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Invariance principle for inertial-scale behavior of scalar fields in Kolmogorov-type turbulence

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Abstract

We prove limit theorems for small-scale pair dispersion in synthetic velocity fields with power-law spatial spectra and wavenumber dependent correlation times. These limit theorems are related to a family of generalized Richardson's laws with a limiting case corresponding to Richardson's t^3- and 4/3-laws. We also characterize a regime of positive dissipation of passive scalars.

Keywords: Invariance principle; Inertial-scale behavior; Kolmogorov-type turbulence

1. Introduction

The celebrated Richardson's t^3-law [36] states that a pair of particles located at $(x^{(0)}(t), x^{(1)}(t)) \in \mathbb{R}^{2d}$ being transported in the incompressible turbulence satisfies

$$\mathbb{E}|x^{(1)}(t) - x^{(0)}(t)|^2 \approx C_R \bar{\varepsilon} t^3 \quad \text{for } \ell_1 \ll |x^{(1)}(t) - x^{(0)}(t)| \ll \ell_0, \tag{1}$$

where $\bar{\varepsilon}$ is the energy dissipation rate, C_R the Richardson constant and ℓ_0 and ℓ_1 are respectively the integral and viscous scales. Here and below $\mathbb{E}$ stands for the expectations w.r.t. the ensemble of the velocity fields. This law has been confirmed experimentally [22,31,39] and numerically [4,11,18,43]. A stronger statement is that the relative diffusivity of the tracer particles is proportional to the 4/3 power of their momentary separation, and this is called Richardson's 4/3-law [36], see also [1,7,29,32]. This paper presents several small-scale limit theorems (Theorems 1–3) related to the Richardson's laws for a family of colored-noise-in-time velocity fields that have Kolmogorov-type spatial spectra and wavenumber dependent correlation times. The other aspect of the scaling limit concerns the dissipation of the scalar field in the limit of vanishing molecular diffusion (Corollaries 1 and 2).

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The nature of time correlation in fully developed turbulences in the inertial range is not entirely clear (see [30] and the references therein). But it seems reasonable to assume that, to the leading order, the temporal correlation structure of the *Eulerian* velocity field $u(t, x)$ is determined by the energy-containing velocity components above the integral scale, consistent with Taylor's hypothesis commonly used in the fluid flow measurements in the presence of a mean flow or the random sweeping hypothesis in the absence of a mean flow (see [35,40]). In both cases the temporal correlation function on the small scales is anisotropic and depends on external forcing. The more robust features of small-scale turbulence can be revealed by considering the relative velocity field $U(t, x) = u(t, x + x^{(0)}(t)) - u(t, x^{(0)}(t))$, with respect to a reference fluid particle $x^{(0)}(t)$, which tends to preserve invariance properties of the fluid equations. The velocity field $u(t, x + x^{(0)}(t))$ as viewed from a fluid particle, which is a useful tool for turbulence modeling [2,23], is called the quasi-Lagrangian velocity field in the physics literature and is an example of the general notion of the Lagrangian environment process [15,33,34].

We assume [13,16] that the two-time structure function of $U(t, x)$ has the power-law form

$$\begin{aligned}&\mathbb{E}[U(t,x)-U(t,y)]\otimes[U(s,x)-U(s,y)]\\&\quad=\int_{\mathbb{R}^d}2[1-\cos(k\cdot(x-y))]\exp(-a|k|^{2\beta}|t-s|)\mathcal{E}_{(\ell_1,\ell_0)}(\alpha,k)|k|^{1-d}\,\mathrm{d}k,\\&\alpha\in(1,2),\quad\beta>0,\quad a>0\end{aligned}\tag{2}$$

with the energy spectrum

$$\mathcal{E}_{(\ell_1,\ell_0)}(\alpha,k)=\begin{cases}E_0(\mathbb{I}-k\otimes k|k|^{-2})|k|^{1-2\alpha} & \text{for } |k|\in(\ell_0^{-1},\ell_1^{-1}),\\ 0 & \text{for } |k|\notin(\ell_0^{-1},\ell_1^{-1}),\end{cases}\quad \ell_0<\infty,\quad \ell_1>0,\quad E_0>0,\tag{3}$$

where ℓ_1 and ℓ_0 are respectively the viscous and integral scales. The assumed temporally stationary vector field $U(t, x)$ has homogeneous spatial increments and its expectation $\mathbb{E}_s[U(t, x)]$, conditioning on the events up to time $s < t$, is assumed to admit the spectral representation

$$\mathbb{E}_s[U(t,x)-U(t,y)]=\int_{\mathbb{R}}[1-\exp(\mathrm{i}k\cdot(x-y))]\exp(-a|k|^{2\beta}|t-s|)\hat{U}(s,\mathrm{d}k),\qquad s<t,\tag{4}$$

where $\hat{U}(t, k)$ is a time-stationary process with uncorrelated increments over k such that

$$\mathbb{E}[\hat{U}(t,\mathrm{d}k)\hat{U}^*(t,\mathrm{d}k')]=\mathcal{E}_{(\ell_1,\ell_0)}(\alpha,k)\delta(k-k')\,\mathrm{d}k\,\mathrm{d}k'\quad\forall t,k,k'.\tag{5}$$

The exponential form of the temporal correlation in (2) and (4) is not important for us; it can be replaced by a more general one like

$$\rho(a|k|^{2\beta}|t-s|)$$

with an integrable function $\rho(\tau)$ decaying to zero as $\tau \to \infty$. Since the exponential form seems to agree well with the Lagrangian measurements (see [37] for the Reynolds number around 100 and [41] for high Reynolds numbers) we will use it for the sake of simplicity.

Set the rescaled velocity

$$U_\lambda(t,x)\equiv\lambda^{1-\alpha}U(\lambda^{2\beta}t,\lambda x).\tag{6}$$

Then $U_\lambda(t, x)$ has the energy spectrum

$$\mathcal{E}_{(\ell_1\lambda^{-1},\ell_0\lambda^{-1})}(\alpha,k)=\begin{cases}E_0(\mathbb{I}-k\otimes k|k|^{-2})|k|^{1-2\alpha} & \text{for } |k|\in(\ell_0^{-1}\lambda,\ell_1^{-1}\lambda),\\ 0 & \text{else.}\end{cases}\tag{7}$$

However, we do *not* assume in this paper the full scale-invariance, namely,

$$U_\lambda(t,x) \stackrel{\mathrm{d}}{=} U(t,x) \quad \text{for} \;\; \ell_1 = 0, \;\; \ell_0 = \infty, \tag{8}$$

where $=^{\mathrm{d}}$ means the identity of the distributions. Instead, we assume the weaker assumption of the 4th order scale-invariance, i.e. that up to the 4th moments of the velocity field can be estimated in term of the energy spectrum as in the case of Gaussian fields.

The viscous and integral scales ℓ_1 and ℓ_0 can be related to each other via the Reynolds number *Re* as

$$\frac{\ell_0}{\ell_1} \sim Re^{1/(4-2\alpha)}$$

by using the positivity of kinetic energy dissipation of fluid in the limit $Re \to \infty$. The correlation time $a^{-1}|k|^{-2\beta}$ decreases as the wavenumber k increases. The spatial Hurst exponent of the velocity equals $\alpha - 1$ in the inertial range (ℓ_1, ℓ_0). It should be noted that because of the temporal stationarity of the Lagrangian field $u(t, x + x^{(0)}(t))$ [15,42], $U(t, x)$ has the same one-time statistics as the Eulerian velocity $u(t, x)$; in particular they share the same energy spectrum, but their multiple-time statistics are usually different. We could work with the modified von Karman spectrum but it is irrelevant for our purpose since we are concerned with transport in the inertial-convective range.

It is convenient to express the coefficients E_0, a in terms of U_0, the root mean-square *longitudinal* velocity increment over the integral length ℓ_0, as

$$E_0 \approx C_\alpha U_0^2 \ell_0^{2-2\alpha}, \qquad a \approx c_0 \ell_0^{2\beta-1} U_0 \quad \text{as} \;\; \frac{\ell_0}{\ell_1} \to \infty \tag{9}$$

with dimensionless constants c_0 and

$$C_\alpha = \frac{(4\pi)^{d/2} 2^{2\alpha-3} (2\alpha-2) \Gamma(\alpha + d/2)}{(d-1)\Gamma(2-\alpha)}, \tag{10}$$

where $\Gamma(r)$ is the Gamma function.

Assuming that the lifetime (i.e. correlation time $\tau(k) = a^{-1}|k|^{-2\beta}$) of eddy of size $|k|^{-1}$ is same as its turnover time one gets the relation

$$\alpha + 2\beta = 2. \tag{11}$$

Assuming that the energy flux given by $\mathcal{E}_{(\ell_1,\ell_0)}|k|/\tau(k)$ is constant across the scales in the inertial range one gets the relation

$$\alpha - \beta = 1. \tag{12}$$

The values of parameters satisfying both Eqs. (11) and (12) correspond to the Kolmogorov spectrum with $\alpha = 4/3$, $\beta = 1/3$. For the Kolmogorov spectrum, one has the expression, by estimating $\bar{\varepsilon}$ by $U_0^3 \ell_0^{-1}$,

$$E_0 \approx C_\alpha \bar{\varepsilon}^{2/3}, \qquad a \approx c_0 \bar{\varepsilon}^{1/3}. \tag{13}$$

Writing $x(t) = x^{(1)}(t) - x^{(0)}(t)$ and adding the molecular diffusivity κ we have the following Itô's stochastic equation for the pair separation $x(t)$

$$\mathrm{d}x(t) = [u(t, x^{(0)}(t) + x(t)) - u(t, x^{(0)}(t))]\,\mathrm{d}t + \sqrt{\kappa}\,\mathrm{d}w(t) = U(t, x(t))\,\mathrm{d}t + \sqrt{\kappa}\,\mathrm{d}w(t),$$

where $w(t)$ is the standard Brownian motion in $\mathbb{R}^d$. It is also useful to consider the associated backward stochastic flow which is the solution of the backward stochastic differential equation

$$\mathrm{d}\Phi_s^t(x) = -U(s, \Phi_s^t(x))\,\mathrm{d}s + \sqrt{\kappa}\,\mathrm{d}w(t), \quad 0 \le s \le t, \tag{14}$$

$$\Phi_t^t(x) = x. \tag{15}$$

Denote by $\mathbb{M}$ the expectation with respect to the molecular diffusion and consider the scalar field $T(t, x)$

$$T(t, x) \equiv \mathbb{M}[T_0(\Phi_0^t(x))], \tag{16}$$

which satisfies the advection–diffusion equation

$$\frac{\partial T(t, x)}{\partial t} = U(t, x) \cdot \nabla T(t, x) + \frac{\kappa}{2}\Delta T(t, x), \quad T(0, x) = T_0(x). \tag{17}$$

We interpret Eq. (17) in the weak sense

$$\langle T(t, \cdot), \theta\rangle - \langle T_0, \theta\rangle = \frac{\tilde{\kappa}}{2}\int_0^t \langle T(s, \cdot), \Delta\theta\rangle \, \mathrm{d}s - \int_0^t \langle T(s, \cdot), V(s, \cdot) \cdot \nabla\theta\rangle \, \mathrm{d}s \tag{18}$$

for any test function $\theta \in C_c^\infty(\mathbb{R}^d)$, the space of smooth functions with compact supports.

To study the small-scale behavior we introduce the following scaling limit. First we assume that the integral and viscous scales of the field U are $\ell_0 = \varepsilon L$, $\ell_1 = \varepsilon/K$ with L, K tending to ∞ in a way to be specified later. Then we re-scale the variables $x \to \varepsilon x$, $t \to \varepsilon^{2q} t$ amounting to consider the re-scaled pair separation

$$x^\varepsilon(t) = \varepsilon^{-1} x(\varepsilon^{2q} t).$$

The scaling parameter ε will tend to zero, indicating that we are considering the emergent inertial range of scales $\ell_1 \ll |x| \ll \ell_0$ (since $K, L \to \infty$) as a result of a large Reynolds number. We also set

$$\kappa = \varepsilon^{2-2q}\tilde{\kappa} \quad \text{with} \quad \tilde{\kappa} = \tilde{\kappa}(\varepsilon). \tag{19}$$

After re-scaling, the advection–diffusion equation becomes

$$\frac{\partial T^\varepsilon}{\partial t} = \varepsilon^{2q-1} U(\varepsilon^{2q} t, \varepsilon x) \cdot \nabla T^\varepsilon + \frac{\tilde{\kappa}}{2}\Delta T^\varepsilon. \tag{20}$$

We take the initial data $T^\varepsilon(0, x) = T_0(x) \in L^\infty(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. Let

$$V(t, x) = \varepsilon^{1-\alpha} U(\varepsilon^{2\beta} t, \varepsilon x).$$

As before (cf. (7)) the energy spectrum of the rescaled field V is given by

$$\mathcal{E}_{K,L}(\alpha, k) = \begin{cases} E_0(\mathbb{I} - k \otimes k|k|^{-2})|k|^{1-2\alpha} & \text{for } |k| \in (L^{-1}, K), \\ 0 & \text{else.} \end{cases}$$

We rewrite Eq. (20) in terms of V as

$$\frac{\partial T^\varepsilon}{\partial t} = \varepsilon^{2q+\alpha-2} V(\varepsilon^{2(q-\beta)} t, x) \cdot \nabla T^\varepsilon + \frac{\tilde{\kappa}}{2}\Delta T^\varepsilon, \quad T^\varepsilon(0, x) = T_0(x). \tag{21}$$

A simple, non-trivial scaling limit is the white-noise limit when

$$q < \beta \tag{22}$$

and

$$q = 2 - \alpha - \beta \tag{23}$$

resulting from equating $2q + \alpha - 2$ and $q - \beta$. Inequalities (22) and (23) then gives the condition

$$\alpha + 2\beta > 2. \tag{24}$$

Note that for

$$\alpha + \beta < 2 \tag{25}$$

and thus $q > 0$ we have a short-time limit; otherwise, it is a long time (but small spatial scale) limit.

The paper is organized as follows. In Section 2 we state the main results and discuss their implications. In Section 3 we discuss the meaning of solutions for the colored-noise and white-noise models and prove the uniqueness for the latter. In Section 4, we prove Theorem 1: we prove the tightness of the measures in Section 4.1 and, in Section 4.2, identify the limiting measure by the martingale formulation. In Section 5, we prove Theorem 2. The method of proof is the same as that in [14] (see also [5]). We refer the reader to [26] for the full exposition of the perturbed test function method used here. We note that the method of Kunita [24] requires sub-Gaussian behavior and spatial regularity of the velocity field and is not applicable here.

2. Main theorems and interpretation

Let us begin by briefly recalling the Kraichnan model. The model has a white-noise-in-time incompressible velocity field which can be described as the time derivative of a zero mean, isotropic Brownian vector field B_t with the two-time structure function

$$\mathbb{E}[B_t(x) - B_t(y)] \otimes [B_s(x) - B_s(y)] = \min(t, s) \int 2[1 - \cos(k \cdot (x - y))]a^{-1}\bar{\mathcal{E}}_L(\eta + 1, k)|k|^{1-d}\,\mathrm{d}k,$$
$$\eta \in (0, 1) \tag{26}$$

with

$$\bar{\mathcal{E}}_L(\eta + 1, k) = \lim_{K \to \infty} \mathcal{E}_{K,L}(\eta + 1, k).$$

In this paper, we interpret the corresponding advection–diffusion equation for the Kraichnan model in the sense of Stratonovich's integral

$$\mathrm{d}T_t(x) = [\nabla T_t(x)]^\dagger \circ [\mathrm{d}B_t(x) - \mathrm{d}B_t(0)] + \frac{\kappa_0}{2}\Delta T_t(x)\,\mathrm{d}t, \quad \kappa_0 \geq 0, \quad T(0, x) = T_0(x), \tag{27}$$

which can be rewritten as an Itô's SDE

$$\mathrm{d}T_t = \left(\frac{\kappa_0}{2}\Delta + \frac{1}{a}\bar{\mathcal{B}}\right) T_t\,\mathrm{d}t + \sqrt{2}a^{-1/2}\nabla T_t \cdot \mathrm{d}\bar{W}_t^{(1)}, \tag{28}$$

where $\bar{W}_t^{(1)}(x)$ is the Brownian vector field with the spatial covariance

$$\bar{\Gamma}^{(1)}(x, y) = \int [\exp(\mathrm{i}k \cdot x) - 1][\exp(-\mathrm{i}k \cdot y) - 1]\bar{\mathcal{E}}_L(\eta + 1, k)|k|^{1-d}\,\mathrm{d}k, \quad \eta = \alpha + \beta - 1 \tag{29}$$

and the operator $\bar{\mathcal{B}}$ is given by

$$\bar{\mathcal{B}}\phi(x) = \sum_{i,j} \bar{\Gamma}_{ij}^{(1)}(x, x)\frac{\partial^2 \phi(x)}{\partial x^i \partial x^j}, \quad \phi \in C^\infty(\mathbb{R}^d). \tag{30}$$

We will discuss the meaning of solutions for the Kraichnan model and prove the uniqueness property in Section 3. The Kraichnan model for passive scalar has been widely studied to understand turbulent transport in the inertial range because of its tractability (see, e.g., [6,10,12,19,20,28,30,38] and the references therein). The tractability of this model lies in the Gaussian and white-noise nature of the velocity field.

Theorem 1. *Suppose $\alpha+2\beta>2$. Let $L<\infty$ be fixed and let $K=K(\varepsilon)$ such that $\lim_{\varepsilon\to 0}K=\infty$. Let $\tilde{\kappa}=\tilde{\kappa}(\varepsilon)>0$ such that $\lim_{\varepsilon\to 0}\tilde{\kappa}=\kappa_0<\infty$. Let $T_0\in L^\infty(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$. If, additionally, any one of the following conditions is satisfied*:

(i) $\alpha+2\beta>4$;
(ii) $\alpha+2\beta=4$, $\lim_{\varepsilon\to 0}\tilde{\kappa}\varepsilon^2\sqrt{\log K}=0$;
(iii) $3<\alpha+2\beta<4$, $\lim_{\varepsilon\to 0}\tilde{\kappa}\varepsilon^2 K^{4-\alpha-2\beta}=0$;
(iv) $\alpha+2\beta=3$, $\lim_{\varepsilon\to 0}\tilde{\kappa}\varepsilon^2 K=\lim_{\varepsilon\to 0}\varepsilon\sqrt{\log K}=0$;
(v) $2<\alpha+2\beta<3$, $\lim_{\varepsilon\to 0}\tilde{\kappa}\varepsilon^2 K^{4-\alpha-2\beta}=\lim_{\varepsilon\to 0}\varepsilon K^{3-\alpha-2\beta}=0$.

Then for the exponent q given in (23) the solution T_t^ε of (21) converges in distribution, as $\varepsilon\to 0$, in the space $D([0,\infty);L^\infty_{w^*}(\mathbb{R}^d))$ to the scalar field T_t for pair dispersion in the Kraichnan model in the time interval $[0,t_0]\ \forall t_0<\infty$. The limiting Kraichnan model has the spatial covariance given by (29). Here $D([0,\infty);L^\infty_{w^*}(\mathbb{R}^d))$ is the space of $L^\infty(\mathbb{R}^d)$-valued right continuous processes with left limits endowed with the Skorohod metric [3] and $L^\infty_{w^*}(\mathbb{R}^d)$ is the standard space $L^\infty(\mathbb{R}^d)$ endowed with the weak* topology.

Remark 1. In addition to the assumptions stated in Section 1 and in the theorem, we use in the proof of Theorem 1 the assumption

$$\sup_{t<t_0}\int_{|x|\le M}|\tilde{V}_t^\varepsilon(x)|\,\mathrm{d}x=\mathrm{o}\left(\frac{1}{\varepsilon}\right),\qquad \varepsilon\to 0\ \ \forall 0<M<\infty \tag{31}$$

with a random constant possessing a finite moment where

$$\tilde{V}_t^\varepsilon(x)=\frac{1}{\varepsilon^2}\int_t^\infty \mathbb{E}_{t\varepsilon^{-2}}V\left(\frac{s}{\varepsilon^2},x\right)\mathrm{d}s.$$

For Gaussian velocity fields one has

$$M^d\sup_{\substack{|x|\le M\\ t\le t_0}}\left|\tilde{V}\left(\frac{t}{\varepsilon^2},x\right)\right|\le CL^{\alpha+2\beta-2}\log\left[\frac{M^d t_0}{\varepsilon^2}\right]=\mathrm{o}\left(\frac{1}{\varepsilon}\right), \tag{32}$$

where the random constant C has a Gaussian-like tail by Chernoff's bound. Condition (31) allows certain degree of intermittency in the velocity field.

Note that, in Theorem 1, when $\kappa_0>0$ and $2<\alpha+2\beta<3$, $\lim_{\varepsilon\to 0}\tilde{\kappa}\varepsilon^2K^{4-\alpha-2\beta}=0$ implies $\lim_{\varepsilon\to 0}\varepsilon K^{3-\alpha-2\beta}=0$. Also, $\alpha+2\beta<3$ contains the regime $\alpha+\beta<2$ in which the limiting Brownian velocity field is spatially Hölder continuous and has a Hurst exponent $\eta=\alpha+\beta-1\in(1/2,1)$, i.e. the limiting velocity field has a *persistent* spatial correlation.

If we let $L\to\infty$ in the Kraichnan model, we see that it gives rise to a Brownian velocity field $\bar{B}_t$ with the structure function

$$\mathbb{E}[\bar{B}_t(x)-\bar{B}_t(y)]\otimes[\bar{B}_s(x)-\bar{B}_s(y)]=\min(t,s)\int 2[1-\cos(k\cdot(x-y))]a^{-1}\bar{\bar{\mathcal{E}}}(\alpha+\beta,k)|k|^{1-d}\,\mathrm{d}k, \tag{33}$$

where

$$\bar{\bar{\mathcal{E}}}(\alpha+\beta,k)=\lim_{L\to\infty}\bar{\mathcal{E}}_L(\alpha+\beta,k).$$

The spectral integral in (33) is convergent only for $\alpha+\beta<2$. The convergence of the integral in (33) means that the limiting Brownian velocity field $\bar{B}_t$ has spatially homogeneous increments.

We can prove the convergence to the Kraichnan model with velocity field $\bar{B}_t$ in the simultaneous limit of $\varepsilon \to 0$, $K, L \to \infty$ if additional conditions are satisfied.

Theorem 2. *Suppose $\alpha + \beta < 2$ and all the assumptions of Theorem 1 (thus, only regime (v) is relevant) except for the finiteness of L. Instead, let $L = L(\varepsilon) \to \infty$ such that*

$$\lim_{\varepsilon\to 0} L^{2(\alpha+2\beta-2)}\varepsilon = 0. \tag{34}$$

Then the same convergence holds as in Theorem 1. The limiting Brownian velocity field $\bar{B}_t$ has the structure function given by (33).

Remark 2. In addition to the assumptions of Theorem 1 (cf. Remark 1) we use in the proof of Theorem 2 the assumption

$$\sup_{t<t_0}\int_{|x|\le M} |\tilde{V}_t^\varepsilon(x)|^2\,\mathrm{d}x \le CL^{2(\alpha+2\beta-2)}\frac{1}{\varepsilon},\quad \varepsilon\to 0,\ \ L\to\infty\ \ \forall 0<M<\infty \tag{35}$$

with a random constant C possessing a finite moment. For Gaussian velocity fields one has

$$\sup_{t<t_0}\int_{|x|\le M} |\tilde{V}_t^\varepsilon(x)|^2\,\mathrm{d}x \le CL^{2(\alpha+2\beta-2)}\left(\log\frac{1}{\varepsilon}\right)^2,\quad \varepsilon\to 0,\ \ L\to\infty\ \ \forall 0<M<\infty.$$

One sees that condition (35) is in some sense more tolerant of intermittency than (31) is.

Due to the divergence-free property of the velocity field, the pre-limit scalar field satisfies the energy identity [29, Chapter III, Theorem 7.2]

$$\int |T_t^\varepsilon(x)|^2\,\mathrm{d}x + \tilde{\kappa}\int_0^t\int |\nabla T_t^\varepsilon|^2(x)\,\mathrm{d}x\,\mathrm{d}s = \int |T_0(x)|^2\,\mathrm{d}x \tag{36}$$

provided that $T_0 \in L^2(\mathbb{R}^d)$. From (36) we have the estimates

$$\|T_t^\varepsilon\|_2^2 < \|T_0\|^2,\qquad \int_0^t \|T_s^\varepsilon\|_{H^1}^2\,\mathrm{d}s \le \left(t+\frac{1}{\tilde{\kappa}}\right)\|T_0\|_2^2,\quad t>0,$$

where $\|\cdot\|_{H^1}$ is the norm of the standard Sobolev space $H^1(\mathbb{R}^d)$ of square-integrable functions with square-integrable first derivative. Thus the law of T^ε is naturally supported by the space of continuous $L^2(\mathbb{R}^d)$-valued processes which are also in $L^2_{\mathrm{loc}}([0,\infty); H^1(\mathbb{R}^d))$. Following [5] we consider the space

$$\Omega = D([0,\infty); L^2_{\mathrm{w}}(\mathbb{R}^d)\cap L^\infty_{\mathrm{w}^*}(\mathbb{R}^d))\cap L^2_{\mathrm{w,loc}}([0,\infty); H^1_{\mathrm{w}}(\mathbb{R}^d)),$$

where the subscripts w and loc denote the weak and the local topologies, respectively.

In the case of $\tilde{\kappa} > 0$, $\kappa_0 > 0$ the above observation and the tightness argument for Theorems 1 and 2 then imply the tightness of T_t^ε in the space Ω. We have the following corollary.

Corollary 1. *If $\kappa_0 > 0$ and $T_0 \in L^\infty(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$ then the convergence holds in the space Ω in the following regimes:*

Case 1: Let $L < \infty$ be fixed and $K \to \infty$ as $\varepsilon \to 0$.

(i) $\alpha + 2\beta > 4$;
(ii) $\alpha + 2\beta = 4$, $\lim_{\varepsilon\to 0}\varepsilon^2\sqrt{\log K} = 0$;

(iii) $2 < \alpha + 2\beta < 4$, $\lim_{\varepsilon\to 0} \varepsilon^2 K^{4-\alpha-2\beta} = 0$.

Case 2: *Suppose $\alpha + \beta < 2 < \alpha + 2\beta$ and $L, K \to \infty$ as $\varepsilon \to 0$ such that*

$$\lim_{\varepsilon\to 0} \varepsilon^2 K^{4-\alpha-2\beta} = \lim_{\varepsilon\to 0} L^{2(\alpha+2\beta-2)}\varepsilon = 0.$$

In particular,

$$\|T_0\|_2^2 - \limsup_{\varepsilon\to 0} \mathbb{E}[\|T_t^\varepsilon\|_2^2] = \liminf_{\varepsilon\to 0} \tilde{\kappa} \int_0^t \mathbb{E}[\|\nabla T_s^\varepsilon\|_2^2]\,\mathrm{d}s \geq \kappa_0 \int_0^t \mathbb{E}[\|\nabla T_s\|_2^2]\,\mathrm{d}s > 0,$$
$$t > 0, \quad \text{unless} \quad T_s \equiv 0, \quad 0 \leq s \leq t, \tag{37}$$

where T_t is the solution of the corresponding Kraichnan model.

In the case of $\tilde{\kappa} > 0$, $\kappa_0 = 0$ and $T_0 \in L^2 \cap L^\infty$, the limiting Kraichnan model conserves the L^2-norm of T_t. The energy identity (36) then implies

$$\|T_t^\varepsilon\|_2^2 \leq \|T_0\|_2^2 = \|T_t\|_2^2 \quad \forall \varepsilon > 0 \ \ \forall t > 0,$$

which in turn implies $\lim_{\varepsilon\to 0}\|T_t^\varepsilon\|_2 = \|T_t\|_2$. Hence the weak sense of convergence in Theorems 1 and 2 can be strengthened to the strong L^2 convergence.

Corollary 2. *If $\kappa_0 = 0$ and $T_0 \in L^\infty(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ then the convergence holds in the space $D([0,\infty); L^2(\mathbb{R}^d) \cap L^\infty_{w*}(\mathbb{R}^d))$ in the respective regimes listed in Theorems 1 and 2. In particular*,

$$\|T_0\|_2^2 - \lim_{\varepsilon\to 0} \mathbb{E}[\|T_t^\varepsilon\|_2^2] = 0, \qquad t > 0 \ \ a.e.$$

We see that in the context of Corollary 1 there is positive dissipation (37) while there is none in the context of Corollary 2. The conditions of the limit theorems set a constraint for the presence of positive dissipation: on the observation scale ε, if the molecular diffusion κ is of order ε^{2-2q}, then there is always positive dissipation no matter how slow ℓ_1 vanishes. On the other hand, if $\kappa \ll \varepsilon^{2-2q}$ (i.e. $\kappa_0 = 0$) and the dissipation is positive, then

$$\ell_1 = \mathrm{O}(\varepsilon^\nu), \qquad \nu = \frac{4-\alpha-2\beta}{3-\alpha-2\beta}$$

with $\nu \in (2,\infty)$ in the regime $\alpha + \beta < 2 < \alpha + 2\beta$ (cf. (41)). An open question is whether there is a positive dissipation as $\varepsilon, \kappa \to 0$ with $\ell_1 = 0$ at the outset. If there is, then the Kraichnan model (27) is unlikely to be the governing equation of the scaling limit (if exists).

In the case of $\tilde{\kappa} = 0$, a still stronger sense of convergence holds since now Eq. (21) is of first order and any locally bounded measurable function $\phi(T^\varepsilon)$ of the scalar field satisfies the same equation (18) with $\tilde{\kappa} = 0$. The same argument for the proof of Theorems 1 and 2 will then yield the following theorem.

Theorem 3. *Assume the conditions stated in Remarks 1 and 2. Let $\tilde{\kappa} = 0$, $T_0, \phi(T_0) \in L^\infty(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ and ϕ is a locally bounded measurable function from $\mathbb{R}$ to $\mathbb{R}$. Then $T_t^\varepsilon, \phi(T_t^\varepsilon)$ converge in the space $D([0,\infty); L^\infty_{w*}(\mathbb{R}^d) \cap L^2(\mathbb{R}^d))$ to the corresponding Kraichnan model in the following regimes.*

Case 1: *Let $L < \infty$ be fixed and $K \to \infty$ as $\varepsilon \to 0$.*

(i) $\alpha + 2\beta > 3$;
(ii) $\alpha + 2\beta = 3$, $\lim_{\varepsilon\to 0} \varepsilon\sqrt{\log K} = 0$;

(iii) $2 < \alpha + 2\beta < 3$, $\lim_{\varepsilon\to 0} \varepsilon K^{3-\alpha-2\beta} = 0$.

Case 2: Suppose $\alpha + \beta < 2 < \alpha + 2\beta$ and $L, K \to \infty$ as $\varepsilon \to 0$ such that

$$\lim_{\varepsilon\to 0} \varepsilon K^{3-\alpha-2\beta} = \lim_{\varepsilon\to 0} L^{2(\alpha+2\beta-2)}\varepsilon = 0.$$

Remark 3. The assertions of Theorems 1–3 and Corollaries 1 and 2 hold true for random as well as deterministic initial data.

When the parameters are in the regime $\alpha + \beta < 2 < \alpha + 2\beta$, by taking the expectation in the Itô's equation with the Brownian velocity field $\bar{B}_t$ one sees readily that the *longitudinal* relative diffusion coefficient is given by

$$\frac{\kappa_0}{2} + \frac{1}{a}\frac{x}{|x|} \cdot \bar{\bar{\Gamma}}^{(1)}(x, x) \cdot \frac{x}{|x|} \approx \frac{1}{a} C^{-1}_{\alpha+\beta} E_0 |x|^{2\eta} \quad \text{for } \kappa_0 \ll 1, \quad \eta = 1 - q = \alpha + \beta - 1 \in \left(\frac{1}{2}, 1\right) \tag{38}$$

with

$$\bar{\bar{\Gamma}}^{(1)}(x, x) = \lim_{L\to\infty} \bar{\Gamma}^{(1)}(x, x) = C^{-1}_{\alpha+\beta} E_0 |x|^{2\eta}\left[\left(1 + \frac{2(\alpha+\beta-1)}{d-1}\right)\mathbb{I} - \frac{2(\alpha+\beta-1)}{d-1} x \otimes x |x|^{-2}\right],$$

where $C_{\alpha+\beta}$ is defined as in (10), except with α replaced by $\alpha + \beta$. The exponent q is related to the exponent p in the expression for the mean-square pair separation as follows:

$$\mathbb{E}|x|^2(t) \sim a^{-p} E_0^p t^p, \quad p = \frac{1}{q} = \frac{1}{2 - \alpha - \beta} \tag{39}$$

up to a dimensionless constant depending only on $\alpha+\beta$. Expressions (38) and (39) can be viewed as the generalization of Richardson's t^3- and 4/3-laws, respectively. In general, $p \in (2, \infty)$, indicating super-ballistic (i.e. accelerating) motion as a result of a scale-dependent relative diffusivity.

We now remark on the range of scales for which Theorem 2 is proved and Richardson's laws can be reasonably interpreted. Let ε be the scale of dispersion. Then the limit theorem holds in the range

$$\varepsilon \ll \min\left[\ell_1^{\gamma}, \left(\frac{1}{\ell_0}\right)^{2(\alpha+2\beta-2)/(5-2\alpha-4\beta)}\right], \quad \gamma = \begin{cases} \dfrac{3-\alpha-2\beta}{4-\alpha-2\beta} & \text{if } \kappa_0 = 0, \\[2ex] \dfrac{4-\alpha-2\beta}{6-\alpha-2\beta} & \text{if } \kappa_0 > 0. \end{cases} \tag{40}$$

In the usual situation with $\ell_0 = O(1)$ the range of scales covered by the limit theorem has an upper limit of

$$\ell_1^{\gamma} \quad \text{with } \gamma \in \begin{cases} (0, \frac{1}{2}) & \text{if } \kappa_0 = 0 \\ (\frac{1}{3}, \frac{1}{2}) & \text{if } \kappa_0 > 0 \end{cases} \quad \text{for } \alpha + 2\beta > 2 > \alpha + \beta, \tag{41}$$

which is limited to the low end of the inertial range depending on α, β, κ_0. It is not clear whether this is physical or a technical matter. Qualitatively similar restriction of Richardson's laws in synthetic flows has been observed in numerical calculation (cf. [4,18]).

If we stretch the validity of (38) and (39) by taking the limit $\alpha \to 4/3$, $\beta \to 1/3$ from within the valid regime, the resulting exponents are $p = 3$, $2\eta = 4/3$ in accordance with Richardson's laws. On the boundary $\alpha + 2\beta = 2$ the scaling exponent q should be given by

$$q = \beta = 1 - \tfrac{1}{2}\alpha, \tag{42}$$

which also coincides with the limiting value of (23). With (42) and $K, L \to \infty$, the solution of (21) converges to that of the advection–diffusion equation with the molecular diffusivity $\kappa_0 = \lim_{\varepsilon\to 0}\tilde{\kappa}$ and the time-stationary, spatially Hölder continuous velocity field $\bar{V}$ whose two-time correlation function is

$$\mathbb{E}[\bar{V}(t,x)\otimes\bar{V}(s,y)] = \int_{\mathbb{R}^d}[\exp(\mathrm{i}k\cdot x)-1][\exp(-\mathrm{i}k\cdot y)-1]\exp(-a|k|^{2-\alpha}|t-s|)\bar{\bar{\mathcal{E}}}(\alpha,k)|k|^{1-d}\,\mathrm{d}k,$$
$$\alpha\in(1,2),$$

which has the self-similar structure

$$\mathbb{E}[\bar{V}(\lambda^{2\beta}t,\lambda x)\otimes\bar{V}(\lambda^{2\beta}s,\lambda y)] = \lambda^{2\alpha-2}\mathbb{E}[\bar{V}(t,x)\otimes\bar{V}(s,y)].$$

In view of the 4th order scale-invariance property it is reasonable to postulate the temporal self-similarity on the mean-square relative dispersion as $\kappa_0 \to 0$

$$\mathbb{E}|x(t)|^2 = f(E_0,a)t^{1/\beta},$$

which has the same exponent as the limiting case of (39) as $\alpha+2\beta\to 2$, where the unknown function f satisfies the relation

$$f(E_0,\lambda a)\lambda^{-1/\beta} = f(\lambda^{-2}E_0,a)\quad \forall\lambda>0.$$

Dimensional analysis with (9) then leads to the relation

$$\mathbb{E}|x(t)|^2 = \bar{C}_{\mathrm{R}}C_\alpha^{-1/2\beta}E_0^{1/2\beta}t^{1/\beta},$$

where $\bar{C}_{\mathrm{R}}$ is the generalized Richardson constant. For $\beta = 1/3$ the exponent p is 1/3 as predicted by Richardson's t^3-law. However, since the limiting velocity field is non-white-in-time, the notion of relative diffusivity is not strictly well defined. Therefore the temporal memory persists on small or intermediate time scales and the notion of relative diffusivity does not describe accurately the process of relative dispersion on the boundary $\alpha+2\beta = 2$ (cf., e.g., [18,21,30]).

Let us consider the regime $\alpha+2\beta<2$. The correct scaling is to set

$$2q+\alpha-2=0\quad\text{or}\quad q=1-\tfrac{1}{2}\alpha. \tag{43}$$

Then the exponent $2(q-\beta)$ of the temporal scaling in (21) is positive due to $\alpha+2\beta<2$, meaning the time variable is slowed down as $\varepsilon\to 0$. It is easy to see by a regular perturbation argument that the solution T_t^ε converges in the sense described in Theorem 1 to the solution $\bar{T}_t$ of the following equation:

$$\frac{\partial\bar{T}_t}{\partial t} = V(0,x)\cdot\nabla\bar{T}_t + \frac{\kappa_0}{2}\Delta\bar{T}_t,\qquad \bar{T}_0 = T_0\in L^\infty(\mathbb{R}^d)$$

if $\kappa_0>0$. If, however, $\kappa_0=0$, the above equation probably have multiple solutions for a given initial condition. The relation (43) is consistent with the numerical simulation using two-dimensional frozen velocity fields with Kolmogorov-type spectrum [11].

Unlike the previous regime, for either $\alpha+2\beta=2$ or $\alpha+2\beta<2$ there is no restriction on the vanishing rate of ℓ_1.

3. Formulation

From the general theory of parabolic partial differential equations [17], for any fixed $\tilde{\kappa} > 0$, $\varepsilon > 0$, there is a unique $C^{2+\eta}$-solution $T^\varepsilon_t(x)$, $0 < \forall\eta < \alpha - 1$. But the solutions T^ε_t may lose all the regularity as $\tilde{\kappa} \to 0$, $\varepsilon \to 0$. So we consider the weak formulation of the equation:

$$\langle T^\varepsilon_t, \theta\rangle - \langle T_0, \theta\rangle = \frac{\tilde{\kappa}}{2}\int_0^t \langle T^\varepsilon_s, \Delta\theta\rangle\, \mathrm{d}s - \frac{1}{\varepsilon}\int_0^t \left\langle T^\varepsilon_s, V\left(\frac{s}{\varepsilon^2}, \cdot\right)\cdot\nabla\theta\right\rangle \mathrm{d}s \tag{44}$$

for any test function $\theta \in C^\infty_c(\mathbb{R}^d)$, the space of smooth functions with compact support. On the other hand, the energy identity (36) implies $T^\varepsilon_t \in L^2([0, t_0]; H^1(\mathbb{R}^d))$ if $T_0 \in L^2(\mathbb{R}^d)$. Hence for L^2 initial data the pre-limit measure $\mathbb{P}^\varepsilon$ is supported in the space $L^2([0, t_0]; H^1(\mathbb{R}^d))$ and, by the tightness result (Section 4.1), the limiting measure $\mathbb{P}$ is supported in $L^2_{\mathrm{w}}([0, t_0]; H^1_{\mathrm{w}}(\mathbb{R}^d))$.

As in (14) and (16) the solutions T^ε_t can be represented as

$$T^\varepsilon_t = \mathbb{M}[T_0(\Phi^{t,\varepsilon}_0(x))], \tag{45}$$

where $\Phi^{t,\varepsilon}_s(x)$ is the unique stochastic flow satisfying

$$\mathrm{d}\Phi^{t,\varepsilon}_s(x) = -\frac{1}{\varepsilon}V\left(\frac{s}{\varepsilon^2}, \Phi^{t,\varepsilon}_s(x)\right)\mathrm{d}s + \sqrt{\tilde{\kappa}}\,\mathrm{d}w(t), \quad 0 \le s \le t, \tag{46}$$

$$\Phi^{t,\varepsilon}_t(x) = x. \tag{47}$$

In the case of $\tilde{\kappa} = 0$, $\Phi^{t,\varepsilon}_0(x)\ \forall t$, is almost surely a diffeomorphism of $\mathbb{R}^d$ and $T^\varepsilon_t = T_0(\Phi^{t,\varepsilon}_0(x))$. Moreover, for any locally bounded measurable function $\phi : \mathbb{R} \to \mathbb{R}$, $\phi(T^\varepsilon_t(x)) = (\phi \circ T_0)(\Phi^{t,\varepsilon}_0(x))$.

In view of the averaging in the representation (45) we have the following proposition.

Proposition 1.

$$\|T^\varepsilon_t\|_\infty \le \|T_0\|_\infty \quad a.s.$$

Clearly, Proposition 1 holds for the case of $\tilde{\kappa} = 0$ as well.

For tightness as well as identification of the limit, the following infinitesimal operator $\mathcal{A}^\varepsilon$ will play an important role. Let $V^\varepsilon_t \equiv V(t/\varepsilon^2, \cdot)$. Let $\mathcal{F}^\varepsilon_t$ be the σ-algebras generated by $\{V^\varepsilon_s, s \le t\}$ and $\mathbb{E}^\varepsilon_t$ the corresponding conditional expectation w.r.t. $\mathcal{F}^\varepsilon_t$. Let $\mathcal{M}^\varepsilon$ be the space of measurable function adapted to $\{\mathcal{F}^\varepsilon_t\ \forall t\}$ such that $\sup_{t<t_0}\mathbb{E}|f(t)| < \infty$. We say $f(\cdot) \in \mathcal{D}(\mathcal{A}^\varepsilon)$, the domain of $\mathcal{A}^\varepsilon$, and $\mathcal{A}^\varepsilon f = g$ if $f, g \in \mathcal{M}^\varepsilon$ and for $f^\delta(t) \equiv \delta^{-1}[\mathbb{E}^\varepsilon_t f(t+\delta) - f(t)]$ we have

$$\sup_{t,\delta}\mathbb{E}|f^\delta(t)| < \infty, \qquad \lim_{\delta\to 0}\mathbb{E}|f^\delta(t) - g(t)| = 0 \quad \forall t.$$

For $f(t) = \phi(\langle T^\varepsilon_t, \theta\rangle)$, $f'(t) = \phi'(\langle T^\varepsilon_t, \theta\rangle)\ \forall\phi \in C^\infty(\mathbb{R})$ we have the following expression from (44) and the chain rule:

$$\mathcal{A}^\varepsilon f(t) = \frac{\tilde{\kappa}}{2}f'(t)\langle T^\varepsilon_t, \Delta\theta\rangle - \frac{1}{\varepsilon}f'(t)\langle T^\varepsilon_t, \mathcal{V}^\varepsilon_t(\theta)\rangle, \tag{48}$$

where

$$\mathcal{V}^\varepsilon_t(\theta) \equiv V^\varepsilon_t \cdot \nabla\theta. \tag{49}$$

A main property of $\mathcal{A}^\varepsilon$ is that

$$f(t) - \int_0^t \mathcal{A}^\varepsilon f(s)\,\mathrm{d}s \quad \text{is an } \mathcal{F}_t^\varepsilon\text{-martingale } \forall f \in \mathcal{D}(\mathcal{A}^\varepsilon). \tag{50}$$

Also,

$$\mathbb{E}_s^\varepsilon f(t) - f(s) = \int_0^t \mathbb{E}_s^\varepsilon \mathcal{A}^\varepsilon f(\tau)\,\mathrm{d}\tau \quad \forall s < t \text{ a.s.} \tag{51}$$

(see [25]).

Likewise we formulate the solutions for the Kraichnan model (28) as the solutions to the corresponding martingale problem. Find a measure $\mathbb{P}$ (of T_t) on the space $D([0,\infty); L^\infty_{w^*}(\mathbb{R}^d))$ such that

$$f(\langle T_t,\theta\rangle) - \int_0^t \left\{ f'(\langle T_s,\theta\rangle)\left[\frac{\kappa_0}{2}\langle T_s,\Delta\theta\rangle + \frac{1}{a}\langle T_s,\bar{\mathcal{B}}^*\theta\rangle\right] + \frac{1}{a} f''(\langle T_s,\theta\rangle)\langle\theta,\bar{\mathcal{K}}^{(1)}_{T_s}\theta\rangle\right\}\mathrm{d}s$$
$$\text{is a martingale w.r.t. the filtration of a cylindrical Wiener process, for each } f \in C^\infty(\mathbb{R}), \tag{52}$$

where $\bar{\mathcal{B}}^*$ is the adjoint of $\bar{\mathcal{B}}$ and

$$\langle\theta,\bar{\mathcal{K}}^{(1)}_{T_s}\theta\rangle = \iint T_s(x)T_s(y)\nabla\theta(x)\cdot\bar{\Gamma}^{(1)}(x,y)\cdot\nabla\theta(y)\,\mathrm{d}y \tag{53}$$

with $\bar{\Gamma}^{(1)}(x,y)$ given, respectively, by (29) and

$$\bar{\Gamma}^{(1)}(x,y) = \int [\exp(\mathrm{i}k\cdot x)-1][\exp(-\mathrm{i}k\cdot y)-1]\bar{\bar{\mathcal{E}}}(\eta+1,k)|k|^{1-d}\,\mathrm{d}k, \qquad \eta = \alpha+\beta-1 \tag{54}$$

for $L < \infty$ and $L = \infty$. To identify the limit for the proof of convergence one needs the uniqueness of solution to the martingale problem (52) which can be easily obtained as follows.

Taking expectation of (52) with $f(r) = r^n$, $n \in \mathbb{N}$ we get for the n-point correlation function

$$F_n^t(x_1,x_2,x_3,\dots,x_n) \equiv \mathbb{E}_{T_0}[T_t(x_1)T_t(x_2)\cdots T_t(x_n)]$$

the equation

$$\langle F_n^t,\otimes^n\theta\rangle - \langle F_n^0,\otimes^n\theta\rangle$$
$$= \int_0^t \left[\sum_j \frac{\kappa_0}{2}\langle F_n^s,\theta(x_1)\cdots\Delta\theta(x_j)\cdots\theta(x_n)\rangle + \sum_j \frac{1}{a}\langle F_n^s,\theta(x_1)\cdots\bar{B}^*\theta(x_j)\cdots\theta(x_n)\rangle \right.$$
$$\left. + \sum_{i<j}\frac{2}{a}\langle F_n^s,\bar{\Gamma}^{(1)}(x_i,x_j):\theta(x_1)\cdots\nabla\theta(x_i)\cdots\nabla\theta(x_j)\cdots\theta(x_n)\rangle\right]\mathrm{d}s,$$

which induces a weakly continuous (hence strongly continuous) sub-Markovian semigroup on $L^p(\mathbb{R}^{nd})\,\forall p \in (1,\infty)$. The sub-Markovianity property is inherited from the pre-limit process T_t^ε. The generator of the semigroup is given formally as

$$\mathcal{L}_n\Phi(x_1,\dots,x_n) \equiv \frac{\kappa_0}{2}\sum_{j=1}^n \Delta_{x_j}\Phi + \frac{1}{a}\sum_{i,j=1}^n \bar{\Gamma}^{(1)}(x_i,x_j):\nabla_{x_i}\nabla_{x_j}\Phi, \quad \Phi \in C_c^\infty(\mathbb{R}^{nd}),\ \kappa_0 \geq 0 \tag{55}$$

with the spatial covariance tensor $\bar{\Gamma}^{(1)}(x_i,x_j)$ given by (29) and (54), respectively, for $L < \infty$ and $L = \infty$. Note that the symmetric operator $\mathcal{L}_n$ (55) is an essentially self-adjoint positive operator on $C_c^\infty(\mathbb{R}^N)$, $N = nd$ which then induces a *unique* symmetric Markov semigroup of contractions on $L^2(\mathbb{R}^N)$. The essential self-adjointness is due to the sub-Lipschitz growth of the square-root of $\bar{\Gamma}^{(1)}(x_1, x_2)$ at large $|x_1|, |x_2|$ (hence no escape to infinity) [8].

By Theorem 1.4.1 of [9] this semigroup induces a sub-Markovian C_0-semigroup on $L^p(\mathbb{R}^N)$, $p \in [1, \infty)$. The uniqueness holds for these semigroups in their respective space as well but we will not pursue it here.

4. Proof of Theorem 1

4.1. Tightness

In the sequel we will adopt the following notation

$$f(t) \equiv f(\langle T_t^\varepsilon, \theta\rangle), \qquad f'(t) \equiv f'(\langle T_t^\varepsilon, \theta\rangle), \qquad f''(t) \equiv f''(\langle T_t^\varepsilon, \theta\rangle) \quad \forall f \in C^\infty(\mathbb{R}).$$

Namely, the prime stands for the differentiation w.r.t. the original argument (not t) of f, f', etc.

A family of processes $\{T^\varepsilon, 0 < \varepsilon < 1\} \subset D([0,\infty); L^\infty_{w*}(\mathbb{R}^d))$ is tight if and only if the family of processes $\{\langle T^\varepsilon, \theta\rangle, 0 < \varepsilon < 1\} \subset D([0,\infty); L^\infty_{w*}(\mathbb{R}^d))$ is tight for all $\theta \in C_c^\infty(\mathbb{R}^d)$. We use the tightness criterion of Kushner [28, Chapter 3, Theorem 4], namely, we will prove: firstly,

$$\lim_{N\to\infty} \limsup_{\varepsilon\to 0} \mathbb{P}\{\sup_{t<t_0} |\langle T^\varepsilon, \theta\rangle| \geq N\} = 0 \quad \forall t_0 < \infty. \tag{56}$$

Secondly, for each $f \in C^\infty(\mathbb{R})$ there is a sequence $f^\varepsilon(t) \in \mathcal{D}(\mathcal{A}^\varepsilon)$ such that for each $t_0 < \infty\{\mathcal{A}^\varepsilon f^\varepsilon(t), 0 < \varepsilon < 1, 0 < t < t_0\}$ is uniformly integrable and

$$\lim_{\varepsilon\to 0} \mathbb{P}\{\sup_{t<t_0} |f^\varepsilon(t) - f(\langle T^\varepsilon, \theta\rangle)| \geq \delta\} = 0 \quad \forall \delta > 0. \tag{57}$$

Then it follows that the laws of $\{\langle T^\varepsilon, \theta\rangle, 0 < \varepsilon < 1\}$ are tight in the space of $D([0,\infty); L^\infty_{w*}(\mathbb{R}^d))$.

Condition (56) is satisfied as a result of Proposition 1. Let

$$f_1^\varepsilon(t) \equiv \frac{1}{\varepsilon}\int_t^\infty \mathbb{E}_t^\varepsilon f'(t)\langle T_t^\varepsilon, \mathcal{V}_s^\varepsilon(\theta)\rangle\, \mathrm{d}s$$

be the first perturbation of $f(t)$. We obtain

$$f_1^\varepsilon(t) = \frac{\varepsilon}{a} f'(t)\langle T_t^\varepsilon, \tilde{\mathcal{V}}_t^\varepsilon(\theta)\rangle \tag{58}$$

with

$$\tilde{\mathcal{V}}_t^\varepsilon(\theta) = \tilde{V}_t^\varepsilon \cdot \nabla\theta, \tag{59}$$

$$\tilde{V}_t^\varepsilon \equiv \tilde{V}\left(\frac{t}{\varepsilon^2}, \cdot\right) \equiv \frac{1}{\varepsilon^2}\int_t^\infty \mathbb{E}_t^\varepsilon V_s^\varepsilon\, \mathrm{d}s, \tag{60}$$

where $\tilde{V}$ has the power spectrum $\mathcal{E}_{K,L}(\alpha + 2\beta, k)$ by the spectral representation

$$\mathbb{E}_t^\varepsilon V_s^\varepsilon = \int [\mathrm{e}^{\mathrm{i}x\cdot k} - 1]\, \mathrm{e}^{-a|k|^{2\beta}|s-t|\varepsilon^{-2}}\, \hat{V}_t^\varepsilon(\mathrm{d}k) \quad \forall s \geq t. \tag{61}$$

Note that while V_t^ε loses differentiability as $K \to \infty$, $\tilde{V}_t^\varepsilon$ is almost surely a $C^{1,\eta}$-function in the limit with

$$0 < \forall \eta < \alpha + 2\beta - 2$$

and has uniformly bounded local $W^{1,p}$-norm, $p \geq 1$.

Proposition 2.

$$\lim_{\varepsilon\to 0}\sup_{t<t_0}\mathbb{E}|f_1^\varepsilon(t)|=0,\qquad \lim_{\varepsilon\to 0}\sup_{t<t_0}|f_1^\varepsilon(t)|=0\quad \textit{in probability}.$$

Proof. By Proposition 1 we have

$$\mathbb{E}[|f_1^\varepsilon(t)|]\le \frac{\varepsilon}{a}\|f'\|_\infty\|T_0\|_\infty\|\theta\|_\infty\int_{|x|\le M}\mathbb{E}|\tilde V_t^\varepsilon|\,\mathrm{d}x \tag{62}$$

and

$$\sup_{t<t_0}|f_1^\varepsilon(t)|\le \frac{\varepsilon}{a}\|f'\|_\infty\|T_0\|_\infty\|\theta\|_\infty\sup_{t<t_0}\int_{|x|\le M}|\tilde V_t^\varepsilon|\,\mathrm{d}x. \tag{63}$$

By the temporal stationarity of $\tilde V_t^\varepsilon$ we can replace the terms $\mathbb{E}|\tilde V_t^\varepsilon(x)|$ in (62) by $\mathbb{E}|\tilde V(0,x)|$. By assumption (cf. (31), Remark 1), we have the desired estimate. Proposition 2 now follows from (31), (62) and (63). □

Set $f^\varepsilon(t)=f(t)-f_1^\varepsilon(t)$. A straightforward calculation yields

$$\begin{aligned}\mathcal{A}^\varepsilon f_1^\varepsilon &= -\frac{\tilde\kappa\varepsilon}{2a}f''(t)\langle T_t^\varepsilon,\Delta\theta\rangle\langle T_t^\varepsilon,\tilde{\mathcal{V}}_t^\varepsilon(\theta)\rangle+\frac{\tilde\kappa\varepsilon}{2a}f'(t)\langle T_t^\varepsilon,\Delta\tilde{\mathcal{V}}_t^\varepsilon(\theta)\rangle+\frac{1}{a}f''(t)\langle T_t^\varepsilon,\mathcal{V}_t^\varepsilon(\theta)\rangle\langle T_t^\varepsilon,\tilde{\mathcal{V}}_t^\varepsilon(\theta)\rangle\\ &\quad -\frac{1}{a}f'(t)\langle T_t^\varepsilon,\mathcal{V}_t^\varepsilon(\tilde{\mathcal{V}}_t^\varepsilon(\theta))\rangle-\frac{1}{\varepsilon}f'(t)\langle T_t^\varepsilon,\mathcal{V}_t^\varepsilon(\theta)\rangle\end{aligned}$$

and, hence

$$\begin{aligned}\mathcal{A}^\varepsilon f^\varepsilon(t) &= \frac{\tilde\kappa}{2}f'(t)\langle T_t^\varepsilon,\Delta\theta\rangle+\frac{1}{a}f'(t)\langle T_t^\varepsilon,\mathcal{V}_t^\varepsilon(\tilde{\mathcal{V}}_t^\varepsilon(\theta))\rangle-\frac{1}{a}f''(t)\langle T_t^\varepsilon,\mathcal{V}_t^\varepsilon(\theta)\rangle\langle T_t^\varepsilon,\tilde{\mathcal{V}}_t^\varepsilon(\theta)\rangle\\ &\quad +\frac{\tilde\kappa\varepsilon}{2a}[f''(t)\langle T_t^\varepsilon,\Delta\theta\rangle\langle T_t^\varepsilon,\mathcal{V}_t^\varepsilon(\theta)\rangle-f'(t)\langle T_t^\varepsilon,\Delta\tilde{\mathcal{V}}_t^\varepsilon(\theta)\rangle]=A_1^\varepsilon(t)+A_2^\varepsilon(t)+A_3^\varepsilon(t)+A_4^\varepsilon(t),\end{aligned} \tag{64}$$

where $A_2^\varepsilon(t)$ and $A_3^\varepsilon(t)$ are the O(1) statistical coupling terms.

For the tightness criterion stated in the beginning of the section, it remains to show the folowing proposition.

Proposition 3. $\{\mathcal{A}^\varepsilon f^\varepsilon\}$ *are uniformly integrable and*

$$\lim_{\varepsilon\to 0}\sup_{t<t_0}\mathbb{E}|A_4^\varepsilon(t)|=0.$$

Proof. We show that $\{A_i^\varepsilon\}$, $i=1,2,3,4$ are uniformly integrable. To see this, we have the following estimates:

$$|A_1^\varepsilon(t)|=\frac{\tilde\kappa}{2}|f'(t)\langle T_t^\varepsilon,\Delta\theta\rangle|\le\frac{\tilde\kappa}{2}\|f'\|_\infty\|T_0\|_\infty\|\Delta\theta\|_1.$$

Thus A_1^ε is uniformly integrable since it is uniformly bounded:

$$|A_2^\varepsilon(t)|=\frac{1}{a}|f'(t)\langle T_t^\varepsilon,\mathcal{V}_t^\varepsilon(\tilde{\mathcal{V}}_t^\varepsilon(\theta))\rangle|\le\frac{C}{a}\|f'\|_\infty\|T_0\|_\infty\left[\int_{|x|<M}|V_t^\varepsilon|^2\,\mathrm{d}x\right]^{1/2}\left[\int_{|x|<M}|\nabla\tilde V_t^\varepsilon|^2\,\mathrm{d}x\right]^{1/2}.$$

Similarly,

$$|A_3^\varepsilon(t)|=\frac{1}{a}|f''(t)\langle T_t^\varepsilon,\mathcal{V}_t^\varepsilon(\theta)\rangle\langle T_t^\varepsilon,\tilde{\mathcal{V}}_t^\varepsilon(\theta)\rangle|\le\frac{C}{a}\|f'\|_\infty\|T_0\|_\infty^2\left[\int_{|x|<M}|V_t^\varepsilon|^2\,\mathrm{d}x+\int_{|x|<M}|\tilde V_t^\varepsilon|^2\,\mathrm{d}x\right].$$

Thus A_2^ε and A_3^ε are uniformly integrable in view of the uniform boundedness of the 4th moment of $V_t^\varepsilon, \tilde{V}_t^\varepsilon$ and $\nabla\tilde{V}_t^\varepsilon$ as $L < \infty$ is fixed and $K \to \infty$ (the 4th order scale-invariance):

$$\begin{aligned}|A_4^\varepsilon| &= \frac{\tilde{\kappa}\varepsilon}{2a}|f''(t)\langle T_t^\varepsilon, \Delta\theta\rangle\langle T_t^\varepsilon, \tilde{\mathcal{V}}_t^\varepsilon(\theta)\rangle - f'(t)\langle T_t^\varepsilon, \Delta\tilde{\mathcal{V}}_t^\varepsilon(\theta)\rangle \\ &\le \frac{C\tilde{\kappa}\varepsilon}{2a} \times \left[\|f''\|_\infty\|T_0\|_\infty^2\left[\int_{|x|<M}|\tilde{V}_t^\varepsilon|^2\,\mathrm{d}x\right]^{1/2} + \|f'\|_\infty\|T_0\|_\infty \right.\\ &\quad\left.\times\left[\int_{|x|<M}|\tilde{V}_t^\varepsilon|^2\,\mathrm{d}x + \int_{|x|<M}|\nabla\tilde{V}_t^\varepsilon|^2\,\mathrm{d}x + \int_{|x|<M}|\Delta\tilde{V}_t^\varepsilon|^2\,\mathrm{d}x\right]^{1/2}\right].\end{aligned} \tag{65}$$

The most severe term in the above argument as a result of $K \to \infty$ is

$$\frac{\tilde{\kappa}\varepsilon}{2a}|f'(t)\langle T_t^\varepsilon, \Delta\tilde{\mathcal{V}}_t^\varepsilon(\theta)\rangle|,$$

whose second moment can be bounded as

$$\begin{aligned}\frac{\tilde{\kappa}\varepsilon}{2a}\sqrt{\mathbb{E}|f'(t)\langle T_t^\varepsilon, \Delta\tilde{\mathcal{V}}_t^\varepsilon(\theta)\rangle|^2} &\le C_1\frac{\tilde{\kappa}\varepsilon}{2a}\|f'\|_\infty\|T_0\|_\infty\left(\int_{|x|<M}\mathbb{E}[|\Delta\tilde{V}_t^\varepsilon|^2]\,\mathrm{d}x\right)^{1/2} \\ &\le C_2\tilde{\kappa}\varepsilon\begin{cases}K^{3-\alpha-2\beta} & \text{for } \alpha+2\beta<3,\\ \sqrt{\log K} & \text{for } \alpha+2\beta=3,\\ 1 & \text{for } \alpha+2\beta>3,\end{cases}\end{aligned} \tag{66}$$

and, thus, vanishes in the limit by the assumptions of the theorem. The 4th moment behaves the same way by the 4th order scale-invariance. Hence A_4^ε is uniformly integrable. Clearly

$$\lim_{\varepsilon\to 0}\sup_{t<t_0}\mathbb{E}|A_4^\varepsilon(t)| = 0. \qquad \square$$

4.2. *Identification of the limit*

Once the tightness is established we can use another result in [28, Chapter 3, Theorem 2] to identify the limit. Let $\mathcal{A}$ be a diffusion or jump diffusion operator such that there is a unique solution ω_t in the space $D([0,\infty); L^\infty_{w*}(\mathbb{R}^d))$ such that

$$f(\omega_t) - \int_0^t \mathcal{A}f(\omega_s)\,\mathrm{d}s \tag{67}$$

is a martingale. We shall show that for each $f \in C^\infty(\mathbb{R})$ there exists $f^\varepsilon \in \mathcal{D}(\mathcal{A}^\varepsilon)$ such that

$$\sup_{t<t_0,\varepsilon}\mathbb{E}|f^\varepsilon(t) - f(\langle T_t^\varepsilon, \theta\rangle)| < \infty, \tag{68}$$

$$\lim_{\varepsilon\to 0}\mathbb{E}|f^\varepsilon(t) - f(\langle T_t^\varepsilon, \theta\rangle)| = 0 \quad \forall t < t_0, \tag{69}$$

$$\sup_{t<t_0,\varepsilon}\mathbb{E}|\mathcal{A}^\varepsilon f^\varepsilon(t) - \mathcal{A}f(\langle T_t^\varepsilon, \theta\rangle)| < \infty, \tag{70}$$

$$\lim_{\varepsilon\to 0}\mathbb{E}|\mathcal{A}^\varepsilon f^\varepsilon(t) - \mathcal{A}f(\langle T_t^\varepsilon, \theta\rangle)| = 0 \quad \forall t < t_0. \tag{71}$$

Then the aforementioned theorem implies that any tight processes $\langle T_t^\varepsilon, \theta\rangle$ converge in law to the unique process generated by $\mathcal{A}$. As before we adopt the notation $f(t) = f(\langle T_t^\varepsilon, \theta\rangle)$.

For this purpose, we introduce the next perturbations f_2^ε, f_3^ε. Let

$$A_2^{(1)}(\phi) \equiv \langle \theta, \mathcal{K}_\phi^{(1)} \theta \rangle, \tag{72}$$

$$A_3^{(1)}(\phi) \equiv \langle \phi, \mathbb{E}[\mathcal{V}_t^\varepsilon(\tilde{\mathcal{V}}_t^\varepsilon(\theta))] \rangle, \tag{73}$$

where the positive-definite operator $\mathcal{K}_\phi^{(1)}$ is defined as

$$\mathcal{K}_\phi^{(1)} \theta = \int \theta(y) \nabla\phi(x) \cdot \Gamma^{(1)}(x, y) \nabla\phi(y)\, \mathrm{d}y, \tag{74}$$

$$\Gamma^{(1)}(x, y) = \int [\exp(\mathrm{i}k \cdot x) - 1][\exp(-\mathrm{i}k \cdot y) - 1] \mathcal{E}_{K,L}(\alpha + \beta, k)|k|^{1-d}\, \mathrm{d}k \tag{75}$$

such that

$$\langle \theta_1, \mathcal{K}_{T_t}^{(1)} \theta_2 \rangle = \iint \phi(x)\phi(y) G_{\theta_1,\theta_2}^{(1)}(x, y)\, \mathrm{d}x\, \mathrm{d}y, \tag{76}$$

$$G_{\theta_1,\theta_2}^{(1)} \equiv \sum_{i,j} \frac{\partial^2}{\partial x^i \partial y^j} [\theta_1(x)\theta_2(y) \Gamma_{ij}^{(1)}(x, y)] \tag{77}$$

(cf. (53)).

It is easy to see that

$$A_2^{(1)}(\phi) = \mathbb{E}[\langle \phi, \mathcal{V}_t^\varepsilon(\theta) \rangle \langle \phi, \tilde{\mathcal{V}}_t^\varepsilon(\theta) \rangle], \tag{78}$$

$$A_3^{(1)}(\phi) = \langle \mathcal{B}\phi, \theta \rangle, \tag{79}$$

where the operator $\mathcal{B}$ is given by

$$\mathcal{B}\phi(x) = \sum_{i,j} \Gamma_{ij}^{(1)}(x, x) \frac{\partial^2 \phi(x)}{\partial x^i \partial x^j}.$$

Define

$$f_2^\varepsilon(t) \equiv \frac{1}{a} f''(t) \int_t^\infty \mathbb{E}_t^\varepsilon[\langle T_t^\varepsilon, \mathcal{V}_s^\varepsilon(\theta) \rangle \langle T_t^\varepsilon, \tilde{\mathcal{V}}_s^\varepsilon(\theta) \rangle - A_2^{(1)}(T_t^\varepsilon)]\, \mathrm{d}s,$$

$$f_3^\varepsilon(t) \equiv \frac{1}{a} f'(t) \int_t^\infty \mathbb{E}_t^\varepsilon[\langle T_t^\varepsilon, \mathcal{V}_s^\varepsilon(\tilde{\mathcal{V}}_s^\varepsilon(\theta)) \rangle - A_3^{(1)}(T_t^\varepsilon)]\, \mathrm{d}s.$$

Let

$$G_{\theta_1,\theta_2}^{(2)}(x, y) \equiv \sum_{i,j} \Gamma_{ij}^{(2)}(x, y) \frac{\partial \theta_1(x)}{\partial x^i} \frac{\partial \theta_2(y)}{\partial y^j}, \qquad \langle \theta_1, \mathcal{K}_\phi^{(2)} \theta_2 \rangle \equiv \iint \phi(x)\phi(y) G_{\theta_1,\theta_2}^{(2)}(x, y)\, \mathrm{d}x\, \mathrm{d}y,$$

where the covariance function $\Gamma^{(2)}(x, y) \equiv \mathbb{E}[\tilde{V}_t^\varepsilon(x) \otimes \tilde{V}_t^\varepsilon(y)]$ has the spectral density $\mathcal{E}_{K,L}(\alpha + 2\beta, k)$. Let

$$A_2^{(2)}(\phi) \equiv \langle \theta, \mathcal{K}_\phi^{(2)} \theta \rangle, \qquad A_3^{(2)}(\phi) \equiv \langle \phi, \mathbb{E}[\tilde{\mathcal{V}}_t^\varepsilon(\tilde{\mathcal{V}}_t^\varepsilon(\theta))] \rangle.$$

Noting that

$$\mathbb{E}_t^\varepsilon[V_s^\varepsilon(x) \otimes \tilde{V}_s^\varepsilon(y)] = \iint [\mathrm{e}^{\mathrm{i}x\cdot k} - 1][\mathrm{e}^{-\mathrm{i}y\cdot k'} - 1]\, \mathrm{e}^{-a|k|^{2\beta}|s-t|\varepsilon^{-2}}\, \mathrm{e}^{-a|k'|^{2\beta}|s-t|\varepsilon^{-2}}\, \hat{V}_t^\varepsilon(\mathrm{d}k) \otimes \hat{\tilde{V}}_t^{\varepsilon *}(\mathrm{d}k')$$
$$+ \int [\mathrm{e}^{\mathrm{i}x\cdot k} - 1][\mathrm{e}^{-\mathrm{i}y\cdot k} - 1][1 - \mathrm{e}^{-2a|k|^{2\beta}|s-t|\varepsilon^{-2}}] \mathcal{E}_{K,L}(\alpha + \beta, k)\, \mathrm{d}k, \tag{80}$$

we then have

$$f_2^\varepsilon(t) = \frac{\varepsilon^2}{2a^2} f''(t)[\langle T_t^\varepsilon, \tilde{\mathcal{V}}_t^\varepsilon(\theta)\rangle^2 - A_2^{(2)}(T_t^\varepsilon)] \tag{81}$$

and similarly

$$f_3^\varepsilon(t) = \frac{\varepsilon^2}{2a^2} f'(t)[\langle T_t^\varepsilon, \tilde{\mathcal{V}}_t^\varepsilon(\tilde{\mathcal{V}}_t^\varepsilon(\theta))\rangle - A_3^{(2)}(T_t^\varepsilon)]. \tag{82}$$

Proposition 4.

$$\lim_{\varepsilon\to 0} \sup_{t<t_0} \mathbb{E}|f_2^\varepsilon(t)| = 0, \qquad \lim_{\varepsilon\to 0} \sup_{t<t_0} \mathbb{E}|f_3^\varepsilon(t)| = 0.$$

Proof. We have the bounds

$$\sup_{t<t_0} \mathbb{E}|f_2^\varepsilon(t)| \le \sup_{t<t_0} \frac{\varepsilon^2}{2a^2} \|f''\|_\infty \|T_0\|_\infty^2 \|\nabla\theta\|_\infty^2 \left[\int_{|x|<M} \mathbb{E}|\tilde{V}_t^\varepsilon|^2(x)\,\mathrm{d}x + \int_{|x|<M} |\Gamma^{(2)}(x,x)|\,\mathrm{d}x\right] \le C_1\varepsilon^2,$$

$$\sup_{t<t_0} \mathbb{E}|f_3^\varepsilon(t)| \le \sup_{t<t_0} \frac{\varepsilon^2}{2a^2} \|f'\|_\infty \|T_0\|_\infty \left[\|\nabla\theta\|_\infty \int_{|x|<M} \mathbb{E}|\tilde{V}_t^\varepsilon|^2(x)\,\mathrm{d}x + \|\theta\|_\infty \left[\int_{|x|<M} \mathbb{E}|\tilde{V}_t^\varepsilon|^2(x)\,\mathrm{d}x\right]^{1/2}\right.$$

$$\left.\times \left[\int_{|x|<M} \mathbb{E}|\nabla\tilde{V}_t^\varepsilon|^2(x)\,\mathrm{d}x\right]^{1/2}\right] \le C_2\varepsilon^2 K^{2-\alpha-2\beta}$$

both of which tend to zero. □

We have

$$\mathcal{A}^\varepsilon f_2^\varepsilon(t) = \frac{1}{a} f''(t)[-\langle T_t^\varepsilon, \mathcal{V}_t^\varepsilon(\theta)\rangle\langle T_t^\varepsilon, \tilde{\mathcal{V}}_t^\varepsilon(\theta)\rangle - A_2^{(1)}(T_t^\varepsilon)] + R_2^\varepsilon(t),$$

$$\mathcal{A}^\varepsilon f_3^\varepsilon(t) = \frac{1}{a} f'(t)[-\langle T_t^\varepsilon, \mathcal{V}_t^\varepsilon(\tilde{\mathcal{V}}_t^\varepsilon(\theta))\rangle - A_3^{(1)}(T_t^\varepsilon)] + R_3^\varepsilon(t)$$

with

$$\begin{aligned} R_2^\varepsilon(t) = {} & \frac{f'''(t)}{2}\left[\frac{\varepsilon^2\tilde{\kappa}}{2a^2}\langle T_t^\varepsilon, \Delta\theta\rangle - \frac{\varepsilon}{a^2}\langle T_t^\varepsilon, \mathcal{V}_t^\varepsilon(\theta)\rangle\right][\langle T_t^\varepsilon, \tilde{\mathcal{V}}_t^\varepsilon(\theta)\rangle^2 - A_2^{(2)}(T_t^\varepsilon)] \\ & + f''(t)\langle T_t^\varepsilon, \tilde{\mathcal{V}}_t^\varepsilon(\theta)\rangle\left[\frac{\tilde{\kappa}\varepsilon^2}{2a^2}\langle T_t^\varepsilon, \Delta\tilde{\mathcal{V}}_t^\varepsilon(\theta)\rangle - \frac{\varepsilon}{a^2}\langle T_t^\varepsilon, \mathcal{V}_t^\varepsilon(\tilde{\mathcal{V}}_t^\varepsilon(\theta))\rangle\right] \\ & - f''(t)\left[\frac{\tilde{\kappa}\varepsilon^2}{4a^2}\langle T_t^\varepsilon, \Delta G_\theta^{(2)} T_t^\varepsilon\rangle - \frac{\varepsilon}{a^2}\langle T_t^\varepsilon, \mathcal{V}_t^\varepsilon(G_\theta^{(2)} T_t^\varepsilon)\rangle\right], \end{aligned} \tag{83}$$

where $G_\theta^{(2)}$ denotes the operator

$$G_\theta^{(2)}\phi \equiv \int G_{\theta,\theta}^{(2)}(x,y)\phi(y)\,\mathrm{d}y,$$

and similarly

$$\begin{aligned} R_3^\varepsilon(t) = {} & f''(t)\left[\frac{\tilde{\kappa}\varepsilon^2}{4a^2}\langle T_t^\varepsilon, \Delta\theta\rangle - \frac{\varepsilon}{2a^2}\langle T_t^\varepsilon, \mathcal{V}_t^\varepsilon(\theta)\rangle\right][\langle T_t^\varepsilon, \tilde{\mathcal{V}}_t^\varepsilon(\tilde{\mathcal{V}}_t^\varepsilon(\theta))\rangle - A_3^{(2)}(T_t^\varepsilon)] \\ & + f'(t)\left[\frac{\tilde{\kappa}\varepsilon^2}{4a^2}\langle T_t^\varepsilon, \Delta\tilde{\mathcal{V}}_t^\varepsilon(\tilde{\mathcal{V}}_t^\varepsilon(\theta))\rangle - \frac{\varepsilon}{2a^2}\langle T_t^\varepsilon, \mathcal{V}_t^\varepsilon(\tilde{\mathcal{V}}_t^\varepsilon(\tilde{\mathcal{V}}_t^\varepsilon(\theta)))\rangle\right] \\ & - f'(t)\left[\frac{\tilde{\kappa}\varepsilon^2}{4a^2}\langle T_t^\varepsilon, \Delta\mathbb{E}[\tilde{\mathcal{V}}_t^\varepsilon(\tilde{\mathcal{V}}_t^\varepsilon(\theta))]\rangle + \frac{\varepsilon}{2a^2}\langle T_t^\varepsilon, \mathcal{V}_t^\varepsilon(\mathbb{E}[\tilde{\mathcal{V}}_t^\varepsilon(\tilde{\mathcal{V}}_t^\varepsilon(\theta))])\rangle\right]. \end{aligned}$$

Proposition 5.

$$\lim_{\varepsilon\to 0}\sup_{t<t_0}\mathbb{E}|R_2^\varepsilon(t)|=0,\qquad \lim_{\varepsilon\to 0}\sup_{t<t_0}\mathbb{E}|R_3^\varepsilon(t)|=0.$$

Proof. The argument is entirely analogous to that for Proposition 4. The most severe term without the prefactor $\tilde{\kappa}$ occurs in the expression for $R_3^\varepsilon(t)$ and can be bounded as

$$\varepsilon\mathbb{E}|\langle T_t^\varepsilon,\mathcal{V}_t^\varepsilon(\tilde{\mathcal{V}}_t^\varepsilon(\tilde{\mathcal{V}}_t^\varepsilon(\theta)))\rangle| \le \varepsilon\|T_0\|_\infty\mathbb{E}|\mathcal{V}_t^\varepsilon(\tilde{\mathcal{V}}_t^\varepsilon(\tilde{\mathcal{V}}_t^\varepsilon(\theta)))| \le C_1\varepsilon\|T_0\|_\infty\left(\int_{|x|<M}\mathbb{E}|V_t^\varepsilon|^2\,\mathrm{d}x\right)^{1/2} \times\left(\int_{|x|<M}\{\mathbb{E}[|\tilde{V}_t^\varepsilon|^4]\mathbb{E}[|\nabla^2\tilde{V}_t^\varepsilon|^4]\}^{1/2}\,\mathrm{d}x+\int_{|x|<M}\mathbb{E}[|\nabla\tilde{V}_t^\varepsilon|^4]\,\mathrm{d}x\right)^{1/2} \tag{84}$$

by assumption. The right-hand side of the above tends to zero if either

$$\alpha+2\beta>3$$

or

$$\alpha+2\beta=3,\qquad \lim_{\varepsilon\to 0}\varepsilon\sqrt{\log K}=0 \tag{85}$$

or

$$\alpha+2\beta<3,\qquad \lim_{\varepsilon\to 0}\varepsilon K^{3-\alpha-2\beta}=0 \tag{86}$$

is satisfied. The term involving $\varepsilon\langle T_t^\varepsilon,\mathcal{V}_t^\varepsilon(G_\theta^{(2)}T_t^\varepsilon)\rangle$ can be similarly estimated.

The most severe term involving the prefactor $\tilde{\kappa}$ occurs in R_3^ε and can be bounded as

$$\tilde{\kappa}\varepsilon^2\mathbb{E}|\langle T_t^\varepsilon,\Delta\tilde{\mathcal{V}}_t^\varepsilon(\tilde{\mathcal{V}}_t^\varepsilon(\theta))\rangle| \le C\tilde{\kappa}\varepsilon^2\|T_0\|_\infty\left(\int_{|x|<M}\mathbb{E}[|\nabla^3\tilde{V}_t^\varepsilon|^2]\right)^{1/2}\sim\begin{cases}\tilde{\kappa}\varepsilon^2 & \text{for }\ \alpha+2\beta>4,\\ \tilde{\kappa}\varepsilon^2\sqrt{\log K} & \text{for }\ \alpha+2\beta=4,\\ \tilde{\kappa}\varepsilon^2K^{4-\alpha-2\beta} & \text{for }\ \alpha+2\beta<4,\end{cases} \tag{87}$$

the right-hand side of which tends to zero if either

$$\alpha+2\beta>4$$

or

$$\alpha+2\beta=4,\qquad \lim_{\varepsilon\to 0}\tilde{\kappa}\varepsilon^2\sqrt{\log K}=0$$

or

$$3<\alpha+2\beta<4,\qquad \lim_{\varepsilon\to 0}\tilde{\kappa}\varepsilon^2K^{4-\alpha-2\beta}=0 \tag{88}$$

or

$$2<\alpha+2\beta<3,\qquad \lim_{\varepsilon\to 0}\tilde{\kappa}\varepsilon^2K^{4-\alpha-2\beta}=\lim_{\varepsilon\to 0}\varepsilon K^{3-\alpha-2\beta}=0.$$

Note that for $\alpha+2\beta\le 2$ the condition (85) or (86) implies that

$$\lim_{\varepsilon\to 0}\varepsilon^2K^{4-\alpha-2\beta}=0.$$

□

Set

$$R^{\varepsilon}(t) = A_4^{\varepsilon}(t) - R_2^{\varepsilon}(t) - R_3^{\varepsilon}(t).$$

It follows from Propositions 3 and 5 that:

$$\lim_{\varepsilon\to 0}\sup_{t<t_0} \mathbb{E}|R^{\varepsilon}(t)| = 0.$$

Recall that

$$M_t^{\varepsilon}(\theta) = f^{\varepsilon}(t) - \int_0^t \mathcal{A}^{\varepsilon} f^{\varepsilon}(s)\,\mathrm{d}s = f(t) - f_1^{\varepsilon}(t) - f_2^{\varepsilon}(t) - f_3^{\varepsilon}(t) - \int_0^t \frac{\tilde{\kappa}}{2} f'(t)\langle T_t^{\varepsilon}, \Delta\theta\rangle\,\mathrm{d}s - \int_0^t \frac{1}{a}[f''(s)A_2^{(1)}(T_s^{\varepsilon}) + f'(s)A_3^{(1)}(T_s^{\varepsilon})]\,\mathrm{d}s - \int_0^t R^{\varepsilon}(s)\,\mathrm{d}s$$

is a martingale. Now that (68)–(71) are satisfied we can identify the limiting martingale to be

$$M_t(\theta) = f(t) - \int_0^t \left\{ f'(s)\left[\frac{\kappa_0}{2}\langle T_s, \Delta\theta\rangle + \frac{1}{a}\bar{A}_3^{(1)}(T_s)\right] + \frac{1}{a}f''(s)\bar{A}_2^{(1)}(T_s)\right\}\mathrm{d}s, \tag{89}$$

where

$$\bar{A}_2^{(1)}(\phi) = \lim_{K\to\infty} A_2^{(1)}(\phi), \qquad \bar{A}_3^{(1)}(\phi) = \lim_{K\to\infty} A_3^{(1)}(\phi)$$

(cf. (72) and (79)).

Since $\langle T_t^{\varepsilon}, \theta\rangle$ is uniformly bounded

$$|\langle T_t^{\varepsilon}, \theta\rangle| \le \|T_0\|_{\infty}\|\theta\|_1,$$

we have the convergence of the second moment

$$\lim_{\varepsilon\to 0}\mathbb{E}\{\langle T_t^{\varepsilon}, \theta\rangle^2\} = \mathbb{E}\{\langle T_t, \theta\rangle^2\}.$$

Use $f(r) = r$ and r^2 in (89)

$$M_t^{(1)}(\theta) = \langle T_t, \theta\rangle - \int_0^t \left[\frac{\kappa_0}{2}\langle T_s, \Delta\theta\rangle + \frac{1}{a}\bar{A}_3^{(1)}(T_s)\right]\mathrm{d}s$$

is a martingale with the quadratic variation

$$[M^{(1)}(\theta), M^{(1)}(\theta)]_t = \frac{2}{a}\int_0^t \bar{A}_2^{(1)}(T_s)\,\mathrm{d}s = \frac{2}{a}\int_0^t \langle\theta, \bar{\mathcal{K}}_{T_s}^{(1)}\theta\rangle\,\mathrm{d}s,$$

where $\bar{\mathcal{K}}_{T_t}^{(1)}$ is a positive-definite operator given formally as

$$\bar{\mathcal{K}}_{T_t}^{(1)}\theta = \int \theta(y)\nabla T_t(x)\cdot\bar{\Gamma}^{(1)}(x,y)\nabla T_t(y)\,\mathrm{d}y, \tag{90}$$

(cf. (74)). Therefore,

$$M_t^{(1)} = \sqrt{\frac{2}{a}}\int_0^t \sqrt{\bar{\mathcal{K}}_{T_s}^{(1)}}\,\mathrm{d}W_s,$$

where W_s is a cylindrical Wiener process (i.e. $\mathrm{d}W_t(x)$ is a space–time white-noise field) and $\sqrt{\bar{\mathcal{K}}^{(1)}_{T_s}}$ the square-root of the positive-definite operator given in (90). From (72) and (79) we see that the limiting process T_t is the (assumed unique) distributional solution to the martingale problem (52) of the Itô's equation

$$\mathrm{d}T_t = \left(\frac{\kappa_0}{2}\Delta + \frac{1}{a}\bar{\mathcal{B}}\right) T_t\,\mathrm{d}t + \sqrt{2a^{-1}\bar{\mathcal{K}}^{(1)}_{T_t}}\,\mathrm{d}W_t = \left(\frac{\kappa_0}{2}\Delta + \frac{1}{a}\bar{\mathcal{B}}\right) T_t\,\mathrm{d}t + \sqrt{2}a^{-1/2}\nabla T_t \cdot \mathrm{d}\bar{W}^{(1)}_t,$$

where the operator $\bar{\mathcal{B}}$ is given by (30) and $\bar{W}^{(1)}_t$ is the Brownian vector field with the spatial covariance $\bar{\Gamma}^{(1)}(x, y)$.

5. Proof of Theorem 2

As we let $L \to \infty$ along with $\varepsilon \to 0$ the proof of the uniform integrability of $\mathcal{A}^\varepsilon[f(t) - f_1^\varepsilon(t)]$ (the first part of Proposition 3) breaks down. In this case, we work with the perturbed test function

$$f^\varepsilon(t) = f(t) - f_1^\varepsilon(t) + f_2^\varepsilon(t) + f_3^\varepsilon(t).$$

Proposition 6.

$$\lim_{\varepsilon\to 0}\sup_{t<t_0} \mathbb{E}|f_j^\varepsilon(t)| = 0, \qquad \lim_{\varepsilon\to 0}\sup_{t<t_0} |f_j^\varepsilon(t)| = 0 \quad \textit{in probability } \forall j = 1, 2, 3. \tag{91}$$

Proof. The argument for the case of $f_1^\varepsilon(t)$ is the same as Proposition 2. For $f_2^\varepsilon(t)$ and $f_3^\varepsilon(t)$ we have the bounds

$$\begin{aligned}
\sup_{t<t_0} \mathbb{E}|f_2^\varepsilon(t)| &\le \sup_{t<t_0} \frac{\varepsilon^2}{2a^2}\|f''\|_\infty \|T_0\|^2_\infty \|\nabla\theta\|^2_\infty \left[\int_{|x|<M} \mathbb{E}|\tilde{V}^\varepsilon_t|^2(x)\,\mathrm{d}x + \int_{|x|<2M} |\Gamma^{(2)}(x, x)|\,\mathrm{d}x\right] \\
&\le C_1\varepsilon^2 L^{2(\alpha+2\beta)-4}, \\
\sup_{t<t_0} \mathbb{E}|f_3^\varepsilon(t)| &\le \sup_{t<t_0} \frac{\varepsilon^2}{2a^2}\|f'\|_\infty \|T_0\|_\infty \left[\|\nabla\theta\|_\infty \int_{|x|<M} \mathbb{E}|\tilde{V}^\varepsilon_t|^2(x)\,\mathrm{d}x + \|\theta\|_\infty \left[\int_{|x|<M} \mathbb{E}|\tilde{V}^\varepsilon_t|^2(x)\,\mathrm{d}x\right]^{1/2}\right. \\
&\quad \left. \times \left[\int_{|x|<M} \mathbb{E}|\nabla\tilde{V}^\varepsilon_t|^2(x)\,\mathrm{d}x\right]^{1/2}\right] \le C_2\varepsilon^2 L^{2(\alpha+2\beta)-4}
\end{aligned}$$

both of which vanish under the assumptions of the theorem. Here we have used the fact that

$$\int_{|x|<M} \mathbb{E}|\tilde{V}^\varepsilon_t|^2(x)\,\mathrm{d}x = \mathrm{O}(L^{2(\alpha+2\beta)-4}), \quad L \to \infty.$$

As for estimating $\sup_{t<t_0}|f_j^\varepsilon(t)|$, $j = 2, 3$, we can use

$$M^d \int_{|x|<M} |\tilde{V}^\varepsilon_t|^2(x)\,\mathrm{d}x \quad \text{in place of} \quad \int_{|x|<M} \mathbb{E}|\tilde{V}^\varepsilon_t|^2(x)\,\mathrm{d}x$$

in the above bounds and obtain by assumption (cf. (35), Remark 2) the desired estimate which have a similar order of magnitude with an additional factor of $1/\varepsilon$ and a random constant possessing a finite moment. □

We have

$$\mathcal{A}^\varepsilon f^\varepsilon(t) = \frac{\tilde{\kappa}}{2} f'(t)\langle T^\varepsilon_t, \Delta\theta\rangle - \frac{1}{a} f''(t) A_2^{(1)}(T^\varepsilon_t) - \frac{1}{a} f'(t) A_3^{(1)}(T^\varepsilon_t) + R_1^\varepsilon(t) + R_2^\varepsilon(t) + R_3^\varepsilon(t) \tag{92}$$

with

$$R_1^\varepsilon(t) = \frac{\tilde{\kappa}\varepsilon}{2a}[f''(t)\langle T_t^\varepsilon, \Delta\theta\rangle\langle T_t^\varepsilon, \mathcal{V}_t^\varepsilon(\theta)\rangle - f'(t)\langle T_t^\varepsilon, \Delta\tilde{\mathcal{V}}_t^\varepsilon(\theta)\rangle] \tag{93}$$

and $R_2^\varepsilon(t)$, $R_3^\varepsilon(t)$ as before.

Proposition 7.

$$\lim_{\varepsilon\to 0}\sup_{t<t_0}\mathbb{E}|R_j^\varepsilon(t)| = 0, \quad j = 1, 2, 3.$$

Proof. The proof is similar to that of Proposition 5 with the additional consideration due to $L \to \infty$. These additional terms can all be estimated by

$$C_1\varepsilon\int_{|x|<M}\mathbb{E}[|\tilde{V}_t^\varepsilon(x)\tilde{V}_t^\varepsilon(x)|]\,\mathrm{d}x \le C_2\varepsilon L^{2(\alpha+2\beta-2)},$$

which tends to zero under the assumptions of the theorem. □

For the tightness it remains to show the following proposition.

Proposition 8. $\{\mathcal{A}^\varepsilon f^\varepsilon\}$ *are uniformly integrable.*

Proof. We shall prove that each term in the expression (92) is uniformly integrable.

The first three terms are clearly bounded under the assumption of $\alpha+\beta < 2$. The last three terms can be estimated as in Proposition 7 by

$$C_1\varepsilon\sup_{t<t_0}\int_{|x|<M}|\tilde{V}_t^\varepsilon(x)\tilde{V}_t^\varepsilon(x)|$$

whose second moment behaves like $\varepsilon^2 L^{4(\alpha+2\beta-2)}$, by the 4th order scale-invariance property, and tends to zero. □

Now we have all the estimates needed to identify the limit as in the proof of Theorem 1.

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