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Dissipation time and decay of correlations

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Abstract
We consider the effect of noise on the dynamics generated by volume-preserving maps on a d-dimensional torus. The quantity we use to measure the irreversibility of the dynamics is the *dissipation time*. We focus on the asymptotic behaviour of this time in the limit of small noise. We derive universal lower and upper bounds for the dissipation time in terms of various properties of the map and its associated propagators: spectral properties, local expansivity, and global mixing properties. We show that the dissipation is slow for a general class of non-weakly-mixing maps; on the other hand, it is fast for a large class of exponentially mixing systems, which include uniformly expanding maps and Anosov diffeomorphisms.

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1. Introduction

The origin of irreversibility in dynamical systems is often modelled by small stochastic perturbations of the otherwise reversible dynamics. These perturbations may be attributed to uncontrolled interactions with the 'environment', or with internal variables neglected in the equations. In experimental or numerical investigations, stochasticity or 'noise' is introduced, respectively, by finite precision of the preparation and measurement apparatus, and by rounding-off errors due to finite computer precision.

One may take these stochastic perturbations explicitly into account by adding a term of Langevin type in the evolution equations, or equivalently introducing some diffusion term in the Fokker–Planck equation. In this paper, we choose to deal with discrete-time dynamics on some compact phase space, namely the d-dimensional torus. All the maps we study preserve the Lebesgue measure, which is therefore the 'natural' invariant measure associated with the dynamics. The noise is implemented through a convolution operator, the kernel of which has a (small) width $\epsilon > 0$. For a given map F, two types of stochastic perturbations will be considered: the noise operator may be applied at each step of evolution, resulting in a 'noisy evolution'; on the other hand, we may choose to introduce some stochasticity only at the initial (preparation) and final (measurement) steps, resulting in a 'coarse-grained evolution'.

In general, the influence of the noise on the long-time evolution of the system depends on the dynamical properties of the map. Typically, the effect is stronger if the map is 'chaotic' than if it is 'regular' [21, 3, 25]. We will try to give a quantitative version of this statement. To this end, we will characterize the effect of noise on a map F through a single observable, the *dissipation time*. By dissipation, we mean the damping (through the noisy or coarse-grained evolution) of the density fluctuations, while the total probability remains constant. Mathematically, this damping is expressed by the decay of the L^2-norm of the density (or equivalently, the L^2-norm of the density fluctuations) with respect to the Lebesgue measure. The decrease of this norm due to dissipation may be interpreted as an 'entropy increase' of the system, up to the state of maximal entropy, which is the uniform density (here the word 'entropy' denotes the Boltzmann entropy of the probability density, unrelated with the topological or Kolmogorov–Sinai entropies associated with the map). We define the dissipation time as the time needed for the evolution to bring fluctuations under a fixed threshold (i.e. reduce the L^2-norm by a fixed factor) [14]. We are especially interested in the behaviour of this time in the limit of small noise. We will show that this behaviour drastically depends on the dynamics generated by the noiseless map F: in short, the dissipation will be 'fast' for a chaotic dynamics, as opposed to 'slow' for a regular dynamics, the difference being ever more striking as the level of noise decreases. Our main results exhibit this opposition by means of quantitative estimates on the asymptotics of the dissipation time.

Since the dissipation time is defined in terms of an L^2-norm, it is naturally related to the spectral properties of the propagator associated with the map, acting on the space of square-integrable functions. In section 3, we analyse the links between, on the one hand, the spectrum and pseudospectrum of the noisy or noiseless propagators and, on the other, their dissipation time. The relevance of the pseudospectrum for time evolution problems has been recently proved in the context of non-unitary continuous-time dynamical systems [12]. These spectral relationships will be mainly used to analyse the case of 'regular', precisely non-weakly-mixing maps. In contrast, they are of little help for more chaotic ones.

In the following sections, we connect the dissipation time to more 'dynamical' properties of the map F (under some smoothness assumptions on F). We first obtain lower bounds for the dissipation time from the *local expansion* properties of the map: a weak expansion, or absence of local expansion will imply slow dissipation (section 4). Next, we obtain upper bounds for the dissipation time using information on mixing properties of the map (i.e. the time decay of correlations between observables). Strong (e.g. exponential) mixing is a typical characteristic of chaotic behaviour and can be easily measured in numerical simulations [4]. The main conclusion is that exponential or stronger mixing implies fast dissipation, for both the noisy and the coarse-grained evolution.

In section 6, we describe several families of volume-preserving maps on the torus, for which the results obtained above yield useful information concerning the dissipation. The two main families are expanding (non-invertible) maps and Anosov (or at least partially hyperbolic) diffeomorphisms of the torus. All these maps are at least exponentially mixing, so the dissipation is 'fast'. In the case of Anosov maps, we collect our results in theorem 4. For illustration, we also analyse in detail examples of *linear* mixing maps, for which exact asymptotics of the dissipation times can be obtained, and therefore give an idea of the sharpness of the bounds obtained earlier (the results concerning linear automorphisms had been obtained in [14]).

From these developments one can see that the dissipation time provides a robust characterization of the chaoticity of a given dynamical system. As opposed to, e.g. the decay rate of dynamical correlation, the dissipation time has the same asymptotics (up to a constant factor) among Anosov diffeomorphisms while the decorrelation may be exponential (generic Anosov case) or super-exponential depending on the particular map (cf results regarding hyperbolic toral automorphisms [3] and corollary 5).

2. Set-up and notation

2.1. Evolution operators

Let $(\mathbb{T}^d, \mathcal{B}(\mathbb{T}^d), m)$ denote the d-dimensional torus, equipped with its σ-field of Borel sets and the Lebesgue measure m. Let $F : \mathbb{T}^d \to \mathbb{T}^d$ be a map on the torus preserving the Lebesgue measure: for any set $B \in \mathcal{B}(\mathbb{T}^d)$ we have $m(F^{-1}(B)) = m(B)$. In general, F is not supposed to be invertible. In the following, we call such a map 'volume preserving' with implicit reference to the Lebesgue measure.

The map F generates a discrete-time dynamics on $\mathbb{T}^d$, which in terms of a pathwise description can be represented by the forward trajectory $\{F^n(\boldsymbol{x}_0), n \in \mathbb{N}\}$ of any initial point (particle) $\boldsymbol{x}_0 \in \mathbb{T}^d$. However, instead of looking at the evolution of a single particle, one can consider the statistical description of the dynamics, which is the evolution of a density (more generally a measure) describing the initial statistical configuration of the system.

Let $\mathcal{M}(\mathbb{T}^d)$ denote the set of all Borel measures on $\mathbb{T}^d$. For any $\mu \in \mathcal{M}(\mathbb{T}^d)$ and $f \in C^0(\mathbb{T}^d)$ we write

$$\mu(f) = \int_{\mathbb{T}^d} f(\boldsymbol{x})\,\mathrm{d}\mu(\boldsymbol{x}).$$

The map F induces a map F^* on $\mathcal{M}(\mathbb{T}^d)$ given by

$$(F^*\mu)(f) = \mu(f \circ F), \qquad \text{for all } f \in C^0(\mathbb{T}^d).$$

This map can also be defined as follows:

$$(F^*\mu)(B) = \mu(F^{-1}(B)), \qquad \text{for all } B \in \mathcal{B}(\mathbb{T}^d).$$

In particular, if $\mu = \delta_{\boldsymbol{x}_0}$ then $F^*(\mu) = \delta_{F(\boldsymbol{x}_0)}$ and one recovers the pathwise description.

If μ is absolutely continuous w.r.t. m, then $F^*(\mu)$ preserves this property (since the measure-preserving map F is nonsingular w.r.t. m, see [23, p 42]). The corresponding densities $g = \mathrm{d}\mu/\mathrm{d}m \in L^1(\mathbb{T}^d)$ are transformed by the Frobenius–Perron or transfer operator P_F [3]:

$$P_F\left(\frac{\mathrm{d}\mu}{\mathrm{d}m}\right) = \frac{\mathrm{d}(F^*\mu)}{\mathrm{d}m}.$$

If the map F is invertible, P_F is given explicitly by

$$(P_F g)(\boldsymbol{x}) = (g \circ F^{-1})(\boldsymbol{x}) \frac{\mathrm{d}F^*m}{\mathrm{d}m}(\boldsymbol{x}) = g \circ F^{-1}(\boldsymbol{x}).$$

If the map F is differentiable, and the pre-image set of $\boldsymbol{x}$ is finite for all $\boldsymbol{x}$, the Perron–Frobenius operator is given by

$$(P_F g)(\boldsymbol{x}) = \sum_{\boldsymbol{y}|F(\boldsymbol{y})=\boldsymbol{x}} \frac{g(\boldsymbol{y})}{|J_F(\boldsymbol{y})|},$$

where $J_F(\boldsymbol{y})$ is the Jacobian of F at $\boldsymbol{y}$.

On the other hand, one can consider the dual of the Frobenius–Perron operator, called the Koopman operator, which governs the evolution of observables $f \in L^\infty(\mathbb{T}^d)$ instead of that of densities $g \in L^1$. The Koopman operator U_F is defined as

$$U_F f = f \circ F. \tag{1}$$

Due to the nonseparability of the Banach space $L^\infty(\mathbb{T}^d)$, it is often more convenient to consider its closure in some weaker L^p-norm, which yields larger (but separable) spaces of observables $L^p(\mathbb{T}^d)$. In this paper, we will be mainly concerned with the space $L^2(\mathbb{T}^d)$ and its codimension-1 subspace of zero-mean functions $L^2_0(\mathbb{T}^d) = \{f \in L^2(\mathbb{T}^2) : m(f) = 0\}$. This subspace is obviously invariant under U_F and P_F, due to the assumption $F^*m = m$. Throughout the paper, $\| \cdot \|$ will always refer to the L^2-norm (and corresponding operator norm) on $L^2_0(\mathbb{T}^d)$ (any other norm will carry an explicit subscript).

For any measure-preserving map F, the operator U_F is isometric on $L^2(\mathbb{T}^d)$ and $L^2_0(\mathbb{T}^d)$. When F is invertible, U_F is unitary on these spaces, and satisfies $U_F = P_F^{-1} = P_{F^{-1}}$.

2.1.1. Additional notation. Although the operators introduced in previous sections were mostly defined on the space $L^2_0(\mathbb{T}^d)$, it will be useful to consider other function spaces, which we define now in some detail. For any $m \in \mathbb{N}$, we denote by $C^m(\mathbb{T}^d)$ the space of m-times continuously differentiable functions, with the norm

$$\|f\|_{C^m} = \sum_{|\alpha|_1 \leqslant m} \|D^\alpha f\|_\infty$$

(we use the norm $|\boldsymbol{\alpha}|_1 = \alpha_1 + \cdots + \alpha_d$ for the multi-index $\boldsymbol{\alpha} \in \mathbb{N}^d$). For any $s = m + \eta$ with $m = [s] \in \mathbb{N}$, $\eta \in (0, 1)$, let $C^s(\mathbb{T}^d)$ denote the space of C^m functions for which the m-derivatives are η-Hölder continuous; this space is equipped with the norm

$$\|f\|_{C^s} = \|f\|_{C^m} + \sum_{|\boldsymbol{\alpha}|_1 = m} \sup_{\boldsymbol{x} \neq \boldsymbol{y}} \frac{|D^\alpha f(\boldsymbol{x}) - D^\alpha f(\boldsymbol{y})|}{|\boldsymbol{x} - \boldsymbol{y}|^\eta}.$$

The Fourier transforms of functions $g \in L^1(\mathbb{R}^d)$ and $f \in L^1(\mathbb{T}^d)$ are defined as follows:

$$\forall \boldsymbol{\xi} \in \mathbb{R}^d, \qquad \hat{g}(\boldsymbol{\xi}) = \int_{\mathbb{R}^d} g(\boldsymbol{x}) \mathrm{e}^{-2\pi \mathrm{i} \boldsymbol{x} \cdot \boldsymbol{\xi}} \, \mathrm{d}\boldsymbol{x}, \tag{2}$$

$$\forall \boldsymbol{k} \in \mathbb{Z}^d, \qquad \hat{f}(\boldsymbol{k}) = \int_{\mathbb{T}^d} f(\boldsymbol{x}) \mathrm{e}^{-2\pi \mathrm{i} \boldsymbol{x} \cdot \boldsymbol{k}} \, \mathrm{d}\boldsymbol{x} = \langle \boldsymbol{e}_{\boldsymbol{k}}, f \rangle. \tag{3}$$

In the above expression we used the Fourier modes on the torus $\boldsymbol{e}_{\boldsymbol{k}}(\boldsymbol{x}) := \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{x} \cdot \boldsymbol{k}}$. For any $s \geqslant 0$, we denote by $H^s(\mathbb{T}^d)$ and $H^s(\mathbb{R}^d)$ the Sobolev spaces of s-times weakly differentiable L^2-functions equipped with the norms $\| \cdot \|_{H^s}$ defined, respectively, by

$$\|g\|^2_{H^s(\mathbb{R}^d)} = \int_{\boldsymbol{\xi} \in \mathbb{R}^d} (1 + |\boldsymbol{\xi}|^2)^s |\hat{g}(\boldsymbol{\xi})|^2 \, \mathrm{d}\boldsymbol{\xi},$$

$$\|f\|^2_{H^s(\mathbb{T}^d)} = \sum_{\boldsymbol{k} \in \mathbb{Z}^d} (1 + |\boldsymbol{k}|^2)^s |\hat{f}(\boldsymbol{k})|^2.$$

Finally, for any of these spaces, adding the subscript 0 will mean that we consider the (U_F-invariant) subspace of functions with zero average, e.g. $C^j_0(\mathbb{T}^d) = \{f \in C^j(\mathbb{T}^d), m(f) = 0\}$.

2.2. Noise operator

To construct the noise operator we first define the *noise generating density* i.e. an arbitrary probability density function $g \in L^1(\mathbb{R}^d)$ with even parity w.r.t. the origin: $g(\boldsymbol{x}) = g(-\boldsymbol{x})$. The *noise width* (or *noise level*) will be given by a single non-negative parameter, which we call ϵ. To each $\epsilon > 0$ corresponds the noise kernel on $\mathbb{R}^d$:

$$g_\epsilon(\boldsymbol{x}) = \frac{1}{\epsilon^d} g\left(\frac{\boldsymbol{x}}{\epsilon}\right),$$

with the convention that $g_0 = \delta_0$. The noise kernel on the torus is obtained by periodizing g_ϵ, yielding the periodic kernel

$$\tilde{g}_\epsilon(\boldsymbol{x}) = \sum_{\boldsymbol{n}\in\mathbb{Z}^d} g_\epsilon(\boldsymbol{x}+\boldsymbol{n}). \tag{4}$$

We remark that the Fourier transform of $\tilde{g}_\epsilon$ is related to that of g by the identities $\hat{\tilde{g}}_\epsilon(\boldsymbol{k}) = \hat{g}_\epsilon(\boldsymbol{k}) = \hat{g}(\epsilon\boldsymbol{k})$.

The noise operator G_ϵ is defined on any function $f \in L^2_0(\mathbb{T}^d)$ as the convolution:

$$G_\epsilon f = \tilde{g}_\epsilon * f.$$

As a convolution operator defined by an L^1 density, G_ϵ is compact on $L^2_0(\mathbb{T}^d)$ (if g is square-integrable, G_ϵ is Hilbert–Schmidt). The Fourier modes $\{\boldsymbol{e_k},\ \boldsymbol{k} \in \mathbb{Z}^d \setminus \{0\}\}$ form an orthonormal basis of eigenvectors of G_ϵ, yielding the following spectral decomposition:

$$\forall f \in L^2_0(\mathbb{T}^d), \qquad G_\epsilon f = \sum_{0\neq\boldsymbol{k}\in\mathbb{Z}^d} \hat{g}(\epsilon\boldsymbol{k})\langle \boldsymbol{e_k}, f\rangle\, \boldsymbol{e_k}. \tag{5}$$

This formula shows that the eigenvalue associated with $\boldsymbol{e_k}$ is $\hat{g}(\epsilon\boldsymbol{k})$. Since g is a symmetric function, this eigenvalue is real, so that G_ϵ is a self-adjoint operator. Its spectral radius $r_{\mathrm{sp}}(G_\epsilon)$ is, therefore, given by

$$r_{\mathrm{sp}}(G_\epsilon) = \|G_\epsilon\| = \sup_{0\neq\boldsymbol{k}\in\mathbb{Z}^d} |\hat{g}(\epsilon\boldsymbol{k})|. \tag{6}$$

Since the density g is positive, $\hat{g}$ attains its maximum, $\hat{g}(0) = 1$, at the origin and nowhere else. Besides, because $g \in L^1(\mathbb{R}^d)$, $\hat{g}$ is a continuous function vanishing at infinity. As a result, for small enough $\epsilon > 0$, the supremum on the RHS of (6) is reached at some point $\epsilon\boldsymbol{k}$ close to the origin, and this maximum is strictly smaller than 1. This shows that the operator G_ϵ is strictly contracting on $L^2_0(\mathbb{T}^d)$:

$$\forall\epsilon > 0, \qquad \|G_\epsilon\| = r_{\mathrm{sp}}(G_\epsilon) < 1. \tag{7}$$

In the next section we study this noise operator more precisely, starting from appropriate assumptions on the noise generating density.

2.3. Noise kernel estimates

In this subsection, we present some estimates regarding the noise operator, which will be used throughout the paper to estimate the dissipation time.

We will be interested in the behaviour of the system in the limit of small noise level, that is the limit $\epsilon \to 0$. Therefore, it will be useful to introduce the following asymptotic notation. Given two variables a_ϵ, b_ϵ depending on $\epsilon > 0$, we write

$$a_\epsilon \lesssim b_\epsilon \qquad \text{if } \limsup_{\epsilon\to 0} \frac{a_\epsilon}{b_\epsilon} < \infty, \tag{8}$$

$$a_\epsilon \approx b_\epsilon \qquad \text{if } \lim_{\epsilon\to 0} \frac{a_\epsilon}{b_\epsilon} = 1, \tag{9}$$

$$a_\epsilon \sim b_\epsilon \qquad \text{if } a_\epsilon \lesssim b_\epsilon \quad \text{and} \quad b_\epsilon \lesssim a_\epsilon. \tag{10}$$

In order to obtain interesting estimates on the noise operator G_ϵ, it will be necessary to impose some additional conditions on its generating density g, regarding, e.g. its rate of decay at infinity, or the behaviour of its Fourier transform near the origin.

The weakest condition considered in this paper is the existence of some positive moment of g, by which we mean that for some $\alpha \in (0, 2]$,

$$M_\alpha = \int_{\mathbb{R}^d} |\boldsymbol{x}|^\alpha g(\boldsymbol{x})\, d\boldsymbol{x} < \infty, \tag{11}$$

(we take the length $|\boldsymbol{x}| = (x_1^2 + \cdots + x_d^2)^{1/2}$ on $\mathbb{R}^d$). This condition implies the following properties of the Fourier transform $\hat{g}$ (proved in appendix A.1):

Lemma 1. *For any $\alpha \in (0, 2]$ there exists a universal constant C_α such that, if a normalized density g satisfies (11), then the following inequalities hold:*

$$\forall \boldsymbol{\xi} \in \mathbb{R}^d, \qquad 0 \leqslant 1 - \hat{g}(\boldsymbol{\xi}) \leqslant C_\alpha M_\alpha |\boldsymbol{\xi}|^\alpha. \tag{12}$$

If (11) holds with $\alpha = 2$, we have the more precise behaviour:

$$1 - \hat{g}(\boldsymbol{\xi}) \sim |\boldsymbol{\xi}|^2 \qquad \textit{in the limit } \boldsymbol{\xi} \to 0.$$

In the case $\alpha < 2$, we will sometimes assume a stronger property than (12), namely that

$$1 - \hat{g}(\boldsymbol{\xi}) \sim |\boldsymbol{\xi}|^\alpha \quad \text{in the limit } \boldsymbol{\xi} \to 0. \tag{13}$$

Note that this behaviour implies a uniform bound $1 - \hat{g}(\boldsymbol{\xi}) \leqslant C|\boldsymbol{\xi}|^\gamma$ for any $\gamma \leqslant \alpha$ and C independent of γ.

Typical examples of noise kernels satisfying (13) include the Gaussian kernel and more general symmetric α-stable kernels [27, p 152] defined for $\alpha \in (0, 2]$:

$$g_{\epsilon,\alpha}(\boldsymbol{x}) := \sum_{\boldsymbol{k} \in \mathbb{Z}^d} \mathrm{e}^{-(\boldsymbol{Q}(\epsilon \boldsymbol{k}))^{\alpha/2}} \boldsymbol{e}_{\boldsymbol{k}}(\boldsymbol{x}), \tag{14}$$

where $\boldsymbol{Q}$ denotes an arbitrary positive definite quadratic form. For the values of α indicated, the function $g_{\epsilon,\alpha}(x)$ is positive on $\mathbb{R}^d$.

In view of equation (6), the properties (11) or (13) directly constrain the rate at which G_ϵ contracts on $L_0^2(\mathbb{T}^d)$. For instance, (13) implies that in the limit $\epsilon \to 0$,

$$1 - \|G_\epsilon\| \sim \epsilon^\alpha. \tag{15}$$

The following proposition describes the effect of the noise on various types of observables, in the limit of small noise level. The proofs are given in appendix A.2.

Proposition 1.

(i) For any noise generating density $g \in L^1(\mathbb{R}^d)$ and any observable $f \in L_0^2(\mathbb{T}^d)$, one has

$$\|G_\epsilon f - f\| \overset{\epsilon \to 0}{\to} 0. \tag{16}$$

To obtain information on the speed of convergence, we need to impose constraints on both the noise kernel and the observable.

(ii) If for some $\alpha \in (0, 2]$ the kernel g satisfies (11) or (13), then for any $\gamma > 0$ there exists a constant $C > 0$ such that for any observable $f \in H^\gamma(\mathbb{T}^d)$,

$$\|G_\epsilon f - f\| \leqslant C\epsilon^{\gamma \wedge \alpha} \|f\|_{H^{\gamma \wedge \alpha}}, \tag{17}$$

where $\gamma \wedge \alpha := \min\{\gamma, \alpha\}$. If $f \in C^1(\mathbb{T}^d)$, the above upper bound can be replaced by

$$\|G_\epsilon f - f\| \leqslant C\epsilon^{1\wedge\alpha} \|\nabla f\| \leqslant C\epsilon^{1\wedge\alpha} \|\nabla f\|_\infty. \tag{18}$$

Using the noise operator, we are now in a position to define the noisy (resp. the coarse-grained) dynamics generated by a measure-preserving map F.

2.4. Noisy evolution operator and dissipation time

The noisy evolution through the map F is constructed by successively applying the Koopman operator U_F and the noise operator G_ϵ, therefore by taking powers of the *noisy propagator*

$$T_\epsilon = G_\epsilon U_F.$$

In general, the operator T_ϵ is not normal, but satisfies $r_{\text{sp}}(T_\epsilon) \leqslant \|T_\epsilon\| = \|G_\epsilon\|$.

We will also consider a *coarse-grained dynamics* defined by the application of the noise kernel only at the beginning and the end of the evolution. Hence, we define the following family of operators:

$$\tilde{T}_\epsilon^{(n)} = G_\epsilon U_F^n G_\epsilon, \qquad n \in \mathbb{N}.$$

In view of the contracting properties of G_ϵ, the inequalities $\|T_\epsilon^n\| \leqslant \|G_\epsilon\|^n$, $\|\tilde{T}_\epsilon^{(n)}\| \leqslant \|G_\epsilon\|^2$ imply that both noisy and coarse-grained operators are strictly contracting on $L_0^2(\mathbb{T}^d)$. Our aim is to characterize the speed of contraction of these two evolutions, that is the behaviour of the norms $\|T_\epsilon^n\|$, $\|\tilde{T}_\epsilon^{(n)}\|$ in the joint limits $\epsilon \to 0$ and $n \to \infty$. This characterization will be connected with dynamical properties of the map F.

There are many ways to measure this speed of contraction. For instance, for fixed $\epsilon > 0$, the long-time decay of $\|T_\epsilon^n\|$ may be super-exponential (in which case T_ϵ is quasi-nilpotent) or exponential, governed by the largest eigenvalue of T_ϵ. However, such exponential behaviour may appear only after a transient time. In this paper, we will characterize the noisy dynamics through a single, robust characteristic, namely the *dissipation time*. In its general form, the dissipation time τ_* is defined in terms of the norm $\|\cdot\|_{p,0}$ on the space $L_0^p(\mathbb{T}^d)$, and an arbitrary threshold $\eta \in (0, 1)$:

$$\tau_*^{p,\eta}(\epsilon) := \min\{n \in \mathbb{N} : \|T_\epsilon^n\|_{p,0} < \eta\}, \qquad 1 \leqslant p \leqslant \infty. \tag{19}$$

We will be concerned with the behaviour of the dissipation time when the level of noise becomes small (as we prove in proposition 2, this time diverges in this limit). In [14], it was shown that this asymptotic behaviour is independent of the choice of $0 < \eta < 1$ and $1 < p < \infty$. Therefore, in this paper we will consider the computationally convenient choice $p = 2$ and $\eta = e^{-1}$. We will henceforth drop the superscripts, and the dependence of τ_* on ϵ will always be implicit:

$$\tau_* := \tau_*^{2,e^{-1}}(\epsilon) = \min\{n \in \mathbb{N} : \|T_\epsilon^n\| < e^{-1}\}. \tag{20}$$

A similar dissipation time will be defined for the coarse-grained evolution:

$$\tilde{\tau}_* := \min\{n \in \mathbb{N} : \|\tilde{T}_\epsilon^{(n)}\| < e^{-1}\}. \tag{21}$$

The dissipation time does not depend on whether the dynamics is applied to densities (i.e. by the Frobenius–Perron operator) or to observables (by the Koopman operator). Indeed, the norm of an operator equals the norm of its adjoint [30, p 195], so that

$$\|\tilde{T}_\epsilon^{(n)}\| = \|G_\epsilon U_F^n G_\epsilon\| = \|(G_\epsilon U_F^n G_\epsilon)^*\| = \|G_\epsilon P_F^n G_\epsilon\|,$$

and similarly for the noisy operator T_ϵ. In particular, for invertible maps the dissipation time does not depend on the direction of time.

We will distinguish two qualitatively different asymptotic behaviours of dissipation time in the limit $\epsilon \to 0$. We say that the operator T_ϵ (or the map F associated with it), respectively, has

(I) *simple* or *power-law* dissipation time if there exists $\beta > 0$ such that

$$\tau_* \sim \frac{1}{\epsilon^\beta},$$

(II) *fast* or *logarithmic* dissipation time if

$$\tau_* \sim \ln\left(\frac{1}{\epsilon}\right).$$

We will also talk about *slow* dissipation time whenever there exists some $\beta > 0$ s.t.

$$\tau_* \gtrsim \frac{1}{\epsilon^\beta}.$$

In the case of a logarithmic dissipation time, the dissipation rate constant R_*, when it exists, is defined as

$$R_* = \lim_{\epsilon\to 0} \frac{\tau_*}{\ln(1/\epsilon)}. \tag{22}$$

A similar terminology will be applied to the coarse-grained dissipation time $\tilde{\tau}_*$.

3. Spectral properties and dissipation time of non-weakly-mixing maps

In this section, we investigate the connection between the dissipation time of the noisy propagator T_ϵ and its pseudospectrum together with some spectral properties of U_F and G_ϵ. All the operators considered in this section are defined on $L^2_0(\mathbb{T}^d)$. In the framework of continuous-time dynamics, connections have been obtained between, on the one hand, the pseudospectrum of the (non-self-adjoint) generator A, and, on the other, the norm of the evolution operator e^{tA} [12]. We will obtain results of the same flavour, yet the proofs seem easier here than in the case of continuous time.

3.1. Definitions and general bounds

Let us start with the definition of the pseudospectrum of a bounded operator [29].

Definition 1. *Let T be a bounded linear operator on a Hilbert space $\mathcal{H}$ (we note $T \in \mathcal{L}(\mathcal{H})$). For any $\delta > 0$, the δ-pseudospectrum of T (denoted by $\sigma_\delta(T)$) can be defined in the following three equivalent ways:*

(I) $\sigma_\delta(T) = \{\lambda \in \mathbb{C} : \|(\lambda - T)^{-1}\| \geqslant \delta^{-1}\}$,
(II) $\sigma_\delta(T) = \{\lambda \in \mathbb{C} : \exists v \in \mathcal{H},\ \|v\| = 1,\ \|(T-\lambda)v\| \leqslant \delta\}$,
(III) $\sigma_\delta(T) = \{\lambda \in \mathbb{C} : \exists B \in \mathcal{L}(\mathcal{H}),\ \|B\| \leqslant \delta,\ \lambda \in \sigma(T+B)\}$.

We will apply these definitions to the operator T_ϵ. For brevity, the resolvent of this operator will be denoted by $R_\epsilon(\lambda) = (\lambda - T_\epsilon)^{-1}$. We call S^r the circle $\{\lambda \in \mathbb{C} : |\lambda| = r\}$ in the complex plane, and define the following *pseudospectrum distance function*:

$$d_\epsilon(r) := \inf\{\delta > 0 : \sigma_\delta(T_\epsilon) \cap S^r \neq \emptyset\}.$$

From the definition (I) of the pseudospectrum, one easily shows that this distance is also given by

$$d_\epsilon^{-1}(r) = \sup_{|\lambda|=r} \|R_\epsilon(\lambda)\|. \tag{23}$$

We first establish general (abstract) bounds for the dissipation time in terms of the spectral properties of G_ϵ and T_ϵ. In a second step, we relate these properties to dynamical properties of the underlying map F.

Theorem 1. *For any isometric operator U on $L^2_0(\mathbb{T}^d)$ and noise operator G_ϵ, the dissipation time of the noisy evolution operator $T_\epsilon = G_\epsilon U$ satisfies the following estimates:*

$$\frac{1-e^{-1}}{d_\epsilon(1)} \leqslant \tau_* \leqslant \frac{1}{|\ln(\|G_\epsilon\|)|} + 1, \tag{24}$$

$$\tau_* \leqslant \inf_{r_{\mathrm{sp}}(T_\epsilon)<r<1} \frac{1}{|\ln(r)|} \ln\left(\frac{e}{d_\epsilon(r)}\right). \tag{25}$$

We note that the first upper bound does not depend on U at all, but only on the noise. Using the estimate (15), we obtain the following obvious corollary:

Corollary 1. *If the noise generating density satisfies the estimate (13) for some $\alpha \in (0, 2]$, then for any measure-preserving map F the noisy dissipation time is bounded from above by:*

$$\tau_* \lesssim \epsilon^{-\alpha}.$$

Proof of the theorem.

1. *Lower bound*. We use the following series expansion of the resolvent [30, p 211] valid for any $|\lambda| > r_{\mathrm{sp}}(T_\epsilon)$:

$$R_\epsilon(\lambda) = \sum_{n=0}^{\infty} \lambda^{-n-1} T_\epsilon^n. \tag{26}$$

Considering that $r_{\mathrm{sp}}(T_\epsilon) \leqslant \|G_\epsilon\| < 1$, we may take $|\lambda| = 1$, and split this sum into two parts:

$$R_\epsilon(\lambda) = \sum_{n=0}^{\tau_*-1} \lambda^{-n-1} T_\epsilon^n + \lambda^{-\tau_*} T_\epsilon^{\tau_*} R_\epsilon(\lambda).$$

Taking norms and applying the triangle inequality, we get

$$\begin{aligned} \|R_\epsilon(\lambda)\| &\leqslant \left\| \sum_{n=0}^{\tau_*-1} \lambda^{-n-1} T_\epsilon^n \right\| + |\lambda|^{-\tau_*} \|T_\epsilon^{\tau_*}\| \|R_\epsilon(\lambda)\| \\ &\leqslant \tau_* + e^{-1} \|R_\epsilon(\lambda)\| \\ \Longrightarrow \|R_\epsilon(\lambda)\|&(1 - e^{-1}) \leqslant \tau_*. \end{aligned}$$

Taking the supremum over $\lambda \in S_1$ yields the lower bound.

2. *Upper bounds*. To get both upper bounds, we use the following trivial lemma.

Lemma 2. *Assume that (for some value of ϵ) the powers of T_ϵ satisfy*

$$\forall n \in \mathbb{N}, \qquad \|T_\epsilon^n\| \leqslant \Gamma(n),$$

where the function $\Gamma(n)$ is strictly decreasing, and $\Gamma(n) \overset{n\to\infty}{\to} 0$. Then, the dissipation time is bounded from above by

$$\tau_* \leqslant \Gamma^{(-1)}(e^{-1}) + 1,$$

where $\Gamma^{(-1)}$ is the inverse function of Γ. In particular, for the geometric decay $\Gamma(n) = Cr^n$ with $r \in (0, 1)$, $C \geqslant 1$, one obtains $\tau_* \leqslant \ln(eC)/|\ln r| + 1$.

The upper bound in equation (24) comes from the obvious estimate

$$\|T_\epsilon^n\| \leqslant \|G_\epsilon\|^n,$$

on which we apply the lemma with $C = 1$, $r = \|G_\epsilon\|$.

To prove the second upper bound, we use the representation of T_ϵ^n in terms of the resolvent:

$$T_\epsilon^n = \frac{1}{2\pi \mathrm{i}} \int_{S^r} \lambda^n R_\epsilon(\lambda)\, \mathrm{d}\lambda$$

valid for any $r > r_{\mathrm{sp}}(T_\epsilon)$. Thus, for all $r \in (r_{\mathrm{sp}}(T_\epsilon), 1)$, one has

$$\|T_\epsilon^n\| \leqslant \frac{1}{2\pi} \int_{S^r} |\lambda|^n \|R_\epsilon(\lambda)\| |\mathrm{d}\lambda| \leqslant \sup_{|\lambda|=r} \|R_\epsilon(\lambda)\| r^{n+1} = \frac{1}{d_\epsilon(r)} r^{n+1}.$$

We then apply lemma 2 on the geometric decay for any radius $r_{\mathrm{sp}}(T_\epsilon) < r < 1$, with $C = r/d_\epsilon(r) \geqslant 1$. ■

3.2. Consequences

We now use theorem 1 in the case where $U = U_F$ is the Koopman operator for some measure-preserving map F on $\mathbb{T}^d$ with some specific dynamical properties. We recall [10] that the map F is ergodic (resp. weakly-mixing) iff 1 is not an eigenvalue of U_F (resp. iff U_F has no eigenvalue) on $L_0^2(\mathbb{T}^d)$.

Proposition 2. *For any measure-preserving map F and any noise generating function g, the dissipation time of T_ϵ diverges in the small-noise limit $\epsilon \to 0$.*

Proof. We drop the subscript F to simplify the notation. We only use the fact that $U = U_F$ is an isometry. We shall prove by induction the following strong convergence of operators:

$$\forall f \in L_0^2(\mathbb{T}^d), \quad \forall n \in \mathbb{N}, \qquad \|T_\epsilon^n f - U^n f\| \stackrel{\epsilon\to 0}{\to} 0.$$

From proposition 1(*i*), this limit holds in the case $n = 1$. Let us assume it holds at the rank $n - 1$. Then, we write

$$T_\epsilon^n f = U T_\epsilon^{n-1} f + (G_\epsilon - I) U T_\epsilon^{n-1} f.$$

From the inductive hypothesis, $T_\epsilon^{n-1} f \stackrel{\epsilon\to 0}{\to} U^{n-1} f$, so that the first term on the RHS converges to $U^n f$. Applying proposition 1(*i*) to the function $U^n f$, we see that the second term vanishes in the limit $\epsilon \to 0$. From the isometry of U, we obtain that for any $n > 0$, $\|T_\epsilon^n\| \stackrel{\epsilon\to 0}{\to} 1$, so that $\tau_* \stackrel{\epsilon\to 0}{\to} \infty$. ■

This 'non-finiteness' of the noisy dissipation time allows us to prove another general result concerning the pseudospectrum of T_ϵ (this corollary is proved in appendix A.3):

Corollary 2. *For any isometry U and noise generating function g, one has*

$$d_\epsilon(1) \stackrel{\epsilon\to 0}{\to} 0. \tag{27}$$

This means that for any fixed $\delta > 0$, the pseudospectrum $\sigma_\delta(T_\epsilon)$ will intersect the unit circle for small enough ϵ.

In order to better control the growth of τ_*, we need more precise information on the noise and the dynamics. In the present section, we restrict ourselves to the dynamical property of weak-mixing.

Corollary 3. *Assume that the noise generating density g satisfies the estimates (11) or (13) with exponent* $\alpha \in (0, 2]$. *If F is not weakly-mixing and at least one eigenfunction of* U_F *belongs to* $H^\gamma(\mathbb{T}^d)$ *for some* $\gamma > 0$, *then* T_ϵ *has a slow dissipation time:*

$$\epsilon^{-(\alpha\wedge\gamma)} \lesssim \tau_*.$$

Proof. Let $h \in H^\gamma(\mathbb{T}^d)$ be a normalized eigenfunction of U_F with eigenvalue λ. Applying proposition 1(*ii*), we get

$$\|(\lambda - T_\epsilon)h\| = \|(I - G_\epsilon)h\| \leqslant K\epsilon^{\gamma\wedge\alpha}$$

for some constant $K > 0$ depending on g and h. This implies that $\|R_\epsilon(\lambda)\| \geqslant 1/K\epsilon^{\gamma\wedge\alpha}$; therefore, taking the supremum over $|\lambda| = 1$ yields $d_\epsilon(1)^{-1} \geqslant 1/K\epsilon^{\gamma\wedge\alpha}$. The lower bound in theorem 1 then implies

$$\frac{1 - e^{-1}}{K\epsilon^{\gamma\wedge\alpha}} \leqslant \tau_*. \tag{28}$$

■

Remark 1. Recall that if g satisfies (13) with exponent α, then the dissipation time is also bounded from above, as shown in corollary 1. If one eigenfunction h has regularity H^γ with $\gamma \geqslant \alpha$, then both corollaries imply that the dissipation is simple, of exponent α.

Remark 2. The above results can be stated in a more general form: U_F does not need to be a Koopman operator associated with a map F. The result holds true for any isometric operator U on L_0^2 with an eigenfunction of Sobolev regularity.

The dependence of the lower bound in (28) on γ can be intuitively explained as follows. In the case of non-weakly-mixing maps the eigenfunctions of U_F are, in general, responsible for slowing down the dissipation. The rate of dissipation is affected by the regularity of the smoothest eigenfunction. In principle, irregular functions undergo faster dissipation giving rise to slower asymptotics of τ_*. It is not clear, however, whether the actual asymptotics of the dissipation time will be slower than power law asymptotics in the case when all eigenfunctions of U_F on $L_0^2(\mathbb{T}^d)$ are outside any space $H^\gamma(\mathbb{T}^d)$ with $\gamma > 0$.

In corollary 2 we have shown that for any map F and arbitrary small $\delta > 0$, the pseudospectrum $\sigma_\delta(T_\epsilon)$ intersects the unit circle for sufficiently small $\epsilon > 0$. If F is not weakly-mixing, the spectral radius of T_ϵ (i.e. the modulus of its largest eigenvalue) is believed to converge to 1 when $\epsilon \to 0$, and the associated eigenstate h_ϵ should converge to a 'noiseless eigenstate' h. This 'spectral stability' has been discussed for several cases in the continuous-time as well as for discrete-time maps on $\mathbb{T}^2$ [21, 25].

In contrast, if F is an Anosov map on $\mathbb{T}^2$ (see section 6.3), the spectrum of T_ϵ does not approach the unit circle, but stays away from it uniformly: $r_{\mathrm{sp}}(T_\epsilon)$ is smaller than some $r_0 < 1$ for any $\epsilon > 0$ [7]. Simultaneously, $\|T_\epsilon\| \to 1$, so we have here a clear manifestation of the *non-normality* of T_ϵ for such a map. In some cases (see [25] and the linear examples of section 6), the operator T_ϵ is even quasi-nilpotent, meaning that $r_{\mathrm{sp}}(T_\epsilon) = 0$ for all $\epsilon > 0$. For such an Anosov map, the spectral radius of T_ϵ is, therefore, 'unstable' or 'discontinuous' in the limit $\epsilon \to 0$, while in the same limit the (radius of its) pseudospectrum $\sigma_\delta(T_\epsilon)$ (for $\delta > 0$ fixed) is 'stable'.

4. Universal lower bounds for the dissipation time of C^1 maps

In this section, we consider both noisy and coarse-grained evolutions. We start our discussion with general properties of the coarse-grained dissipation time.

Proposition 3. *Let F be a measure-preserving map:*

(i) For an arbitrary noise kernel, the coarse-grained dissipation time diverges as $\epsilon \to 0$.
(ii) If F is not weakly-mixing, then $\tilde{\tau}_* = \infty$ for small enough $\epsilon > 0$.

Proof.

(i) Similarly as in proposition 2 we have

$$\|\tilde{T}_\epsilon^n f - U^n f\| = \|G_\epsilon U_F^n (G_\epsilon - I) f + (G_\epsilon - I) U^n f\| \leqslant \|(G_\epsilon - I) f\| + \|(G_\epsilon - I) U^n f\| \to 0.$$

(ii) Let $h \in L_0^2(\mathbb{T}^d)$ be a normalized eigenfunction of U_F, then

$$\|\tilde{T}_\epsilon^{(n)} h\| = \|G_\epsilon U_F^n (h + (G_\epsilon - I) h)\| \geqslant \|G_\epsilon h\| - \|G_\epsilon U_F^n (G_\epsilon - I) h\| \geqslant 1 - 2\|(G_\epsilon - I) h\|.$$

Since the RHS above is independent of n, we see that $\|\tilde{T}_\epsilon^{(n)}\|$ is close to 1 for all times and sufficiently small $\epsilon > 0$. ∎

As opposed to the noisy case (see proposition 1), the coarse-grained evolution through a non-weakly mixing map does not dissipate. Therefore, there exists no general upper bound for $\tilde{\tau}_*$.

In contrast, we will prove below a general *lower* bound for both coarse-grained and noisy evolutions, valid for any measure-preserving map F of regularity C^1. We note that propositions 2 and 3(*i*) (which are valid independently of any regularity assumption) do not provide an explicit lower bound.

First, we introduce some notation. For any map $F \in C^1$, $DF(\boldsymbol{x})$ is the tangent map of F at the point $\boldsymbol{x} \in \mathbb{T}^d$, mapping a tangent vector at $\boldsymbol{x}$ to a tangent vector at $F(\boldsymbol{x})$. Selecting the canonical (i.e. Cartesian) basis and metrics on $T(\mathbb{T}^d)$, this map can be represented as a $d \times d$ matrix. The metrics naturally yield a norm $\boldsymbol{v} \in T_{\boldsymbol{x}}(\mathbb{T}^d) \mapsto |v|$ on the tangent space, and therefore a norm on this matrix: $|DF(\boldsymbol{x})| = \max_{|\boldsymbol{v}|=1} |DF(\boldsymbol{x}) \cdot \boldsymbol{v}|$. We are now in position to define the maximal expansion rate of F:

$$\mu_F = \limsup_{n\to\infty} \|DF^n\|_\infty^{1/n}, \qquad \text{where } \|DF^n\|_\infty = \sup_{\boldsymbol{x}\in\mathbb{T}^d} |(DF^n)(\boldsymbol{x})|.$$

Since F preserves the Lebesgue measure, the Jacobian $J_F(\boldsymbol{x})$ satisfies $|J_F(\boldsymbol{x})| \geqslant 1$ at all points. In the Cartesian basis, $J_F(\boldsymbol{x}) = \det(DF(\boldsymbol{x}))$, so that we have $\|DF^n(\boldsymbol{x})\| \geqslant 1$ for all $\boldsymbol{x} \in \mathbb{T}^d$, $n \geqslant 0$. One can actually prove the following remark.

Remark 3. Although $|(DF^n)(\boldsymbol{x})|$ and $\|DF\|_\infty$ may depend on the choice of the metrics, μ_F does not, and satisfies $1 \leqslant \mu_F \leqslant \|DF\|_\infty$.

From the definition of μ_F, for any $\mu > \mu_F$ there exists a constant $A \geqslant 1$ such that

$$\forall n \in \mathbb{N}, \qquad \|DF^n\|_\infty \leqslant A\mu^n. \tag{29}$$

In some cases one may take $\mu = \mu_F$ in the RHS. In the case $\mu_F = 1$, $\|DF^n\|_\infty$ can sometimes grow as a power-law:

$$\|DF^n\|_\infty \leqslant An^\beta, \qquad n \in \mathbb{N} \tag{30}$$

for some $\beta > 0$, or even be uniformly bounded by a constant ($\beta = 0$).

The relationship between the local expansion of the map F, on the one hand, and the dissipation time, on the other, can be intuitively understood as follows. A lack of expansion ($\|DF\|_\infty = 1$) results in the transformation of 'soft' or 'long-wavelength' oscillations into 'soft oscillations', both being little affected by the noise operator G_ϵ. In contrast, a locally strictly expansive map ($\|DF\|_\infty > 1$) will quickly transform soft oscillations into 'hard' or 'short-wavelength' oscillations, the latter being much more damped by the noise.

The following theorem precisely measures this relationship, in terms of *lower bounds* for the dissipation times.

Theorem 2. *Let F be a measure-preserving C^1 map on $\mathbb{T}^d$, and assume that the noise generating density g satisfies (11) or (13) for some $\alpha \in (0, 2]$.*

(i) If $\|DF\|_\infty > 1$, resp. $\mu_F > 1$, then there exists a constant c, resp. constants $\mu \geqslant \mu_F$ and $\tilde{c}$, such that for small enough ϵ,

$$\tau_* \geqslant \frac{\alpha \wedge 1}{\ln(\|DF\|_\infty)} \ln(\epsilon^{-1}) + c, \qquad \textit{resp. } \tilde{\tau}_* \geqslant \frac{\alpha \wedge 1}{\ln \mu} \ln(\epsilon^{-1}) + \tilde{c}. \tag{31}$$

If F is a C^1 diffeomorphism, then (31) holds with $\|DF\|_\infty$ replaced by $\|DF\|_\infty \wedge \|D(F^{-1})\|_\infty$, resp. with some $\mu \geqslant \mu_F \wedge \mu_{F^{-1}}$.

(ii) If $\|DF\|_\infty = 1$, then T_ϵ has slow dissipation time, $\tau_* \gtrsim \epsilon^{-(\alpha\wedge 1)}$. If the noise kernel satisfies the condition (13) for $\alpha \in (0, 1]$, then the dissipation time is simple, $\tau_* \sim \epsilon^{-\alpha}$.

(iii) If $\mu_F = 1$ and $\|DF^n\|_\infty$ grows as a power-law as in equation (30) with $\beta > 0$, then $\tilde{\tau}_* \gtrsim \epsilon^{-(\alpha\wedge 1)/\beta}$. If $\|DF^n\|_\infty$ is uniformly bounded above by a constant, then $\tilde{\tau}_* = \infty$ for small enough ϵ.

Remark 4. This theorem shows that classical systems on $\mathbb{T}^d$ (i.e. C^1 diffeomorphisms) cannot have a dissipation time growing slower than $C \ln(\epsilon^{-1})$. In view of the results for toral automorphisms (cf proposition 4), this lower bound on the dissipation time is sharp and consistent with Kouchnirenko's upper bound on the entropy of the classical systems, namely all classical systems have a finite (possibly zero) Kolmogorov–Sinai entropy (theorem 12.35. in [1], see also [2, 22]).

Proof of the theorem. We will need the following trivial lemma (similar to lemma 2).

Lemma 3. *Assume that there exists some $\alpha > 0$ and a strictly increasing function $\gamma(n)$, $\gamma(0) = 0$ such that*

$$\forall n \geqslant 1, \qquad \|T_\epsilon^n\| \geqslant 1 - \epsilon^\alpha \gamma(n). \tag{32}$$

Then, the dissipation time is bounded from below as:

$$\tau_* \geqslant \gamma^{(-1)} \left(\frac{1 - e^{-1}}{\epsilon^\alpha} \right), \tag{33}$$

where $\gamma^{(-1)}$ is the inverse function of γ.

The same statement holds for the coarse-grained version.

Our task is, therefore, to bound $\|T_\epsilon^n\|$ (resp. $\|\tilde{T}_\epsilon^{(n)}\|$) from below. A simple computation shows that for any $f \in C^0(\mathbb{T}^d)$, $\|G_\epsilon f\|_\infty \leqslant \|f\|_\infty$. Since convolution commutes with differentiation, for $f \in C^1$ we also have $\|\nabla(G_\epsilon f)\|_\infty \leqslant \|\nabla f\|_\infty$. We use this fact to estimate the gradient of $T_\epsilon f$:

$$\begin{aligned} \|\nabla(T_\epsilon f)\|_\infty &= \|\nabla(G_\epsilon U_F f)\|_\infty \\ &\leqslant \|\nabla(f \circ F)\|_\infty = \|(\nabla f) \circ F \cdot DF\|_\infty \\ &\leqslant \|(\nabla f) \circ F\|_\infty \|DF\|_\infty = \|\nabla f\|_\infty \|DF\|_\infty. \end{aligned}$$

Repeating the above procedure m times, we get

$$\|\nabla(T_\epsilon^m f)\|_\infty \leqslant \|\nabla f\|_\infty \|DF\|_\infty^m, \qquad \|\nabla(U_F T_\epsilon^m f)\|_\infty \leqslant \|\nabla f\|_\infty \|DF\|_\infty^{m+1}. \tag{34}$$

We now choose some arbitrary $f \in C_0^1(\mathbb{T}^d)$, with $\|f\| = 1$. We first apply the triangle inequality:

$$\|T_\epsilon^n f\| = \|G_\epsilon U_F T_\epsilon^{n-1} f\| \geqslant \|U_F T_\epsilon^{n-1} f\| - \|(G_\epsilon - I) U_F T_\epsilon^{n-1} f\|.$$

To estimate the second term on the RHS we use the bound (18) and the estimate (34) to obtain

$$\|T_\epsilon^n f\| \geqslant \|T_\epsilon^{n-1} f\| - C\epsilon^{\alpha\wedge 1} \|\nabla f\|_\infty \|DF\|_\infty^n.$$

Applying the same procedure iteratively to the first term on the RHS, we finally get (remember $\|f\| = 1$):

$$\|T_\epsilon^n\| \geqslant \|T_\epsilon^n f\| \geqslant 1 - C\epsilon^{\alpha\wedge 1} \|\nabla f\|_\infty \sum_{m=1}^{n} \|DF\|_\infty^m. \tag{35}$$

The computations in the case of the coarse-grained operator are even simpler:

$$\begin{aligned} \|\tilde{T}_\epsilon^{(n)} f\| &= \|G_\epsilon U_F^n G_\epsilon f\| \\ &\geqslant 1 - C\epsilon^{\alpha\wedge 1} \|\nabla f\|_\infty - C\epsilon^{\alpha\wedge 1} \|\nabla(G_\epsilon f)\|_\infty \|DF^n\|_\infty \\ &\geqslant 1 - 2C\epsilon^{\alpha\wedge 1} \|\nabla f\|_\infty \|DF^n\|_\infty. \end{aligned} \tag{36}$$

Note that from the assumptions on f, $\|\nabla f\|_\infty$ cannot be made arbitrary small, but is necessarily larger than some positive constant. We choose some arbitrary function, say $f = \boldsymbol{e_k}$ with $\boldsymbol{k} = (1, 0)$ which satisfies $\|\nabla f\|_\infty = 2\pi$.

The estimate (35) has the form given in lemma 3. The growth of the function $\gamma(n)$ depends on whether $\|DF\|_\infty$ is equal to or larger than 1, which explains why the lower bounds are qualitatively different in the two cases.

In the case when $\|DF\|_\infty$ is strictly larger than 1, the function $\gamma(n)$ grows like an exponential; therefore, the lower bound is of the type (31). For the coarse-grained version, a growth of $\|DF\|_\infty$ of the type (29) yields the lower bound for $\tilde{\tau}_*$ in (31).

In the case $\|DF\|_\infty = 1$, $\gamma(n)$ is a linear function, so that $\tau_* \geqslant ((1 - e^{-1})/C\|\nabla f\|_\infty)\epsilon^{-(\alpha\wedge 1)}$.

In the coarse-grained version, if $\mu_F = 1$ and $\|DF^n\|_\infty$ grows like in (30) with $\beta > 0$, the dissipation is slow: $\tilde{\tau}_* \geqslant C\epsilon^{-(\alpha\wedge 1)/\beta}$. In the case where $\|DF^n\|_\infty$ is uniformly bounded by some constant, the norm of the coarse-grained propagator stays larger than some positive constant for all times, so that for small enough noise $\tilde{\tau}_*$ is infinite. ■

5. An upper bound of the dissipation time for mixing maps

For any two functions $f, h \in L_0^2(\mathbb{T}^d)$, the dynamical correlation function for the map F is defined as the following function of $n \in \mathbb{N}$ (see, e.g. [3]):

$$C_{f,h}(n) = C_{f,h}^0(n) = m(f U_F^n h) = \langle \bar{f}, U_F^n h\rangle = \langle P_F^n \bar{f}, h\rangle.$$

The same quantity may be defined for the noisy evolution:

$$C_{f,h}^\epsilon(n) = m(f T_\epsilon^n h).$$

We recall that a map F is *mixing* iff for any $f,\ h \in L_0^2$,

$$C_{f,h}(n) \to 0, \qquad \text{as } n \to \infty.$$

The correlation function can easily be measured in (numerical or real-life) experiments, so it is often used to characterize the dynamics of a system.

To focus our attention, we will only be concerned with maps for which correlations decay in a precise way. We assume that there exist Hölder exponents $s_*, s \in \mathbb{R}_+$, $0 \leqslant s_* \leqslant s$ together with some decreasing function $\Gamma(n) = \Gamma_{s_*,s}(n)$ with $\Gamma(n) \stackrel{n\to\infty}{\to} 0$, such that for any observables $f \in C_0^{s_*}(\mathbb{T}^d)$, $h \in C_0^s(\mathbb{T}^d)$ and for sufficiently small $\epsilon \geqslant 0$ (sometimes only for $\epsilon = 0$),

$$\forall n \in \mathbb{N}, \qquad |C^{\epsilon}_{f,h}(n)| \leqslant \|f\|_{C^{s_*}} \|h\|_{C^s}\, \Gamma(n). \tag{37}$$

In general, such a bound can be proven only if the map F has regularity C^{s+1}. The reason why we do not necessarily take the same norm for the functions f and h will be clear below.

We will be mainly interested in the following two types of decay

(i) Power-law decay: there exists $C > 0$, $\beta > 0$ such that,

$$\Gamma(n) = C n^{-\beta}. \tag{38}$$

This behaviour is characteristic of intermittent maps, e.g. maps possessing one or several neutral orbits [4].

(ii) Exponential decay: there exists $C > 0$, $0 < \sigma < 1$ such that,

$$\Gamma(n) = C\sigma^n. \tag{39}$$

Such a behaviour was proved in the case of uniformly expanding or hyperbolic maps on the torus (see section 6), as well as for many more general cases [4].

The central result of this section is a relationship between the decay of noisy (resp. noiseless) correlations, on the one hand, and the small-noise behaviour of the noisy (resp. coarse-graining) dissipation time on the other. The intuitive picture is similar to the one linking the local expansion rate to the dissipation: namely, a fast decay of correlations is generally due to the transition of 'soft' into 'hard' fluctuations of the observable through the evolution, which is itself induced by large expansion rates of the map. Still, as opposed to what we obtained in the last section, the following theorem and its corollary yield *upper bounds* for the dissipation time.

Theorem 3. *Let F be a volume-preserving map on $\mathbb{T}^d$ with correlations decaying as in equation (37) for some indices s, s_* and decreasing function $\Gamma(n)$, at least in the noiseless limit $\epsilon = 0$. Assume that the noise generating function g is $([s] + 1)$-differentiable, and that all its derivatives of order $|\alpha|_1 \leqslant [s] + 1$ satisfy*

$$|D^{\alpha} g(\boldsymbol{x})| \lesssim \frac{1}{|\boldsymbol{x}|^M}, \qquad |\boldsymbol{x}| \gg 1,$$

with a power $M > d$.

Then, there exist constants $\tilde{C} > 0$, $\epsilon_0 > 0$ such that the coarse-grained propagator satisfies

$$\forall \epsilon \leqslant \epsilon_0, \quad \forall n \geqslant 0, \qquad \|\tilde{T}^{(n)}_{\epsilon}\| \leqslant \tilde{C}\, \frac{\Gamma(n)}{\epsilon^{d+s+s_*}}. \tag{40}$$

If the decay of correlations (37) also holds for sufficiently small $\epsilon > 0$ (and assuming the Perron–Frobenius operator P_F is bounded in $C^s(\mathbb{T}^d)$), then the noisy operator satisfies (for some constants $C > 0$, $\epsilon_0 > 0$):

$$\forall \epsilon \leqslant \epsilon_0, \quad \forall n \geqslant 0, \qquad \|T^n_{\epsilon}\| \leqslant C\, \frac{\Gamma(n)}{\epsilon^{d+s+s_*}}. \tag{41}$$

From these estimates, we straightforwardly obtain the following bounds on both dissipation times (the assumptions on F and the noise generating function g are the same as in the theorem):

Corollary 4.

(I) If the correlation function satisfies the bound (37) for $\epsilon = 0$*, then the coarse-grained dissipation time is well defined (*$\tilde{\tau}_* < \infty$*). Moreover,*

(i) if $\Gamma(n) \sim n^{-\beta}$ *then there exists a constant* $\tilde{C} > 0$ *such that*

$$\tilde{\tau}_* \leqslant \tilde{C}\epsilon^{-((d+s+s_*)/\beta)},$$

(ii) if $\Gamma(n) \sim \sigma^n$ *then there exists a constant* $\tilde{c}$ *such that*

$$\tilde{\tau}_* \leqslant \frac{d+s+s_*}{|\ln\sigma|}\ln(\epsilon^{-1}) + \tilde{c},$$

(II) If equation (37) holds for sufficiently small $\epsilon > 0$*, then*

(i) if $\Gamma(n) \sim n^{-\beta}$*, there exists a constant* $C > 0$ *such that*

$$\tau_* \leqslant C\epsilon^{-((d+s+s_*)/\beta)},$$

(ii) if $\Gamma(n) \sim \sigma^n$*, there exists a constant c such that*

$$\tau_* \leqslant \frac{d+s+s_*}{|\ln\sigma|}\ln(\epsilon^{-1}) + c.$$

Proof of the theorem

1st step. We represent the action of T_ϵ^n (resp. $\tilde{T}_\epsilon^{(n)}$) on an observable $f \in L_0^2(\mathbb{T}^d)$ in terms of the correlation functions $C^\epsilon(n)$ (resp. $C(n)$). To do this we Fourier decompose both $T_\epsilon^{n+2} f$ and $f_1 = U_F f$, and use equation (5):

$$\begin{aligned}
T_\epsilon^{n+2} f &= \sum_{0\neq \boldsymbol{j}\in\mathbb{Z}^d} \langle \boldsymbol{e_j}, G_\epsilon U_F T_\epsilon^n G_\epsilon f_1\rangle \boldsymbol{e_j} \\
&= \sum_{0\neq \boldsymbol{j}\in\mathbb{Z}^d} \sum_{0\neq \boldsymbol{k}\in\mathbb{Z}^d} \hat{f}_1(\boldsymbol{k})\langle G_\epsilon \boldsymbol{e_j}, U_F T_\epsilon^n G_\epsilon \boldsymbol{e_k}\rangle \boldsymbol{e_j} \\
&= \sum_{0\neq \boldsymbol{j}\in\mathbb{Z}^d} \sum_{0\neq \boldsymbol{k}\in\mathbb{Z}^d} \hat{f}_1(\boldsymbol{k})\hat{g}_\epsilon(\boldsymbol{j})\hat{g}_\epsilon(\boldsymbol{k})\langle P_F \boldsymbol{e_j}, T_\epsilon^n \boldsymbol{e_k}\rangle \boldsymbol{e_j}
\end{aligned}$$

(remember that $\hat{g}$ is a real function). A similar computation for the coarse-grained propagator yields:

$$\tilde{T}_\epsilon^{(n)} f = \sum_{0\neq \boldsymbol{j}\in\mathbb{Z}^d} \sum_{0\neq \boldsymbol{k}\in\mathbb{Z}^d} \hat{f}(\boldsymbol{k})\hat{g}_\epsilon(\boldsymbol{j})\hat{g}_\epsilon(\boldsymbol{k})\langle \boldsymbol{e_j}, U_F^n \boldsymbol{e_k}\rangle \boldsymbol{e_j}.$$

Taking the norms on both sides, we get in the noisy case:

$$\begin{aligned}
\|T_\epsilon^{n+2} f\|^2 &= \sum_{0\neq \boldsymbol{j}\in\mathbb{Z}^2} \left| \sum_{0\neq \boldsymbol{k}\in\mathbb{Z}^d} \hat{f}_1(\boldsymbol{k})\langle P_F \boldsymbol{e_j}, T_\epsilon^n \boldsymbol{e_k}\rangle \hat{g}_\epsilon(\boldsymbol{j})\hat{g}_\epsilon(\boldsymbol{k}) \right|^2 \\
&\leqslant \sum_{0\neq \boldsymbol{j}\in\mathbb{Z}^d} \left(\sum_{0\neq \boldsymbol{k}\in\mathbb{Z}^d} |\hat{f}_1(\boldsymbol{k})|^2 \right) \sum_{0\neq \boldsymbol{k}\in\mathbb{Z}^d} |\langle P_F \boldsymbol{e_j}, T_\epsilon^n \boldsymbol{e_k}\rangle|^2 |\hat{g}_\epsilon(\boldsymbol{j})\hat{g}_\epsilon(\boldsymbol{k})|^2
\end{aligned}$$

$$\Longrightarrow \|T_\epsilon^{n+2} f\|^2 \leqslant \|f_1\|^2 \sum_{0\neq \boldsymbol{j},\boldsymbol{k}\in\mathbb{Z}^d} |C^\epsilon_{P_F \boldsymbol{e}_{-\boldsymbol{j}}, \boldsymbol{e_k}}(n)|^2 |\hat{g}(\epsilon \boldsymbol{j})\hat{g}(\epsilon \boldsymbol{k})|^2, \tag{42}$$

and in the coarse-grained case

$$\|\tilde{T}_\epsilon^{(n)} f\|^2 \leqslant \|f\|^2 \sum_{0\neq \boldsymbol{j},\boldsymbol{k}\in\mathbb{Z}^d} |C_{\boldsymbol{e}_{-\boldsymbol{j}}, \boldsymbol{e_k}}(n)|^2 |\hat{g}(\epsilon \boldsymbol{j})\hat{g}(\epsilon \boldsymbol{k})|^2. \tag{43}$$

These two expressions explicitly relate the dissipation with the correlation functions.

2nd step. We now apply the estimates (37) on correlations for the observables $\boldsymbol{e_k}$, $\boldsymbol{e_{-j}}$, $P_F\boldsymbol{e_{-j}}$. In the coarse-grained case, it yields (using simple bounds of the type in equation (59)):

$$\forall \boldsymbol{j},\ \boldsymbol{k} \in \mathbb{Z}^d \setminus \{0\}, \qquad |C_{\boldsymbol{e_{-j}},\boldsymbol{e_k}}(n)| \leqslant C'\ |\boldsymbol{j}|^s |\boldsymbol{k}|^{s_*}\Gamma(n).$$

In the noisy case, we need to assume that the Perron–Frobenius operator P_F is bounded in the space $C^s(\mathbb{T}^d)$. This property is, in general, a prerequisite in the proof of estimates of the type (37), so this assumption is quite natural here.

$$\forall \boldsymbol{j},\ \boldsymbol{k} \in \mathbb{Z}^d \setminus \{0\}, \qquad |C^{\epsilon}_{P_F\boldsymbol{e_{-j}},\boldsymbol{e_k}}(n)| \leqslant C\ \|P_F\boldsymbol{e_{-j}}\|_{C^s}\|\boldsymbol{e_k}\|_{C^{s_*}}\Gamma(n)$$
$$\leqslant C\|P_F\|_{C^s}|\boldsymbol{j}|^s|\boldsymbol{k}|^{s_*}\Gamma(n). \qquad (44)$$

We insert these bounds on the decay of correlations in the expressions (42)–(43); for instance, in the coarse-grained case we get

$$\forall n \geqslant 0, \qquad \|\tilde{T}^{(n)}_{\epsilon}\|^2 \leqslant C\ \Gamma(n)^2\left(\epsilon^{-(s+s_*)}\sum_{0\neq\boldsymbol{k}\in\mathbb{Z}^d}|\epsilon\boldsymbol{k}|^{s+s_*}\hat{g}(\epsilon\boldsymbol{k})^2\right)^2. \qquad (45)$$

3rd step. We finally estimate the ϵ-dependence of the RHS of the above inequality. Up to a factor ϵ^{-d}, the sum in the brackets is a Riemann sum for the integral $\int|\boldsymbol{\xi}|^{s+s_*}\hat{g}(\boldsymbol{\xi})^2\mathrm{d}\boldsymbol{\xi} < \infty$. A precise connection is given in the following lemma and proved in appendix A.4:

Lemma 4. *Let $f \in C^0(\mathbb{R}^d)$ be symmetric w.r.t. the origin and decaying at infinity as $|f(\boldsymbol{x})| \lesssim |\boldsymbol{x}|^{-M}$ with $M > d$. Then, the following estimate holds in the limit $\epsilon \to 0$:*

$$\epsilon^d\sum_{\boldsymbol{k}\in\mathbb{Z}^d}\hat{f}(\epsilon\boldsymbol{k})^2 = \int_{\mathbb{R}^d}\hat{f}(\boldsymbol{\xi})^2\,\mathrm{d}\boldsymbol{\xi} + \mathcal{O}(\epsilon^M). \qquad (46)$$

Let $m \in \mathbb{N}$ satisfy $2m \leqslant s + s_* \leqslant 2m + 2$ (note that $m \leqslant [s]$ since we assumed $s_* \leqslant s$). From the obvious inequality

$$\forall x > 0, \qquad x^{s+s_*} \leqslant x^{2m} + x^{2m+2},$$

we may replace in the RHS of (45) the factor $|\epsilon\boldsymbol{k}|^{s+s_*}$ by $|\epsilon\boldsymbol{k}|^{2m} + |\epsilon\boldsymbol{k}|^{2m+2}$. Applying lemma 4 to the derivatives of g of order m and $m + 1$, we end up with the following upper bound, which proves the first part of the theorem:

$$\|\tilde{T}^{(n)}_{\epsilon}\|^2 \leqslant C\ \Gamma(n)^2\left(\frac{1}{\epsilon^{d+s+s_*}}\int_{\mathbb{R}^d}\left(|\boldsymbol{\xi}|^{2m} + |\boldsymbol{\xi}|^{2(m+1)}\right)\hat{g}(\boldsymbol{\xi})^2 d\boldsymbol{\xi} + \mathcal{O}(\epsilon^M)\right)^2$$
$$\leqslant C'\ \frac{\Gamma(n)^2}{\epsilon^{2(d+s+s_*)}}\ \|g\|^4_{H^{m+1}}.$$

The computations follow identically for the case of the noisy operator, yielding the second part of the theorem. ∎

6. Some examples of mixing maps

In this section, we apply the results of the last two sections to several classes of Lebesgue measure-preserving maps which have been proven to be mixing, with various types of correlation decays.

Remark. In general, an expanding or hyperbolic map on the torus does not preserve the Lebesgue measure, so the first step is to specify precisely with respect to which invariant measure one wants to study the ergodic properties. In the 'nice' cases, one can prove the existence and uniqueness of a 'physical measure', which is ergodic for the map F, and then study the (noisy or noiseless) mixing properties with respect to this measure. As pointed out in the introduction, in this paper, we only consider maps for which the physical measure is the Lebesgue measure.

6.1. Decays of correlations

We briefly summarize the results concerning mixing maps, mostly relying on the review [4] and the book [3].

Let us first consider the noiseless correlations. A common route to proving that a map F is *exponentially* mixing consists in identifying an invariant space $\mathcal{B}$ of densities, on which the Perron–Frobenius has the following spectral structure: except for the eigenvalue 1 associated with the constant density, the rest of the spectrum (on the subspace $\mathcal{B}_0$ of zero-mean densities) is contained inside a disc of radius $\sigma < 1$ centred at the origin. More precisely, the spectrum on $\mathcal{B}_0$ may consist of isolated eigenvalues (called *resonances*) $\{\lambda_i\}$ and of some essential spectrum in the disc of radius $r_{ess} < \sigma$. P_F is said to be *quasicompact* on $\mathcal{B}$. This spectral structure implies that there is some constant $C > 0$ such that for any $f \in \mathcal{B}_0$, $h \in \mathcal{B}_0^*$,

$$\forall n \geqslant 0, \qquad |C_{f,h}(n)| = |\langle h, P_F^n f\rangle_{\mathcal{B}^*,\mathcal{B}}| \leqslant \|h\|_{\mathcal{B}^*}\|P_F^n\|_{\mathcal{B}_0}\|f\|_{\mathcal{B}} \leqslant C\,\|h\|_{\mathcal{B}^*}\|f\|_{\mathcal{B}}\,\sigma^n. \tag{47}$$

For the maps we study, the spectrum of P_F on L_0^2 intersects the unit circle, so $\mathcal{B}$ cannot simply be the Hilbert space L^2. Depending on the properties of the map, $\mathcal{B}$ can be a Fréchet space of analytic functions, a Banach space of bounded variation, Hölder or C^s functions (see section 6.2); it may also be a space of generalized functions lying outside L^2 (see section 6.3). No matter how complicated $\mathcal{B}$ is, in general, there exist Hölder exponents $0 \leqslant s_* \leqslant s$ such that C^s (resp. C^{s_*}) embeds continuously in $\mathcal{B}$ (resp. in its dual $\mathcal{B}^*$). As a result, the upper bound (47) can be specialized to functions $f \in C_0^s$, $h \in C_0^{s_*}$ as follows:

$$\forall n \geqslant 0, \qquad |C_{f,h}(n)| \leqslant C\,\|h\|_{C^{s_*}}\|f\|_{C^s}\,\sigma^n. \tag{48}$$

This is the form of the upper bound we used in theorem 3.

The strategy used for this proof has been applied to several types of maps, including the (non-invertible) expanding maps and the Anosov or Axiom-A diffeomorphisms on a compact manifold. Exponential decay of correlations has also been proven (using various methods) for piecewise expanding maps on the interval, some nonuniformly hyperbolic/expanding maps, and some expanding or hyperbolic maps with singularities.

Other types of decay occur as well: for instance, a polynomial decay of correlations $C_{f,h}(n) \lesssim n^{-\beta}$ was shown to be optimal for some 'intermittent' systems, like a one-dimensional map expanding everywhere except at a fixed 'neutral' point (such maps are sometimes called 'almost expanding' or 'almost hyperbolic').

There exist fewer results on the decay of correlations for stochastic perturbations of deterministic maps, like our noisy evolution T_ϵ. In general, one wants to prove strong stochastic stability—that is, stability of the invariant measure and of the rate of decay of the correlations in the small-noise limit. In our case, only the second point needs to be proved, since the Lebesgue measure remains invariant after switching on the noise.

Stochastic stability has been proved for smooth uniformly expanding maps [5] (see next subsection) and some nonuniformly expanding or piecewise expanding maps. It has been shown also for uniformly hyperbolic (Anosov) maps on the two-dimensional torus [7] (see section 6.3). In all these cases, the mixing is exponential, so the stability of the decay (48) means that for small enough $\epsilon > 0$, there exists a radius $\sigma_\epsilon \overset{\epsilon\to 0}{\to} \sigma$ such that for any $f \in C_0^s$, $h \in C_0^{s_*}$,

$$\forall n > 0, \qquad |C_{f,h}^{\epsilon}(n)| \leqslant C\,\|h\|_{C^{s_*}}\,\|f\|_{C^s}\,\sigma_\epsilon^n. \tag{49}$$

Here, the constant $C > 0$ can be taken independent of ϵ.

In the next two sections, we describe in more detail the cases of smooth uniformly expanding maps and Anosov diffeomorphisms on the torus.

6.2. Smooth uniformly expanding maps

Let F be a C^{s+1} map on $\mathbb{T}^d$ (with $s \geqslant 0$). Assume that there exists $\lambda > 1$ such that for any $\boldsymbol{x} \in \mathbb{T}^d$ and any $\boldsymbol{v}$ in the tangent space $T_{\boldsymbol{x}}\mathbb{T}^d$, $\|DF(\boldsymbol{x})\boldsymbol{v}\| \geqslant \lambda\|\boldsymbol{v}\|$ (we assume that λ is the largest such constant). Such a map is called uniformly expanding. In general, it admits a unique absolutely continuous invariant probability measure; here, we restrict ourselves to maps for which this measure is the Lebesgue measure.

Ruelle [26] proved that the Perron–Frobenius operator P_F of such a map is quasicompact on the space $C^s(\mathbb{T}^d)$, and that its essential spectrum is contained inside the disc of radius $r_1 = \lambda^{-s}$. In general, one has little information on the possible discrete spectrum outside this disc (upper bounds on the decay rate have been obtained in the case of an expanding map of regularity $C^{1+\eta}$ [3]). Strong stochastic stability for such maps was proved in [5], with a more general definition of the noise than the one we gave.

For all these cases, one can take $s_* = 0$, since the continuous functions are continuously embedded in any space $(C^s)^*$.

Case of a linear expanding map. We describe the simplest example possible for such a map, namely the angle-doubling map on $\mathbb{T}^1$ defined as $F(x) = 2x \mod 1$. This map is real analytic, with a uniform expansion rate $\lambda = 2$. Due to its linearity, the dynamics of this map (as well as its noisy version) is simple to express in the basis of Fourier modes $\boldsymbol{e}_k(x) = \mathrm{e}^{2\mathrm{i}\pi kx}$:

$$\begin{aligned}
&\forall k \in \mathbb{Z}, \qquad U_F\boldsymbol{e}_k = \boldsymbol{e}_{2k}\\
&\Longrightarrow T_\epsilon\boldsymbol{e}_k = \hat{g}(\epsilon k)\boldsymbol{e}_{2k}\\
&\Longrightarrow T_\epsilon^n\boldsymbol{e}_k = \left[\prod_{j=1}^{n}\hat{g}(\epsilon\, 2^j k)\right]\boldsymbol{e}_{2^n k}.
\end{aligned}$$

The computation is even simpler for the coarse-grained propagator:

$$\tilde{T}_\epsilon^{(n)}\boldsymbol{e}_k = \hat{g}(k)\hat{g}(2^n k)\boldsymbol{e}_{2^n k}.$$

To fix the ideas, we consider the α-stable noise $\hat{g}(\xi) = \mathrm{e}^{-|\xi|^\alpha}$ for some $0 < \alpha \leqslant 2$. One easily checks that for any $n \geqslant 1$,

$$\begin{aligned}
\|T_\epsilon^n\| &= \|T_\epsilon^n\boldsymbol{e}_1\| = \exp\left\{-\epsilon^\alpha\frac{2^{n\alpha}-1}{1-2^{-\alpha}}\right\},\\
\|\tilde{T}_\epsilon^{(n)}\| &= \exp\{-\epsilon^\alpha(2^{n\alpha}+1)\}.
\end{aligned}$$

For any $\epsilon > 0$, these decays are super-exponential: the spectrum of T_ϵ on L_0^2 is reduced to $\{0\}$ for any $\epsilon > 0$ (the spectrum of U_F is the full unit disc). From this explicit expression, we get both dissipation times:

$$\tau_* = \frac{1}{\ln 2}\ln(\epsilon^{-1}) + \mathcal{O}(1), \qquad \tilde{\tau}_* = \frac{1}{\ln 2}\ln(\epsilon^{-1}) + \mathcal{O}(1). \tag{50}$$

For this linear map, $\|DF\|_\infty = \mu_F = 2$, so this estimate is in agreement with the lower bounds (31), the latter being sharp if $\alpha \in [1, 2]$. On the other hand, $\ln 2$ is also equal to the Kolmogorov–Sinai (K–S) entropy $h(F)$ of F. Therefore, for this linear map the dissipation rate constant exists, and is equal to $1/h(F)$.

To compare these exact asymptotics with the upper bounds of corollary 4, we estimate the correlation functions $C_{f,h}(n)$ on the spaces $C^s(\mathbb{T}^1)$. We give below a short proof in the case $s > \frac{1}{2}$. We will use the following Fourier estimates [33]:

$$\exists C > 0, \quad \forall f \in C_0^s(\mathbb{T}^1), \quad \forall k \neq 0, \qquad |\hat{f}(k)| \leqslant C \frac{\|f\|_{C^s}}{|k|^s}.$$

Therefore, writing the correlation function as a Fourier series, we get:

$$\|P_F^n f\|^2 = \sum_{0 \neq k \in \mathbb{Z}} |\hat{f}(2^n k)|^2 \leqslant \sum_{0 \neq k \in \mathbb{Z}} \left(C \frac{\|f\|_{C^s}}{|2^n k|^s} \right)^2$$

$$\Longrightarrow \|P_F^n f\| \leqslant C' \frac{\|f\|_{C^s}}{(2^s)^n}. \tag{51}$$

This estimate yields a decay of the correlation function as in equation (48), with a rate $\sigma = 2^{-s}$ and $s_* = 0$. One can check that this rate is sharp for functions in C^s: indeed, any $z \in \mathbb{C}$, $|z| < 2^{-s}$ is an eigenvalue of P_F on that space. Applying corollary 4, *I(ii)*, we get that for any $s > \frac{1}{2}$, there exists a constant $\tilde{c}$ such that

$$\tilde{\tau}_* \leqslant \frac{1+s}{s \ln 2} \ln(\epsilon^{-1}) + \tilde{c} \tag{52}$$

for sufficiently small ϵ. The exact dissipation rate constant $1/\ln 2$ is recovered only for large s.

A straightforward computation shows that the estimate (51) also holds if one replaces P_F by $P_F \circ G_\epsilon$; hence, the noisy correlation function dynamics satisfies the same uniform upper bound as the noiseless one, with the decay rate $\sigma_\epsilon = 2^{-s}$. As a result, the upper bound on τ_* given by corollary 4, *II(ii)* is the same as in equation (52).

6.3. Anosov diffeomorphisms on the torus

We recall that a diffeomorphism $F : \mathbb{T}^d \mapsto \mathbb{T}^d$ is called Anosov if it is uniformly hyperbolic: there exist constants $A > 0$ and $0 < \lambda_s < 1 < \lambda_u$ such that at each $\boldsymbol{x} \in \mathbb{T}^d$ the tangent space $T_{\boldsymbol{x}}\mathbb{T}^d$ admits the direct sum decomposition $T_{\boldsymbol{x}}\mathbb{T}^d = E_{\boldsymbol{x}}^s \oplus E_{\boldsymbol{x}}^u$ into stable and unstable subspaces such that for every $n \in \mathbb{N}$,

$$(D_{\boldsymbol{x}}F)(E_{\boldsymbol{x}}^s) = E_{F\boldsymbol{x}}^s, \qquad \|(D_{\boldsymbol{x}}F^n)_{|E_{\boldsymbol{x}}^s}\| \leqslant A\lambda_s^n,$$
$$(D_{\boldsymbol{x}}F)(E_{\boldsymbol{x}}^u) = E_{F\boldsymbol{x}}^u, \qquad \|(D_{\boldsymbol{x}}F^{-n})_{|E_{\boldsymbol{x}}^u}\| \leqslant A\lambda_u^{-n}.$$

These inequalities have obvious consequences for the expansion rates of F and F^{-1}; for instance, they imply $\|DF\|_\infty^n \geqslant \|DF^n\|_\infty \geqslant A^{-1}\lambda_u^n$. As a consequence, the quantities of interest in theorem 2, *(i)* satisfy

$$\|DF\|_\infty \wedge \|DF^{-1}\|_\infty \geqslant \lambda_u \wedge \lambda_s^{-1},$$

$$\mu_F \wedge \mu_{F^{-1}} \geqslant \lambda_u \wedge \lambda_s^{-1}.$$

All these expansion rates are > 1, so both noisy and coarse-grained dissipation times admit logarithmic lower bounds as in equation (31).

Exponential mixing has been proved for Anosov diffeomorphisms of regularity $C^{1+\eta}$ $(0 < \eta < 1)$ by Bowen [8], using symbolic dynamics; the exponential decay is then valid for Hölder observables in $C^{\eta'}$ for some $0 < \eta' < \eta$. Because we are also interested in the noisy dynamics, we will refer to a more recent work [7] concerning C^3 Anosov maps on $\mathbb{T}^d$, which bypasses symbolic dynamics. The authors construct an invariant Banach space $\mathcal{B}$ of generalized functions on the phase space, such that the Perron–Frobenius operator is quasicompact on this space. One subtlety (compared with the case of expanding maps) is that $\mathcal{B}$ explicitly depends on the (un)stable foliations of the map F on $\mathbb{T}^d$. Vaguely speaking, the elements of $\mathcal{B}$ are 'smooth' along this unstable direction E_x^u, but can be singular ('dual of smooth') along the stable foliation E_x^s. The space $\mathcal{B}$ is the completion of $C^1(\mathbb{T}^d)$ with respect to a norm $\|\cdot\|_{\mathcal{B}}$ adapted to these foliations. This norm, and therefore $\mathcal{B}$, are defined in terms of a parameter $0 < \beta < 1$, the choice of which depends on the *regularity* of the unstable foliation. In general, the latter is τ-Hölder continuous, for some exponent $0 < \tau < 2$. Then, the authors prove that if one takes $\beta < \tau \wedge 1$, then the essential spectrum of P_F on $\mathcal{B}$ has a radius smaller than $r_\beta = \max(\lambda_u^{-1}, \lambda_s^\beta)$. This upper bound is sharper if β can be taken close to 1, that is, if the foliation is C^1. This is the case for smooth area-preserving Anosov maps on $\mathbb{T}^2$, for which the foliations have regularity $C^{2-\delta}$ for any $\delta > 0$ [19]. The operator P_F may have isolated eigenvalues (resonances) $1 > |\lambda_i| > r_\beta$, corresponding to eigenstates in $\mathcal{B}_0$ which are genuine distributions $\notin L_0^2$. There is (to our knowledge) no simple general upper bound for the largest resonance $|\lambda_1|$ in terms of the expansion parameters (λ_u, λ_s).

By construction, the space $C^1(\mathbb{T}^d)$ embeds continuously in both $\mathcal{B}$ and its dual $\mathcal{B}^*$, so that one can take $s = s_* = 1$ in equation (48). Therefore, for any $\sigma_\beta > \max(|\lambda_1|, r_\beta)$, there is some constant $C > 0$ such that for any $f, h \in C_0^1(\mathbb{T}^d)$,

$$\forall n > 0, \qquad |C_{f,h}(n)| \leqslant C\, \|h\|_{C^1}\, \|f\|_{C^1}\, \sigma_\beta^n. \tag{53}$$

In the proof of theorem 3 (Step 3), for the case $s = s_* = 1$ we only need to assume that the noise generating function g is C^1 with fast-decaying first derivatives. The fast decay implies that the first moment of g is finite (i.e. one can take $\alpha \geqslant 1$).

The noisy propagator $G_\epsilon P_F$ is also analysed in [7]. If the unstable foliation has regularity $C^{1+\eta}$ with $\eta > 0$ (for instance, for any C^3 Anosov diffeomorphism on $\mathbb{T}^2$), and assuming that the noise generating function $g \in C^2(\mathbb{R}^d)$ has *compact support*[3], the authors prove the strong spectral stability of the Perron–Frobenius operator P_F on any space $\mathcal{B}$ defined with a parameter $\beta' < \eta$. Note that this constraint on β' is stronger than in the noiseless case, where one could take any $\beta < 1$; the spectral radius $\sigma_{\beta'}$ may accordingly be larger than σ_β. Modulo the replacement of σ_β by $\sigma_{\beta'}$, the estimate (53), therefore, applies to the noisy correlation function $C_{f,h}^\epsilon(n)$ as long as ϵ is small enough.

Below, we collect the results regarding the dissipation time of C^3 Anosov maps on the torus.

Theorem 4. *Let F be a volume-preserving C^3 Anosov diffeomorphism on $\mathbb{T}^d$, and let the noise generating function be C^1 with fast decay at infinity.*

(I) Then, there exist $\mu \geqslant \lambda_u \wedge \lambda_s^{-1}$, $0 < \tilde{\sigma} < 1$ and $\tilde{C} > 0$ such that the dissipation time of the coarse-grained dynamics satisfies

$$\frac{1}{\ln\mu}\ln(\epsilon^{-1}) - \tilde{C} \leqslant \tilde{\tau}_* \leqslant \frac{d+2}{|\ln\tilde{\sigma}|}\ln(\epsilon^{-1}) + \tilde{C}.$$

(II) If in addition F has $C^{1+\eta}$-regular foliations, and $g \in C^2(\mathbb{R}^d)$ is compactly supported, then there exist $\tilde{\sigma} \leqslant \sigma < 1$ and C such that the dissipation time of the noisy dynamics satisfies

$$\frac{1}{\ln\|DF\|_\infty}\ln(\epsilon^{-1}) - C \leqslant \tau_* \leqslant \frac{d+2}{|\ln\sigma|}\ln(\epsilon^{-1}) + C.$$

[3] The condition of compact support could probably be relaxed to one of fast decrease at infinity (C Liverani, private communication).

6.4. Ergodic linear automorphisms of the torus

In this section, we describe examples of Anosov maps for which the dissipation time can be precisely determined. One can even compute the dissipation rate constant; we will mention the connection of the latter with the Kolmogorov–Sinai entropy. After the simple example of a (generalized) cat map on the two-dimensional torus [1], we recall the results obtained in [14] for d-dimensional ergodic automorphisms.

Throughout this section, we take a noise generating function of the type (14) for a certain $\alpha \in (0, 2]$, with $\boldsymbol{Q} = I$:

$$g_{\epsilon,\alpha}(\boldsymbol{x}) := \sum_{\boldsymbol{k}\in\mathbb{Z}^d} \mathrm{e}^{-|\epsilon\boldsymbol{k}|^\alpha} \boldsymbol{e}_{\boldsymbol{k}}(\boldsymbol{x}). \tag{54}$$

Note that this function is not compactly supported, therefore, it does not satisfy *stricto sensu* the assumptions required in [7] to prove the strong spectral stability of P_F (see the footnote after theorem 4).

Two-dimensional cat map. We take $A \in SL(2, \mathbb{Z})$ with $|\mathrm{Tr}\, A| > 2$ and define the dynamics as $F(\boldsymbol{x}) = F_A(\boldsymbol{x}) = A^t\boldsymbol{x} \bmod 1$ (A^t is the transposed matrix of A). This map is of Anosov type, and preserves the Lebesgue measure. The dynamics is easy to express on the Fourier modes:

$$U_F\, \boldsymbol{e}_{\boldsymbol{k}} = \boldsymbol{e}_{A\boldsymbol{k}},$$
$$T_\epsilon^n\, \boldsymbol{e}_{\boldsymbol{k}} = e^{-\sum_{l=1}^n |\epsilon A^l\boldsymbol{k}|^\alpha} \boldsymbol{e}_{A^n\boldsymbol{k}},$$
$$\Longrightarrow \|T_\epsilon^n\| = \exp\left(-\epsilon^\alpha \min_{0\neq\boldsymbol{k}\in\mathbb{Z}^2} \sum_{l=1}^n |A^l\boldsymbol{k}|^\alpha\right).$$

Similarly,

$$\|\tilde{T}_\epsilon^{(n)}\| = \exp\left(-\epsilon^\alpha \min_{0\neq\boldsymbol{k}\in\mathbb{Z}^2} (|\boldsymbol{k}|^\alpha + |A^n\boldsymbol{k}|^\alpha)\right).$$

Let us call λ and λ^{-1} the eigenvalues of A, with the convention $|\lambda| > 1$. One can easily show that there are two constants $0 < C_1 < C_2$ such that for any $n > 0$,

$$C_1\, |\lambda|^{n\alpha/2} \leqslant \min_{0\neq\boldsymbol{k}\in\mathbb{Z}^2} \sum_{l=1}^n |A^l\boldsymbol{k}|^\alpha \leqslant C_2\, |\lambda|^{n\alpha/2},$$
$$C_1\, |\lambda|^{n\alpha/2} \leqslant \min_{0\neq\boldsymbol{k}\in\mathbb{Z}^2} (|\boldsymbol{k}|^\alpha + |A^n\boldsymbol{k}|^\alpha) \leqslant C_2\, |\lambda|^{n\alpha/2}.$$

The above estimate yields the following asymptotics in the small-ϵ limit:

$$\tau_* = \frac{2}{\ln|\lambda|} \ln(\epsilon^{-1}) + \mathcal{O}(1), \qquad \tilde{\tau}_* = \frac{2}{\ln|\lambda|} \ln(\epsilon^{-1}) + \mathcal{O}(1).$$

As in the case of linear expanding maps, the dissipation rate constant exists, and seems related to the K–S entropy of the linear map $h(F) = \ln|\lambda|$.

Let us compare these exact asymptotics with the bounds obtained in the previous sections. The cat map being linear, one has $\|DF\|_\infty = \|A^t\| \geqslant |\lambda|$, $\|DF^{-1}\|_\infty = \|(A^t)^{-1}\| \geqslant |\lambda|$ (with equality iff A is symmetric). Since A is diagonalizable, we have $\|DF^n\|_\infty = \|(A^t)^n\| \sim |\lambda|^n$, so that $\mu_F = \mu_{F^{-1}} = |\lambda|$. Therefore, the lower bounds for the dissipation times given in theorem 2*(i)* are strictly smaller than the exact rates derived above; they differ from the latter by a factor $\leqslant \frac{1}{2}$.

To estimate the dynamical correlations, we use Fourier decomposition and proceed as in [11]: we construct a 'primitive subset' $\mathcal{S}$ of the Fourier plane $\mathbb{Z}^2 \setminus \{0\}$, such that each Fourier orbit intersects $\mathcal{S}$ once and only once. This subset looks like the union of four angular sectors of the plane, and has the following crucial property:

$$\exists c > 0, \quad \forall \boldsymbol{k} \in \mathcal{S}, \quad \forall n \in \mathbb{Z}, \qquad |A^n \boldsymbol{k}| \geqslant c|\lambda|^{|n|}|\boldsymbol{k}|. \tag{55}$$

The correlation function between two observables $f,\ h \in L_0^2(\mathbb{T}^d)$ may be written as:

$$C_{f,h}(n) = \sum_{0\neq \boldsymbol{k}\in\mathbb{Z}^2} \hat{f}(-\boldsymbol{k})\hat{h}(A^n\boldsymbol{k}) = \sum_{\boldsymbol{k}\in\mathcal{S}}\sum_{l\in\mathbb{Z}} \hat{f}(-A^l\boldsymbol{k})\hat{h}(A^{l+n}\boldsymbol{k}).$$

Let us assume $s > 1$, and take $f \in C_0^s(\mathbb{T}^2)$: its Fourier coefficients decrease for all $0 \neq \boldsymbol{k} \in \mathbb{Z}^2$ as

$$|\hat{f}(\boldsymbol{k})| \leqslant C\frac{\|f\|_{C^s}}{|\boldsymbol{k}|^s}.$$

Using this decrease as well as the estimate (55), we find that

$$|C_{f,h}(n)| \leqslant C^2\, \|f\|_{C^s}\, \|h\|_{C^s} \sum_{\boldsymbol{k}\in\mathcal{S}}\sum_{l\in\mathbb{Z}} \frac{1}{(c^2|\lambda|^{|l+n|+|l|}|k|^2)^s}.$$

The sum over $\boldsymbol{k}$ converges, and the sum over l is bounded from above by $C/|\lambda|^{ns}$. This implies that for any $s > 1$, the correlation function decreases as in equation (48), with the rate $\sigma = |\lambda|^{-s}$. Actually, this decrease holds in the case $0 < s \leqslant 1$ as well, but the proof is different. Then, corollary 4, *I(ii)*, yields the upper bound

$$\tilde{\tau}_* \leqslant \frac{2+2s}{s\ln|\lambda|}\ln(\epsilon^{-1}) + c. \tag{56}$$

This method can be straightforwardly adapted to prove that the noisy correlation function $C^{\epsilon}_{f,h}(n)$ decreases as fast as the noiseless one (in case $s > 1$). As a result, we obtain the same upper bound for τ_* as for $\tilde{\tau}_*$. We note that, as in the case of the angle-doubling map, the constant in the upper bound converges to the exact rate only in the limit $s \gg 1$, while for $s = 1$ (cf the discussion preceding theorem 4), the constant is twice as large.

Toral automorphisms in higher dimensions. We consider toral automorphisms $F = F_A$ given by matrices $A \in SL(d,\mathbb{Z})$; such an automorphism is ergodic iff none of the eigenvalues of A are roots of unity. The K–S entropy $h(F)$ of F is given by the formula [31]

$$h(F) = \sum_{|\lambda_j|>1} \ln|\lambda_j|, \tag{57}$$

where λ_j are the eigenvalues of F. All ergodic toral automorphisms have positive entropy. We denote by P the characteristic polynomial of the matrix A and by $\{P_1,\ldots,P_r\}$ the complete set of its distinct irreducible factors (over $\mathbb{Q}$). Let d_j denote the degree of the polynomial P_j and h_j the K–S entropy of a toral automorphism with the characteristic polynomial P_j. For each P_j we define its dimensionally averaged K–S entropy as

$$\hat{h}_j = \frac{h_j}{d_j}. \tag{58}$$

For the whole matrix F we define its minimal dimensionally averaged entropy (denoted by $\hat{h}(F)$) as

$$\hat{h}(F) = \min_{j=1,\ldots,r} \hat{h}_j.$$

Note that for an *irreducible* matrix A, that is a matrix admitting no proper invariant rational subspace, this quantity reduces to $h(F)/d$.

Our notation regarding the noise parameter slightly differs from the one used in [14] (one has to replace $\varepsilon^{2\alpha}$ by ϵ); the results obtained there read as follows.

Proposition 4 ([14]). *Let $F = F_A$ be a toral automorphism on $\mathbb{T}^d$, and assume the noise kernel to be as in equation (54). Then, in the limit of small noise,*

(i) Both dissipation times have a power-law behaviour iff F is not ergodic.
(ii) Both dissipation times have logarithmic behaviour iff F is ergodic.
(iii) If F is ergodic and A is diagonalizable then

$$\tau_* \approx \tilde{\tau}_* \approx \frac{1}{\hat{h}(F)} \ln(\epsilon^{-1}).$$

We end this section with a remark that the small-noise asymptotics of the dissipation time is insensitive to a super-exponential decay of the correlation functions. We illustrate this fact by the following result on the decay of correlations for d-dimensional toral automorphisms, which can be proved along the same lines as the above proposition.

Proposition 5. *Let F be a diagonalizable ergodic toral automorphism and λ any constant such that $0 < \lambda < \hat{h}(F)$. Then, for any $f, h \in L_0^2(\mathbb{T}^{2d})$*

$$C^{\epsilon}_{f,h}(n) \leqslant \|f\| \, \|h\| \mathrm{e}^{-\epsilon^2 \lambda^n}.$$

Let $f,\ h \in G_\epsilon(L_0^2(\mathbb{T}^{2d}))$ be smooth observables. Then,

$$C_{f,h}(n) \leqslant \|G_\epsilon^{-1} f\| \, \|G_\epsilon^{-1} h\| \mathrm{e}^{-\epsilon^2 \lambda^n}.$$

7. Conclusion

We have investigated the effect of noise or coarse-graining on the dynamics generated by a conservative map, in particular the connection between the speed of dissipation of the noisy dynamics and the spectral and dynamical properties of the underlying map.

We restricted ourselves to the case of volume-preserving maps on the d-dimensional torus. The choice of the torus allowed us to use the Fourier transformation in our proofs, that is, harmonic analysis on that manifold. Most of our results can certainly be generalized to volume-preserving maps on other compact Riemannian manifolds. The choice of the torus was also guided by the existence of simple volume-preserving Anosov maps on it, most notably the linear examples presented in sections 6.2 and 6.4. As explained at the beginning of section 6, one may also want to extend the results to maps which do not leave invariant the Lebesgue measure, but still admit a 'physical measure', as is the case for uniformly expanding or Anosov maps (the physical measure is of SRB type, that is, its projection along the unstable manifold is absolutely continuous [3]). The noisy perturbations of these maps have been analysed as well [5, 7], in particular the strong spectral stability of the Perron–Frobenius operator holds under the same conditions as for the volume-preserving maps. Although the equilibrium measure is more complicated, it might be possible to define and study a dissipation time in this more general framework.

As we explained in corollary 3, the dissipation of a non-weakly-mixing map is governed only by the nontrivial eigenstates of the Koopman operator, the speed of dissipation depending on the smoothness of these eigenstates. The asymptotics of the noisy dissipation time is generally a power law in ϵ^{-1}, while there remains a possibility of faster dissipation if all eigenstates are singular enough (see remarks after corollary 3).

The more interesting results concern the mixing dynamics, in particular when the mixing occurs exponentially fast. We then proved that both noisy and coarse-graining dissipation times behaved as the logarithm of ϵ^{-1} in the small noise limit. This dependence can be understood through the time evolution in Fourier space: a mixing map transforms long wavelength fluctuations into short wavelength ones, the latter being damped fast by the noise operator. This evolution in Fourier space is typical of uniformly expanding/hyperbolic maps; it is already responsible for the fast decay of dynamical correlations. The link between this decay and the dissipation is made explicit in theorem 3, and its application to Anosov maps is given in theorem 4.

In this context, we were unable to solve the problem of the existence and value of the dissipation rate constant (i.e. the prefactor in front of $\ln(\epsilon^{-1})$). We obtained lower, resp. upper, bounds for this constant, in terms of the local expanding rate, resp. the rate of decay of correlations. It is not clear whether this constant can, in general, be related with the measure-theoretic (KS) entropy of the map, as is the case for linear automorphisms (modulo some algebraic subtleties [14]).

Although we used the spectral estimates of theorem 1 mostly for the case of non-weakly-mixing maps, it also makes sense to use that theorem in the reverse direction in the case of mixing maps, that is, deduce (pseudo)spectral properties of the noisy propagators, starting from the dissipation time estimates obtained in corollary 4. The analysis of the pseudospectrum of T_ϵ for mixing maps would complement the spectral one [7, 25]. We did not enter into this aspect in the main text, because our attention was devoted to obtaining information on the dissipation time.

The results of this paper concern classical (i.e. non-quantum) dynamical systems. The quantization of both linear and nonlinear maps on a symplectic (even-dimensional) torus as a phase space has been studied in a number of works [18, 6, 28, 20, 17, 32]. Several recent studies deal with some form of noise, or decoherence, in discrete-time quantum dynamics [9, 15, 24, 16, 25]. While most of these works concentrate on spectral or entropic properties of noiseless/noisy dynamics, the long time behaviour of the quantum system can also be studied from the dissipation time point of view. The quantum setting provides a natural framework for extension of the present work, which we will address in a separate paper [13].

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Appendix A. Proofs of some elementary facts

Appendix A.1. Proof of lemma 1

We use the following upper bound: for any $\alpha \in (0, 2]$, there is a constant C_α such that

$$\forall x \in \mathbb{R}, \qquad 0 \leqslant 1 - \cos(2\pi x) \leqslant C_\alpha |x|^\alpha. \tag{59}$$

Besides, one has the asymptotics $1 - \cos(x) \approx x^2/2$ for small x. We simply apply these estimates to the following integral:

$$\begin{aligned} 1 - \hat{g}(\boldsymbol{\xi}) &= \int_{\mathbb{R}^d} (1 - \cos(2\pi \boldsymbol{x} \cdot \boldsymbol{\xi})) g(\boldsymbol{x}) d\boldsymbol{x} \\ &\leqslant \int_{\mathbb{R}^d} C_\alpha |\boldsymbol{x} \cdot \boldsymbol{\xi}|^\alpha g(\boldsymbol{x}) d\boldsymbol{x} \\ &\leqslant C_\alpha |\boldsymbol{\xi}|^\alpha \int_{\mathbb{R}^d} |\boldsymbol{x}|^\alpha g(\boldsymbol{x}) d\boldsymbol{x} = C_\alpha M_\alpha |\boldsymbol{\xi}|^\alpha. \end{aligned}$$

In the case when g admits a second moment, we have in the limit $\boldsymbol{\xi} \to 0$:

$$\begin{aligned} \int_{\mathbb{R}^d} (1 - \cos(2\pi \boldsymbol{x} \cdot \boldsymbol{\xi})) g(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} &\approx \int_{\mathbb{R}^d} 2\pi^2 (\boldsymbol{x} \cdot \boldsymbol{\xi})^2 g(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \\ &\approx 2\pi^2 |\boldsymbol{\xi}|^2 \int_{\mathbb{R}^d} (\boldsymbol{x} \cdot \hat{\boldsymbol{\xi}})^2 g(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}, \end{aligned}$$

where we have used the notation $\hat{\boldsymbol{\xi}} = \boldsymbol{\xi}/|\boldsymbol{\xi}|$ for any $\boldsymbol{\xi} \neq 0$. ■

Appendix A.2. Proof of proposition 1

The statement (i) is standard in the context of distributions [30, p 157]. In our case, assume that $f \in L^2$ is normalized to unity and consider an arbitrary small $\delta > 0$. Since $f \in L^2(\mathbb{T}^d)$, there exists $K > 0$, s.t. $\sum_{|\boldsymbol{k}| \geqslant K} |\hat{f}(\boldsymbol{k})|^2 < \delta$. Since $\hat{g}$ is continuous and $\hat{g}(0) = 1$, there exists η such that $(1 - \hat{g}(\boldsymbol{\xi}))^2 < \delta$ if $|\boldsymbol{\xi}| < \eta$. Thus, using the spectral decomposition (5) of G_ϵ, we obtain for all $\epsilon < \eta/K$

$$\|G_\epsilon f - f\|^2 = \sum_{\boldsymbol{k} \in \mathbb{Z}^d} (1 - \hat{g}(\epsilon \boldsymbol{k}))^2 |\hat{f}(\boldsymbol{k})|^2 \leqslant \delta \sum_{|\boldsymbol{k}| < K} |\hat{f}(\boldsymbol{k})|^2 + \sum_{|\boldsymbol{k}| > K} |\hat{f}(\boldsymbol{k})|^2 \leqslant 2\delta. \tag{60}$$

To prove the next statement, first note that if g satisfies the estimate (12) for the exponent α, it also satisfies it for the exponent $\gamma \wedge \alpha$. Once again using the spectral decomposition of G_ϵ, and applying the estimate (12) with the latter exponent we get

$$\begin{aligned} \|G_\epsilon f - f\|^2 &\leqslant \sum_{\boldsymbol{k} \in \mathbb{Z}^d} (C_{\gamma\wedge\alpha} M_{\gamma\wedge\alpha} |\epsilon \boldsymbol{k}|^{\gamma\wedge\alpha})^2 |\hat{f}(\boldsymbol{k})|^2 \\ &\leqslant (C_{\gamma\wedge\alpha} M_{\gamma\wedge\alpha})^2 \epsilon^{2(\gamma\wedge\alpha)} \sum_{\boldsymbol{k} \in \mathbb{Z}^d} |\boldsymbol{k}|^{2(\gamma\wedge\alpha)} |\hat{f}(\boldsymbol{k})|^2 \\ &\leqslant (C_{\gamma\wedge\alpha} M_{\gamma\wedge\alpha})^2 \epsilon^{2(\gamma\wedge\alpha)} \|f\|^2_{H^{\gamma\wedge\alpha}}. \end{aligned} \tag{61}$$

To obtain the last statement, we note that any $f \in C^1(\mathbb{T}^d)$ is automatically in $H^1(\mathbb{T}^d)$, and that its gradient satisfies

$$\|\nabla f\|^2_\infty \geqslant \|\nabla f\|^2 = 4\pi^2 \sum_{\boldsymbol{k} \in \mathbb{Z}^d} |\boldsymbol{k}|^2 |\hat{f}(\boldsymbol{k})|^2 \geqslant 4\pi^2 \sum_{\boldsymbol{k} \in \mathbb{Z}^d} |\boldsymbol{k}|^{2(1\wedge\alpha)} |\hat{f}(\boldsymbol{k})|^2.$$

The inequality (61) with $\gamma = 1$ then yields the desired result. ■

Appendix A.3. Proof of corollary 2

We prove the limit $d_\epsilon(1) \overset{\epsilon\to 0}{\to} 0$ by contradiction. Assume that there is some constant $a \in (0, 1)$ such that for all $\epsilon > 0$, the distance $d_\epsilon(1) > a$. We will show that the following triangle inequality holds:

$$\forall \epsilon > 0, \quad d_\epsilon(1 - a/2) > a/2. \tag{62}$$

First of all, note that the assumption $d_\epsilon(1) > a$ means that for any $\lambda \in S^1$, $\|R_\epsilon(\lambda)\| < a^{-1}$. We apply the identity [30]

$$R_\epsilon(\lambda') = R_\epsilon(\lambda)\left\{1 + \sum_{n\geqslant 1}(\lambda - \lambda')^n R_\epsilon(\lambda)^n\right\},$$

with $\lambda' = r\lambda$, for $1 - a < r < 1$. Taking the norm of both sides yields the bound $\|R_\epsilon(\lambda')\| \leqslant 1/(r - (1-a))$, uniformly w.r.t. ϵ. Since this upper bound holds for any $|\lambda'| = r$, it shows that the spectral radius $r_{\rm sp}(T_\epsilon) \leqslant 1 - a$, and proves (62) by taking $r = 1 - a/2$. We can now use (62) in the upper bound (25) of theorem 1: this ϵ-independent upper bound shows that τ_* remains finite in the limit $\epsilon \to 0$, which contradicts proposition 2. ■

Appendix A.4. Proof of lemma 4

Considering its decay at infinity, the function f is automatically in $L^2(\mathbb{R}^d)$. The function $\hat{f}^2$ is the Fourier transform of the self-convolution $f * f$. Therefore, using the parity of f and applying the Poisson summation formula to the LHS of (46) yields

$$\epsilon^d \sum_{\boldsymbol{k}\in\mathbb{Z}^d} \hat{f}(\epsilon\boldsymbol{k})^2 = \int \hat{f}^2(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi} + \sum_{0\neq\boldsymbol{n}\in\mathbb{Z}^d} (f * f)\left(\frac{\boldsymbol{n}}{\epsilon}\right). \qquad (63)$$

A simple computation shows that $(f * f)(\boldsymbol{x})$ also decays as fast as $|\boldsymbol{x}|^{-M}$. This piece of information is now sufficient to control the RHS of (63), yielding the result—namely equation (46). ■

Appendix A.5. Proof of corollary 5

It was shown in [14] that for any $\delta > 0$, in the case of Gaussian noise

$$\|T_\epsilon^n\| \leqslant \mathrm{e}^{-\epsilon^2 \mathrm{e}^{2(1-\delta)\hat{h}(F)n}}. \qquad (64)$$

Using this estimate one immediately gets

$$C_{f,g}^\epsilon(n) = \langle \bar{f}, T_\epsilon^n g\rangle \leqslant \|f\|\|g\|\|T_\epsilon^n\| \leqslant \|f\|\|g\|\mathrm{e}^{-\epsilon^2\lambda^n}.$$

Now, let $f = G_\epsilon f_0$ and $g = G_\epsilon g_0$. Since the estimate (64) holds also in the coarse-grained version, we have

$$\begin{aligned} C_{f,g}(n) &= \langle \bar{f}, U_F^n g\rangle = \langle G_\epsilon \bar{f}_0, U_F^n G_\epsilon g_0\rangle = \langle \bar{f}_0, \tilde{T}_\epsilon^{(n)} g_0\rangle \\ &\leqslant \|f_0\|\|g_0\|\|\tilde{T}_\epsilon^{(n)}\| \leqslant \|f_0\|\|g_0\|\mathrm{e}^{-\epsilon^2\lambda^n}. \end{aligned}$$

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