!\!\!\!\!\int_c^d \frac{\phi(t)}{(t-x)^2}\, dt = \frac{d}{dx}\int_c^d \frac{\phi(t)}{t-x}\, dt \qquad (11)$$

Alternatively, one can define finite part integrals by equation (11) and deduce Definition 2 as property. Thus, for general n, HFP integrals can be defined recursively as follows.

Definition 3 (Finite part integral) *Denote*

$$L^{1+} = \bigcap_{p>1} L^p[c,d]\,.$$

For any $\phi \in C^{n,\alpha}(c,d) \cap L^{1+}$, $c < x < d$, and $n = 1,2,3,\ldots$

$$\neq\!\!\!\!\!\!\int_c^d \frac{\phi(t)}{(t-x)^{n+1}}\, dt := \frac{1}{n}\frac{d}{dx} \neq\!\!\!\!\!\!\int_c^d \frac{\phi(t)}{(t-x)^n}\, dt \qquad (12)$$

with

$$\neq\!\!\!\!\!\!\int_c^d \frac{\phi(t)}{t-x}\, dt := -\!\!\!\!\!\!\int_c^d \frac{\phi(t)}{t-x}\, dt\,.$$

By means of (6) and the (recursive) definition of finite part integrals, one can deduce [15]

Proposition 2

$$\neq\!\!\!\!\!\!\int_c^d \frac{\phi(t)}{(t-x)^n}dt = \int_d^c \frac{\phi(t) - \sum_{j=0}^{n-1}\phi^{(j)}(x)(t-x)^j/j!}{(t-x)^n} + \sum_{j=0}^{n-1}\frac{\phi^{(j)}(x)}{j!}\neq\!\!\!\!\!\!\int_c^d \frac{dt}{(t-x)^{n-j}}\,. \qquad (13)$$

For $\phi \in C^{n,\alpha}(c,d) \cap L^{1+}$, the first integral on the right side of (13) is an ordinary Riemann integral. Also, with (13) in hand, integration by-parts formula holds for finite part integrals [15].

Proposition 3 *For $\phi \in C^{n,\alpha}(c,d) \cap L^{1+}$*

$$\begin{aligned}\neq\!\!\!\!\!\!\int_c^d \frac{\phi'(t)}{(t-x)^n}dt &= n\neq\!\!\!\!\!\!\int_c^d \frac{\phi(t)}{(t-x)^{n+1}}dt \\ &+ \frac{\phi(d)}{(d-x)^n} - \frac{\phi(c)}{(c-x)^n}, \quad n \geq 1\end{aligned}$$

and for $\phi \in C^{\alpha}(c,d) \cap L^{1+}$

$$\begin{aligned}&\int_c^d \phi'(t)\log|t-x|dt \\ &= -\!\!\!\!\!\!\int_c^d \frac{\phi(t)}{t-x}dt + \phi(d)\log|d-x| - \phi(c)\log|c-x|\end{aligned}$$

2.3 Remark on HFP Integrals

The most commonly used integration in mathematical analysis is Lebesgue integration. Not all the properties for Lebesgue integral can be carried onto finite part integral. For example, properties that involve inequality (*e.g.* monotone convergence theorem, Fatou's lemma, bounded convergence theorem) may not be true for finite part integral anymore. A simple demonstration is (see equation (8) and Reference [9])

$$\neq\!\!\!\!\!\!\int_{-1}^1 \frac{dt}{t^2} = -2\,;$$

clearly,

$$\left|\neq\!\!\!\!\!\!\int_c^d f(x)\,dx\right| \leq \neq\!\!\!\!\!\!\int_c^d |f(x)|\,dx$$

is NOT true for finite part integral! This is a very unpleasant outcome from HFP integral; however, fortunately, the most relying formula, integration by-parts, is true for finite part integral.

3 Hypersingular Kernels

For the derivation of hypersingular kernels, we use three basic ingredients:

- Finite part integrals.
- Identity
$$i^n\frac{d^n}{dy^n}\left[\frac{1}{y-i(t-x)}\right] = \frac{d^n}{dx^n}\left[\frac{1}{y-i(t-x)}\right]. \quad (14)$$
- Plemelj formulas [3, 16].
$$\lim_{\epsilon\to 0}\int_c^d \frac{\phi(t)}{(t-x)+i\epsilon}dt = -\!\!\!\!\!\!\int_c^d \frac{\phi(t)}{t-x}dt + \pi i\phi(x)\,, \phi \in L^{1+}.$$

The key point of identity (14) is that it allows one to switch the differentiation from d/dx to d/dy, and *vice versa*; HFP integral has been defined and addressed in previous section; for the sake of completeness, we shall briefly address Plemelj formulas.

3.1 Plemelj Formulas

In general, the Cauchy principal value type of integrals

$$\fint_c^d \frac{\phi(t)}{t-x}\,dt\,, \qquad c<x<d$$

is evaluated indirectly by using complex function theory [11, 13]. Define

$$\Phi(z)=\int_c^d \frac{\phi(t)}{t-z}\,dt\,,$$

with z not on the integration contour. The principal value is then recovered by sending z to the point x on the interval $(c,\,d)$, and the result is different as $z\to x$ from above and below. Say, define

$$\Phi^+(x)=\lim_{y\to 0}\Phi(x+i|y|)\,, \qquad \Phi^-(x)=\lim_{y\to 0}\Phi(x-i|y|)\,,$$

then the limits are

$$\Phi^+(x)=\fint_c^d \frac{\phi(t)}{t-x}\,dt+i\pi\phi(x)\,, \tag{15}$$

and

$$\Phi^-(x)=\fint_c^d \frac{\phi(t)}{t-x}\,dt-i\pi\phi(x)\,. \tag{16}$$

Equations (15) and (16) are Plemelj formulas [11], sometimes called by the Sokhetski formulas. It is (15) that we will be using in the derivation of hypersingular kernels. Notice that $\phi(x)$ can be recovered from Plemelj formulas, *i.e.*

$$\phi(x)=\frac{\Phi^+(x)-\Phi^-(x)}{2\pi i}\,.$$

3.2 Arise of Hypersingular Kernels

To demonstrate how the hypersingular kernels arise, we go back to the PDE (1), and through Forier transform $w(x,y)$ can be expressed as [2]

$$w(x,y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\alpha(\xi)\mathrm{e}^{\lambda(\xi)y}\right]\mathrm{e}^{-ix\xi}d\xi\ , \tag{17}$$

where

$$[\lambda(\xi)]^2=\xi^2+i\beta\xi\,, \tag{18}$$

and $\alpha(\xi)$ is to be determined by the boundary conditions (2). To satisfy the far field boundary condition, $\lim_{y\to\infty}w(x,y)=0$, we choose the root $\lambda(\xi)$ to be the non-positive real part:

$$\frac{-1}{\sqrt{2}}\sqrt{\sqrt{\xi^4+\beta^2\xi^2}+\xi^2}-\frac{i}{\sqrt{2}}\mathsf{sgn}(\beta\xi)\sqrt{\sqrt{\xi^4+\beta^2\xi^2}-\xi^2}\,, \tag{19}$$

where the signum function $\mathsf{sgn}(\cdot)$ is defined as

$$\mathsf{sgn}(\eta)=\begin{cases}1\,, & \eta>0\\ 0\,, & \eta=0\\ -1\,, & \eta<0\,.\end{cases} \tag{20}$$

As the limit of $y\to 0^+$ is taken,

$$\begin{aligned} w(x,0^+) &= \lim_{y\to 0^+}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\alpha(\xi)\mathrm{e}^{\lambda(\xi)y}\right]\mathrm{e}^{-ix\xi}d\xi\\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\alpha(\xi)\mathrm{e}^{-ix\xi}d\xi\,, \end{aligned} \tag{21}$$

that is, $w(x,0^+)$ is the inverse Fourier transform of $\alpha(\xi)$. By inverting the Fourier transform, one obtains

$$\begin{aligned} \alpha(\xi) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}w(x,0^+)\mathrm{e}^{ix\xi}dx\\ &= \frac{1}{\sqrt{2\pi}}\int_c^d w(t,0^+)\mathrm{e}^{it\xi}dt\,, \end{aligned} \tag{22}$$

where the first boundary condition in (2) and a change of dummy variable $(x\leftrightarrow t)$ have been applied.

Defining

$$K(\xi,y)=\lambda(\xi)\mathrm{e}^{\lambda(\xi)y}\ , \tag{23}$$

and using the second boundary condition in (2), one reaches that for $c<x<d$

$$\begin{aligned} &\lim_{y\to 0^+}\frac{G(x)}{2\pi}\int_c^d w(t,0^+)\int_{-\infty}^{\infty}K(\xi,y)\mathrm{e}^{i(t-x)\xi}d\xi dt\\ &\quad=\frac{G(x)}{\pi}\int_c^d w(t,0^+)\mathsf{kernel}(x-t)dt=p(x)\,, \end{aligned} \tag{24}$$

$$2\times\mathsf{kernel}(x-t)=\lim_{y\to 0^+}\int_{-\infty}^{\infty}K(\xi,y)\mathrm{e}^{i(t-x)\xi}d\xi\,. \tag{25}$$

Let $K(\xi)\equiv K(\xi,0^+)=\lambda(\xi)$ and by a step of decomposition

$$K(\xi)=[K(\xi)-K_\infty(\xi)]+K_\infty(\xi)\,, \tag{26}$$

one obtains a closed form expressions of

$$K_\infty(\xi)=-|\xi|-\frac{i\beta}{2}\frac{|\xi|}{\xi}\,. \tag{27}$$

This $K_\infty(\xi)$ gives rise to the quadratic hypersingular (and Cauchy singular) kernels by the following

$$\int_{-\infty}^{\infty}\left[|\xi|\mathrm{e}^{-|\xi|y}\right]\mathrm{e}^{i(t-x)\xi}d\xi \quad \xrightarrow{y\to 0^+} \quad \frac{-2}{(t-x)^2} \tag{28}$$

$$\int_{-\infty}^{\infty}\left[i\frac{|\xi|}{\xi}\mathrm{e}^{-|\xi|y}\right]\mathrm{e}^{i(t-x)\xi}d\xi \quad \xrightarrow{y\to 0^+} \quad \frac{-2}{t-x} \tag{29}$$

3.3 Higher Order Hypersingular Kernels

For a more general and higher order of hypersingular kernels, they can be derived by observing that

$$\begin{aligned} k_n(t-x,y) &:= \frac{1}{2\pi}\int_{-\infty}^{\infty}i^n|\xi|^n\frac{|\xi|}{i\xi}\mathrm{e}^{-|\xi|y+i(t-x)\xi}\ d\xi\\ &= \sqrt{\frac{2}{\pi}}(-i)^n Im\left[\frac{d^n}{dy^n}(y-i(t-x))^{-1}\right]\\ &= (-1)^n\sqrt{\frac{2}{\pi}}Im\left[\frac{d^n}{dx^n}(y-i(t-x))^{-1}\right]\\ &= (-1)^n\sqrt{\frac{2}{\pi}}Re\left[\frac{d^n}{dx^n}(t-x+iy)^{-1}\right]. \end{aligned}$$

Thus,

$$\begin{aligned}
& \lim_{y\to 0^+}\int_{-1}^{1} k_n(t-x,y)\phi(t)\,dt \\
= & \lim_{y\to 0^+}(-1)^n\sqrt{\frac{2}{\pi}}\int_{-1}^{1} Re\left[\frac{d^n}{dx^n}(t-x+iy)^{-1}\right]\phi(t)\;dt \\
= & (-1)^n\sqrt{\frac{2}{\pi}}Re\left[\frac{d^n}{dx^n}\lim_{y\to 0^+}\int_{-1}^{1}(t-x+iy)^{-1}\phi(t)\;dt\right] \\
= & (-1)^n\sqrt{\frac{2}{\pi}}\frac{d^n}{dx^n}\fint_{-1}^{1}\frac{\phi(t)}{t-x}\;dt \\
= & n!(-1)^n\sqrt{\frac{2}{\pi}}\fint_{-1}^{1}\frac{\phi(t)}{(t-x)^{n+1}}\;dt.
\end{aligned}$$

where the Plemelj formula and the definition of finite part integrals have been used. Note that, when n is an odd integer,

$$\begin{aligned}
& \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} i^n\xi^n\frac{|\xi|}{i\xi}e^{-|\xi|y+i(t-x)\xi}\;d\xi \\
& = -\sqrt{\frac{2}{\pi}}Im\left[\frac{d^n}{dx^n}(t-x+iy)^{-1}\right].
\end{aligned}$$

Thus we have

$$\begin{aligned}
& \int_{-1}^{1} dt\phi(t)\lim_{y\to 0^+}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} i^n\xi^n\frac{|\xi|}{i\xi}e^{-|\xi|y+i(t-x)\xi}\;d\xi \\
= & -\sqrt{\frac{2}{\pi}}Im\left[\frac{d^n}{dx^n}\lim_{y\to 0^+}\int_{-1}^{1}\phi(t)(t-x+iy)^{-1}\;dt\right] \\
= & -\sqrt{2\pi}\frac{d^n}{dx^n}\phi(x)\,,
\end{aligned}$$

where the Plemelj formula is used again.

3.4 Strain Gradient Elasticity

In addition to the choice of unknown density function, the underlying elasticity theory also gives rise to higher order hypersingular kernels. For instance, in strain gradient elasticity higher order of singular integral equations often arise [1, 7, 18]. The higher order singularity is actually linked to its governing PDE—a fourth order instead of second order:

$$-\ell^2\nabla^4 w - 2\beta\ell^2\nabla^2\frac{\partial w}{\partial x} + \nabla^2 w - \beta^2\ell^2\frac{\partial^2 w}{\partial x^2} + \beta\frac{\partial w}{\partial x} = 0\,, \tag{30}$$

In a factored form PDE (30) can be written as

$$\left(1 - \beta\ell^2\frac{\partial}{\partial x} - \ell^2\nabla^2\right)\left(\nabla^2 + \beta\frac{\partial}{\partial x}\right)w = 0\,, \tag{31}$$

where ℓ is a length parameter in the strain gradient elasticity theory [14].

After Fourier integral transform and asymptotic analysis, the corresponding hypersingular kernel, as in (27), is

$$-i\ell^2|\xi|\xi - \frac{\ell'}{2}i\xi + \frac{3\beta\ell^2}{2}|\xi| + \frac{\ell'\beta}{2} + \left[\left(\frac{\ell'}{2\ell}\right)^2 + \frac{3\ell^2\beta^2}{8} - 1\right]\frac{i\xi}{|\xi|}\,. \tag{32}$$

Through the analysis as described in section 3.3, we have

$$\begin{aligned}
\int_{-\infty}^{\infty}\left[i\xi|\xi|e^{-|\xi|y}\right]e^{i(t-x)\xi}d\xi &\xrightarrow{y\to 0^+} \frac{4}{(t-x)^3}\,, \\
\int_{-\infty}^{\infty}\left[\xi^2 e^{-|\xi|y}\right]e^{i(t-x)\xi}d\xi &\xrightarrow{y\to 0^+} -2\pi\delta''(t-x)\,, \\
\int_{-\infty}^{\infty}\left[i\xi e^{-|\xi|y}\right]e^{i(t-x)\xi}d\xi &\xrightarrow{y\to 0^+} 2\pi\delta'(t-x)\,, \\
\int_{-\infty}^{\infty}\left[1e^{-|\xi|y}\right]e^{i(t-x)\xi}d\xi &\xrightarrow{y\to 0^+} 2\pi\delta(t-x)\,,
\end{aligned}$$

where $\delta(x)$ denotes the Dirac delta function. Thus one can reach following hypersingular integrodifferential equation:

$$\begin{aligned}
\frac{1}{\pi}\fint_{c}^{d}\Big\{ & \frac{-2\ell^2}{(t-x)^3} - \frac{3\beta\ell^2}{2(t-x)^2} + \frac{1 - 3\beta^2\ell^2/8 - [\ell'/(2\ell)]^2}{t-x} \\
& + k(x,t)\Big\}\,\phi(t)dt + \frac{\ell'}{2}\phi'(x) + \frac{\beta\ell'}{2}\phi(x) = \frac{p(x)}{G}\,.
\end{aligned} \tag{33}$$

The density function $\phi(x)$ used in (33) is the strain function, that is, $\phi(x) = \partial w(x,0)/\partial x$ with $c < x < d$.

Under the case that $\beta = 0$ and $\ell = \ell' = 0$, integral equation (33) has an exact solution [7]. Figure 2 shows that strain is finite at the two crack-tips, which is different from the conventional linear elasticity — strain has $1/\sqrt{r}$ singularity. A crack surface displacement in an infinite nonhomogeneous plane under uniform crack surface shear loading is shown in Figure 3, in which one can see that the tangent line at the two crack-tips has infinite slope.

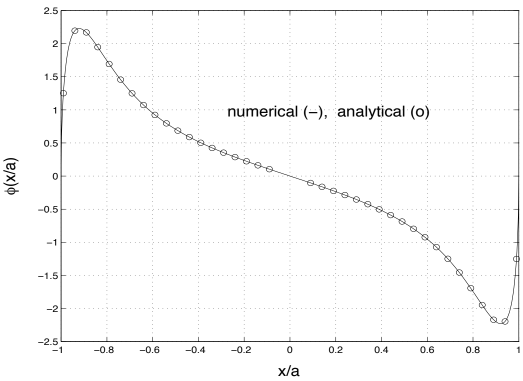

Figure 2: Numerical solution vs. closed form solution for antiplane shear problem under the case $\beta = 0$ and $\ell = \ell' = 0$. All variables have been normalized by the half crack length a.

4 Conclusions

In this paper we have investigated how the hypersingular integral equations arise either due to the choice of unknown density function or the underlying elasticity theory. The hypersingular integral is regularized by the Hadamard finite part integral, and it leads to a very stable numerical approximation.

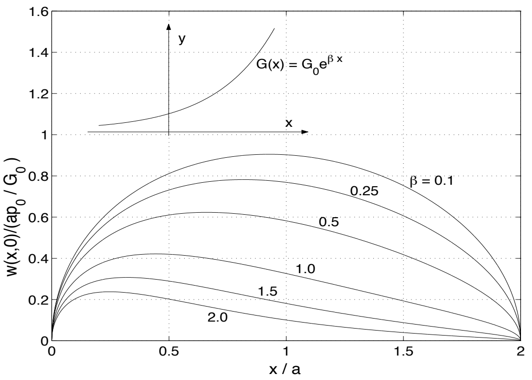

Figure 3: Crack surface displacement in an infinite nonhomogeneous plane under uniform crack surface shear loading $\sigma_{yz}(x,0) = -p_0$ and shear modulus $G(x) = G_0 e^{\beta x}$. Here $a = (d-c)/2$ denotes the half crack length.

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