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DIFFUSIVE AND NONDIFFUSIVE LIMITS OF TRANSPORT IN NONMIXING FLOWS*

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Abstract. We study the passive scalar transport in a class of nonmixing Markovian flows with power-law spectra and correlation times. We establish a new diffusion regime under an optimal condition (convergent Kubo formula) on the spatial/temporal structure of this family of flows. Under such a condition, the Péclet number of the problem may be infinite.

We propose a general criterion for the diffusion regime that takes into account of both the effect of molecular diffusion and the spatial/temporal structure of the velocity field. We conjecture the criterion to be applicable in general for temporally ergodic, reversible Markovian flows. We show heuristically that the violation of this criterion may lead to a nondiffusive scaling limit.

Key words. passive scalar, homogenization

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1. Introduction.

Mass and heat transport in moving fluids is ubiquitous. The study of turbulent transport is fundamental to understanding temperature fields as well as pollutant or tracer particles' movements in the atmosphere and oceans and solute transport in groundwater flows. The simplest model of turbulent transport is the motion of a passive scalar described by Itô's stochastic differential equation

$$d\boldsymbol{x}(t) = \mathbf{V}(t,\boldsymbol{x})dt + \sqrt{2D_0}d\mathbf{w}(t), \quad \mathbf{x}(0) = \mathbf{0}, \tag{1.1}$$

where $\mathbf{V}(t,\mathbf{x}) = (V_1(t,\mathbf{x}),\dots,V_d(t,\mathbf{x}))$ is a random incompressible ($\nabla\cdot\mathbf{V}=0$) velocity field whose statistical properties are given (see below) and $\mathbf{w}(t)$ is the standard Brownian motion. The molecular diffusivity, $D_0 \geq 0$, accounts for the effect of random collisions with ambient fluid moleculars. Our objective is to describe the limit of the rescaled trajectories

$$\mathbf{x}^\varepsilon(t) := \varepsilon\mathbf{x}(t/\varepsilon^{2\delta}), \quad t \geq 0,$$

as $\varepsilon \downarrow 0$, with an appropriate $\delta > 0$. Of particular interest is when the scaling limit is diffusive (i.e., $\delta = 1$) and when it is not.

One criterion (see [2] and [12]) is in terms of the generalized Péclet number defined as follows. Let $\mathbf{V}(t,\boldsymbol{x})$ be a temporally stationary, spatially homogeneous velocity field whose correlation matrix

$$\mathbf{R}(t,\mathbf{x}) = [R_{ij}(t,\mathbf{x})] = [\mathbf{E}\{V_i(s+t,\mathbf{y}+\mathbf{x})V_j(s,\mathbf{y})\}]$$

is given by

$$\mathbf{R}(t,\mathbf{x}) = \int_{R^d} \cos(\mathbf{k}\cdot\mathbf{x})\hat{\mathbf{R}}(t,\mathbf{k})\,d\mathbf{k}, \tag{1.2}$$

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with

$$\hat{\mathbf{R}}(t,\mathbf{k}) := \rho(t,\mathbf{k})\Gamma(\mathbf{k})\mathcal{E}(\mathbf{k})|\mathbf{k}|^{1-d}\, d\mathbf{k}, \quad \Gamma(\mathbf{k}) := \mathbf{I} - \mathbf{k}\otimes\mathbf{k}|\mathbf{k}|^{-2},$$

where $\rho(t,\mathbf{k})$ is the temporal correlation function such that $\rho(0,\cdot) \equiv 1$ and $\mathcal{E}(\cdot)$ is the energy spectrum. Both here and in what follows $\mathbf{E}$ denotes the average with respect to the ensemble of the velocity fields. The Péclet number Pe is defined as

$$\text{(1.3)} \qquad \mathrm{Pe} := D_0^{-1}\sqrt{\int_{R^d} \frac{\mathrm{tr}\ \hat{\mathbf{R}}(0,\mathbf{k})}{|\mathbf{k}|^2}\, d\mathbf{k}} = D_0^{-1}\sqrt{(d-1)\int_{R^d} \frac{\mathcal{E}(\mathbf{k})}{|\mathbf{k}|^{d+1}} d\mathbf{k}}.$$

It has been proved (see [2],[12],[7]) for general random velocity fields that the finiteness of the Péclet number, $\mathrm{Pe} < \infty$, implies the diffusive scaling limit. For time *independent* flows, the condition is in some sense optimal. A finite Péclet number means that the energy of low wavenumbers is sufficiently small and, therefore, that the velocity field has a finite spatial correlation length. Indeed, an associated effective correlation length can be defined as

$$\ell^* := D_0\mathrm{Pe}/\sqrt{\mathbf{E}|\mathbf{V}|^2},$$

which is finite whenever Pe is finite. However, the condition of the finite Péclet number is far from being optimal for time *dependent* flows since the definition of the Péclet number does not take into account the temporal structure of the velocity. For example, we proved in [6] that the scaling limit is diffusive if the velocity is a strongly mixing Markov process; namely, spatial decorrelation is not needed. Such a flow has a finite correlation *time*, and the Péclet number is irrelevant for the diffusive scaling. Moreover, in this case the molecular diffusion was shown to be a regular perturbation in the sense that as $D_0 \to 0$ the effective diffusivity tends to the turbulent diffusivity for $D_0 = 0$. On the other hand, temporal mixing is not always needed for the diffusive limit even when the Péclet number is infinite.

In this paper, we seek to establish a general criterion that takes into account the spatial temporal structure of the flow as well as the effect of molecular diffusion. We shall focus on Markovian velocity fields $\mathbf{V}$ with the power-law energy spectrum

$$\text{(1.4)} \qquad \mathcal{E}(\mathbf{k}) = \mathcal{E}(|\mathbf{k}|), \quad \text{with } \mathcal{E}(k) := a(k)k^{1-2\alpha},$$

and the power-law correlation time

$$\text{(1.5)} \qquad \rho(t,\mathbf{k}) = \rho(t,|\mathbf{k}|), \quad \text{with } \rho(t,k) := \exp\{-k^{2\beta}t\}.$$

The nonnegative function $a(\cdot)$ in (1.4) is the cutoff (infrared or ultraviolet) necessary for the velocity to have a finite averaged energy. In particular, an ultraviolet cutoff is necessary for $\alpha < 1$. We assume that $\alpha < 1$ and that $a(\cdot)$ has a compact support, say in $[0, K_0], K_0 < \infty$.

Under additional assumptions (see section 3), the flow then is strongly mixing in time if and only if $\beta = 0$ (therefore uniformly finite correlation times). If $\beta > 0$, the flow is not strongly mixing and has arbitrarily long correlation times $|\mathbf{k}|^{-2\beta}$ as $|\mathbf{k}| \to 0$. Also, it is easy to check that the flow has a finite Péclet number if and only if $\alpha < 0$. Hence we know from the previous results that the scaling is diffusive for $\alpha < 0$ [2], [12], [7] or $\beta = 0$ [6].

A main result of this paper is the establishment of a new diffusion regime in which the flow neither is mixing nor has finite Péclet number but meets other more suitable (and possibly sharp) conditions. Namely, if $\mathbf{V}(t,\mathbf{x})$, $(t,\mathbf{x}) \in R \times R^d$ is the temporally stationary, spatially homogeneous Markov velocity field with the properties (3.4) and (3.5) of section 3.1 and with the covariance matrix given by (1.2), (1.4), and (1.5), then, for $\alpha+\beta<1, \beta \geq 0$, the finite dimensional distributions of the rescaled trajectories $\mathbf{x}_\varepsilon(t)$ with $\delta=1$ converge weakly, as $\varepsilon \downarrow 0$, to a Wiener measure with the nontrivial covariance matrix $2\mathbf{D} \geq 2D_0\mathbf{I}$. Moreover, the effective diffusivity $\mathbf{D}$ as a function of D_0 satisfies

$$0 < \liminf_{D_0 \to 0} \mathbf{D}(D_0) \leq \limsup_{D_0 \to 0} \mathbf{D}(D_0) < +\infty. \tag{1.6}$$

It is not clear if the limit $\lim_{D_0\to 0}\mathbf{D}(D_0)$ exists; neither is it clear if the diffusive limit holds when D_0 is zero from the outset, except in the mixing case ($\beta=0$) for which we also know that $\lim_{D_0\to 0}\mathbf{D}(D_0)=\mathbf{D}(0)$ [6]. It is worthwhile to point out the example that displays a nonperturbative effect of the molecular diffusion, i.e., $\lim_{D_0\to 0}\mathbf{D}(D_0) > \mathbf{D}(0)$ [10]. We also give a heuristic scaling argument in section 8 to show that, when $\alpha+\beta>1$ but $\alpha+2\beta<2$, the scaling is not diffusive and the limit should be a fractional Brownian motion (FBM). A partial result in this direction is given in [8] (see also [10]). These results suggest the criterion

$$\int_0^\infty \operatorname{Tr}\mathbf{R}(t,\mathbf{0})\,dt < \infty \tag{1.7}$$

as a sufficient condition for the diffusive limit in time dependent flows and that in such a case the molecular diffusion is not needed for homogenization (perhaps in the sense of (1.6)). In contrast to the spatial integration in the definition (1.3), the integration in (1.7) is temporal. We have proved the validity and the sharpness of the criterion (1.7) for a different class of Markovian flows [9].

Finally, we observe that the condition $\mathrm{Pe}<\infty$ and (1.7) can be combined into a single condition

$$\int_0^\infty \mathbf{ME}\left[V_i(t,\sqrt{2D_0}\mathbf{w}(t))\,V_j(0,\mathbf{0})\right]dt < \infty, \tag{1.8}$$

where $\mathbf{M}$ and $\mathbf{E}$ are expectations with respect to the molecular Brownian motion and the ensemble of the velocity, respectively. For $D_0 \to 0$, (1.8) becomes (1.7); for time independent flows except $D_0>0$, (1.8) becomes $\mathrm{Pe}<\infty$. Thus the formulation (1.8) takes into account the effect of molecular diffusion as well as the spatial/temporal structures of the velocity field. In section 7 we explain why (1.8) is a plausible criterion for a diffusive limit in temporary ergodic, reversible Markovian flows.

Although our approach relies on the Markov property of the velocity, it seems likely that (1.8) is also applicable to general (Markovian or non-Markovian) stationary flows whose time correlation functions $\rho(t,\mathbf{k})$ decay *exponentially* in time for each $\mathbf{k}$. For nondecaying $\rho(t,\mathbf{k})$, the condition may be far from optimal (cf. [10]).

2. Outline of the approach. The main object to analyze is the *Lagrangian* velocity process

$$\eta(t,\cdot) \equiv \mathbf{V}(t,\boldsymbol{x}(t)+\cdot)$$

(see section 4.1) in a state space $\mathbb{X}$ defined in section 3. Roughly speaking, η describes the velocity field from the vintage point of an observer sitting at the moving particle.

The particle path can be recovered readily from $\eta(\cdot)$:

$$\boldsymbol{x}(t) = \boldsymbol{x}(0) + \int_0^t \eta(u, \mathbf{0})du.$$

It can be shown that the Lagrangian velocity field is also a Markov process. The complication, however, is that its generator $\mathcal{L}$ is not self-adjoint even when the (Eulerian) velocity process has a self-adjoint generator $\mathcal{A}$. The Lagrangian generator takes the form

$$\mathcal{L} = D_0\Delta + \mathcal{A} + \widetilde{\mathbf{V}} \cdot \nabla,$$

where $\widetilde{\mathbf{V}}$, defined in (3.3), can be thought of as the random value of the sample velocity field at a fixed (space-time) point.

One can decompose the diffusively scaled displacement as follows:

$$\varepsilon x_i\left(t/\varepsilon^2\right) = \sqrt{2D_0}\varepsilon w_i\left(t/\varepsilon^2\right) + \varepsilon \int_0^{t/\varepsilon^2} V_i(\eta(s))ds = R_i^\varepsilon(t) + \varepsilon N_i^\varepsilon\left(t/\varepsilon^2\right) \quad \forall i = 1, \ldots, d,$$

where $N_i^\varepsilon(\cdot)$ is a martingale with respect to the natural filtration corresponding to $\eta(\cdot)$ and $R_i^\varepsilon(\cdot)$ the remainder. The martingale $N^\varepsilon(\cdot) = (N_1^\varepsilon(\cdot), \ldots, N_d^\varepsilon(\cdot))$ is related to the solution $\chi_{\lambda,i}$ (i.e., the corrector) of the abstract cell problem

$$-\mathcal{L}\chi_{\lambda,i} + \lambda\chi_{\lambda,i} = \widetilde{V}_i, \quad i = 1, \ldots, d$$

(cf. (4.3)), by the expression

$$N^\varepsilon(t) := \sqrt{2D_0}w_i(t) + \chi_{\varepsilon^2,i}(\eta(t)) - \chi_{\varepsilon^2,i}(\eta(0)) - \int_0^t \mathcal{L}\chi_{\varepsilon^2,i}(\eta(s))ds,$$

while the remainder R_i^ε has the expression

$$R_i^\varepsilon(t) = \varepsilon^3 \int_0^{t/\varepsilon^2} \chi_{\varepsilon^2,i}(\eta(s))\, ds + \varepsilon\chi_{\varepsilon^2,i}(\eta(0)) - \varepsilon\chi_{\varepsilon^2,i}\left(\eta\left(\frac{t}{\varepsilon^2}\right)\right). \tag{2.1}$$

Thus, if

$$\lim_{\lambda\downarrow 0} \lambda\|\chi_{\lambda,i}\|_{L^2}^2 = 0 \quad \forall\, i = 1, \ldots, d, \tag{2.2}$$

then the remainder R_i^ε vanishes in the limit $\varepsilon \to 0$ and by the martingale central limit theorem $\varepsilon N^\varepsilon(t/\varepsilon^2)$, and, consequently, $\varepsilon\boldsymbol{x}(t/\varepsilon^2)$ converges to a Brownian motion.

In the case of strongly mixing Markov velocity, the generator $\mathcal{A}$ has a spectral gap: $-(\mathcal{A}f, f)_{L^2} \geq c(f, f)_{L^2}$ for some positive constant c. As a consequence, the (Lagrangian) semigroup T^t associated with $\mathcal{L}$ has an exponential decay property [6] so that

$$\psi_i = \int_0^{+\infty} T^t V_i\, dt \quad \forall\, i = 1, \ldots, d \tag{2.3}$$

converges in L^2 and is the *spatially homogeneous* L^2-solution of the corrector equation with $\lambda = 0$. Therefore, the condition (2.2) is automatically satisfied. This case corresponds to either $\beta > 0$ with additional infrared cutoff (i.e., the cutoff $a(\cdot)$ in

(1.4) is supported away from zero) or $\beta = 0$ for both of which the correlation time for any $\mathbf{k}$ is uniformly bounded.

In the nonmixing case, the integral (2.3) diverges in general, and the correctors $\chi_{\lambda,i}$ cease to be spatially homogeneous in the limit of $\lambda \to 0$. In this case the condition (2.2) is more difficult to check. This is the main technical difficulty resolved in this paper.

A standard argument shows that the energy identity (4.9) for the correctors is more or less equivalent to the condition (2.2) (cf. Theorem 1 and Corollary 1). While the energy *inequality* (4.7) is straightforward and holds true in general, the strict equality requires careful estimation of the large scale behavior of the correctors manifest in (3.4).

In section 7 we sketch a plausible generalization of the above argument to general temporally ergodic, reversible Markovian flows satisfying the condition (1.8).

3. The random velocity fields. We will describe the time evolution of the velocity field as a Markov process $\mathbf{V}(t,\cdot)$, with values in the space of d-dimensional divergence-free fields $\mathbb{X}$ endowed with a probability measure μ.

Since the velocity fields are spatially homogeneous elements of the state space of the $\mathbb{X}$, we have to account for a possible growth at ∞. This can be accomplished by introducing a weight function which decays at $\mathbf{x} = \infty$. For the examples considered below, in particular for Gaussian fields, a sufficiently high power-law decaying weight is enough. Therefore, let $\mathbb{X}$ be a Banach space of d-dimensional divergence-free fields $f : R^d \to R^d$, that is, the completion of $\mathcal{S}(R^d, R^d)$ with respect to the norm

$$\left[\int_{R^d} (|f(\mathbf{x})|^p + |\nabla_{\mathbf{x}} f(\mathbf{x})|^p + \cdots + |\nabla_{\mathbf{x}}^m f(\mathbf{x})|^p)(1 + |\mathbf{x}|^2)^{-\rho}\, d\mathbf{x}\right]^{1/p}$$

for $p \in [1, \infty)$, a positive integer m, and $\rho > d/2$. We can choose $m \geq d/p$ to ensure every element is spatially continuous by the Sobolev imbedding theorem.

3.1. General formulation. For spatial homogeneity, the measure μ is assumed to be invariant under the group of translations $\tau_{\mathbf{x}}, \mathbf{x} \in R^d$, acting on $\mathbb{X}$: $\tau_{\mathbf{x}} f(\cdot) := f(\mathbf{x} + \cdot)$ for all $f \in \mathbb{X}$. As a result, the field $f \in \mathbb{X}$ has the Fourier spectral representation

$$f(\mathbf{x}) = \int_{R^d} \left[\hat{f}^c_{d\mathbf{k}} \cos(\mathbf{x}\cdot\mathbf{k}) + \hat{f}^s_{d\mathbf{k}} \sin(\mathbf{x}\cdot\mathbf{k})\right],$$

with

$$\mathbf{k}\cdot\hat{f}^\theta_{d\mathbf{k}} = 0 \quad \forall\, \mathbf{k}, \quad \theta = c, s, \tag{3.1}$$

where $\hat{f}^c_{d\mathbf{k}}$ and $\hat{f}^s_{d\mathbf{k}}$ are, respectively, the Fourier sine and cosine modes of $f(\cdot)$. Equation (3.1) is to ensure the divergence-free property. The translation $\tau_{\mathbf{x}} f$ then has the Fourier modes $(\hat{f}^c_{\mathbf{x},d\mathbf{k}}, \hat{f}^s_{\mathbf{x},d\mathbf{k}})$ with

$$\hat{f}^c_{\mathbf{x},d\mathbf{k}} := \hat{f}^c_{d\mathbf{k}} \cos(\mathbf{k}\cdot\mathbf{x}) + \hat{f}^s_{d\mathbf{k}} \sin(\mathbf{k}\cdot\mathbf{x}), \qquad \hat{f}^s_{\mathbf{x},d\mathbf{k}} := -\hat{f}^c_{d\mathbf{k}} \sin(\mathbf{k}\cdot\mathbf{x}) + \hat{f}^s_{d\mathbf{k}} \cos(\mathbf{k}\cdot\mathbf{x}).$$

It is easy to see that

$$\int \hat{f}^{\theta'}_{d\mathbf{k}'} \otimes \hat{f}^{\theta}_{d\mathbf{k}}\, \mu(df) = \delta_{\theta,\theta'} \delta(\mathbf{k} - \mathbf{k}') \overline{\mathcal{E}}(|\mathbf{k}|) \Gamma(\mathbf{k})\, d\mathbf{k}\, d\mathbf{k}',$$

where $\overline{\mathcal{E}}(k) := \mathcal{E}(k)k^{1-d}$. Notice also the useful fact

$$\int \left| \int_{|\mathbf{k}|\le l_0} \hat{f}^{\theta}_{d\mathbf{k}} \right|^2 \mu(df) = (d-1)\int_{|\mathbf{k}|\le l_0} \overline{\mathcal{E}}(|\mathbf{k}|)\, d\mathbf{k} \sim O(l_0^{2(1-\alpha)}),\ \ \theta \in \{c,s\},\ l_0 \ll 1. \tag{3.2}$$

Here and below we use the short-hand notation $\overline{\mathcal{E}}(k) \equiv \mathcal{E}(k)k^{1-d}$.

Let L^p, L^p_d $1 \le p < +\infty$, be, respectively, the Banach space of all real valued random variables $F : \mathbb{X} \to \mathbb{R}$ satisfying $\|F\|^p_{L^p} := \int |F(f)|^p\,\mu(df) < +\infty$ and the space of all random vectors $F = (F_1, \cdots, F_d)$ such that $\|F\|^p_{L^p_d} := \sum_{m=1}^d \|F_m\|^p_{L^p} < +\infty$. Similarly, we can define the spaces L^∞ and L^∞_d.

We will also need the space H_1 consisting of all first degree homogeneous polynomials, i.e., the L^2-closure of the elements

$$\int (\phi_1(\mathbf{k}) \cdot \hat{f}^c_{d\mathbf{k}} + \phi_2(\mathbf{k}) \cdot \hat{f}^s_{d\mathbf{k}}) \quad \forall\, \phi_1, \phi_2 \in \mathcal{S}(R^d; R^d).$$

Let Π_1 denote the orthogonal projection onto the subspace H_1.

The translation group $\tau_{\mathbf{x}}$ induces a C_0-group of unitary transformations $U^{\mathbf{x}}F(\cdot)$ on $L^p(\mathbb{X})$, $p \in [1, +\infty)$, as given by $U^{\mathbf{x}}F(f) := F(\tau_{\mathbf{x}}(f))$, $\mathbf{x} \in R^d$, $f \in \mathbb{X}$. Let $D_iF := \partial_{x_i} U^{\mathbf{x}}F_{|\mathbf{x}=\mathbf{0}}$, $i = 1, \ldots, d$ be the L^2-generators of $U^{x\mathbf{e}_i}$, $x \in R$. Here $\mathbf{e}_i$ is the i—the vector of the standard basis in R^d. Let C^m_b be the space of those elements $F \in L^\infty$ such that $F(\mathbf{x}, f) := F(\tau_{\mathbf{x}} f)$ are m times differentiable in $\mathbf{x}$, μ a.s., with all derivatives up to order m bounded. Let $\mathcal{C} := \bigcap_{m\ge 1} C^m_b$. For any $p \in [1, \infty]$ and a positive integer m, we define Sobolev spaces $W^{p,m}$ consisting of closures of $\mathcal{C}$ in the norms

$$\|F\|^p_{p,m} := \sum_{m_1+\cdots+m_d \le m} \|D_1^{m_1} \cdots D_d^{m_d} F\|^p_{L^p}.$$

Let $\mathcal{P}(l), l > 0 : L^1 \to L^1$ be the conditional expectation conditioned on $\{f^c_{d\mathbf{k}}, f^s_{d\mathbf{k}} : |\mathbf{k}| > l\}$. It is easy to check that the projection $\mathcal{P}(l), l > 0$, commutes with the operator $U^{\mathbf{x}}, \mathbf{x} \in R^d$, because the conditional measure corresponding to $\mathcal{P}(l)$ is also spatially homogeneous.

The purpose of considering these function spaces on $\mathbb{X}$ is to construct the correctors by solving the abstract cell problems defined on them. To this end, we shall also consider a linear R^d valued functional $\tilde{\mathbf{V}} \in H_1$ that assigns to each realization of the field its value at the origin:

$$\tilde{\mathbf{V}}(f) \equiv f(\mathbf{0}) = \int_{R^d} \hat{f}^c_{d\mathbf{k}} \quad \forall f \in \mathbb{X}. \tag{3.3}$$

Equation (3.3) is well defined since $f \in \mathbb{X}$ is spatially continuous. With $\tilde{\mathbf{V}}$ we can explicitly write down the Lagrangian generator $\mathcal{L}$ (see formula (4.1) below).

We think of the temporal evolution $\mathbf{V}(t), t \ge 0$, as a reversible Markov process on $\mathbb{X}$ with a self-adjoint L^2-generator $\mathcal{A}$ and the invariant measure μ (so that $\mathbf{V}(t)$ is temporally stationary as well as spatially homogeneous with respect to μ). We assume that the generator $\mathcal{A}$ has the following properties:

$$\mathcal{E}_{\mathcal{A}}(F,F) \ge 2\,C_1\,\beta \int_0^{K_0} l^{2\beta-1}\|\mathcal{P}(l)F\|^2_{L^2}\,dl, \quad F \in L^2_0, \tag{3.4}$$

$$\mathcal{E}_{\mathcal{A}}(F,F) \ge C_2\,\mathcal{E}_{\mathcal{A}}(\Pi_1 F, \Pi_1 F), \quad F \in L^2, \tag{3.5}$$

for some positive constants C_1, C_2 independent of F, where $\mathcal{E}_{\mathcal{A}}(F,F) := -(\mathcal{A}F, F)$ is the Dirichlet form associated with $\mathcal{A}$. As we will see in the following sections, as the construction of the velocity field becomes more explicit, (3.4) and (3.5) can often be derived from more intuitive assumptions.

From (3.4) it immediately follows that μ is ergodic with respect to temporal shifts. Indeed, if $F \in L_0^2$ and $\mathcal{E}_{\mathcal{A}}(F,F) = 0$, then from (3.4) we obtain $\mathcal{P}(l)F = 0$ for all $l > 0$, which implies $F = 0$. However, the generator $\mathcal{A}$ does not have a spectral gap; i.e., the temporal evolution is not mixing. And, as we will see more directly in the examples below, the factor $l^{2\beta-1}$ in (3.4) accounts for the time scales of different Fourier modes.

3.2. Velocity with independent Fourier modes. The construction is most straightforward and explicit when each sine or cosine mode $(\hat{\mathbf{V}}^c_{d\mathbf{k}}(t), \hat{\mathbf{V}}^s_{d\mathbf{k}}(t))$ of the process $\mathbf{V}(t)$ evolves *independently.* This is what we assume in this section and in section 3.3. With this and other assumptions, we will derive the properties (3.4) and (3.5).

Because different Fourier modes are independent, $\mathcal{P}(l)$ is the orthogonal projection on L^2 onto the subspace spanned by $\{f^c_{d\mathbf{k}}, f^s_{d\mathbf{k}} : |\mathbf{k}| > l\}$. It is easy to check that the projection $\mathcal{P}(l), l > 0$, commutes with the operator $U^{\mathbf{x}}, \mathbf{x} \in R^d$, because the conditional measure corresponding to $\mathcal{P}(l)$ is also spatially homogeneous. Moreover, the projection $\mathcal{P}(l), l > 0$, commutes with the generator $\mathcal{A}$, and we have

$$\mathcal{E}_{\mathcal{A}}(F,F) \geq \mathcal{E}_{\mathcal{A}}(\mathcal{P}(l)F, \mathcal{P}(l)F). \tag{3.6}$$

To account for the power-law behavior of the correlation times (1.5) we assume that the formal generator $\mathcal{A}$ has the form

$$\mathcal{A}F = \int_{\mathbf{k}\in R^d} |\mathbf{k}|^{2\beta} \widehat{\mathcal{A}}_0(d\mathbf{k}) F(\hat{f}^c_{d\mathbf{l}}, \hat{f}^s_{d\mathbf{l}}; \mathbf{l} \in R^d), \tag{3.7}$$

where $\widehat{\mathcal{A}}_0(d\mathbf{k})$, acting only on $\hat{f}^c_{d\mathbf{k}}, \hat{f}^s_{d\mathbf{k}}$ variables, is the (formal) generator of the strongly mixing, reversible Markov process for each mode $\mathbf{k} \in R^d$. We assume that $\int_{\mathbf{k}\in R^d} \widehat{\mathcal{A}}_0(d\mathbf{k})$ is self-adjoint and satisfies the spectral gap estimate

$$-\int_{\mathbf{k}\in R^d} (\widehat{\mathcal{A}}_0(d\mathbf{k})F, F)_{L^2} \geq c_1 \|F\|^2_{L^2}, \quad c_1 > 0, \tag{3.8}$$

for all $F \in L_0^2$, the square integrable functions with zero mean defined on $\mathbb{X}$.

The factor $|\mathbf{k}|^{2\beta}$ plays the role of time change: it rescales the time for each mode $\mathbf{k} \in R^d$ by the factor $|\mathbf{k}|^{2\beta}$. The generator $\mathcal{A}$ is thus the synthesis of the generators $|\mathbf{k}|^{2\beta}\widehat{\mathcal{A}}_0(d\mathbf{k})$ for different Fourier modes $(\hat{\mathbf{V}}^c_{d\mathbf{k}}(t), \hat{\mathbf{V}}^s_{d\mathbf{k}}(t)), \mathbf{k} \in R^d$. The expression (3.7) is formal and should be defined via the limiting procedure of periodic approximation (cf. [11]).

Set

$$\mathcal{A}_0(k) := \int_{|\mathbf{l}|>k} \hat{\mathcal{A}}_0(d\mathbf{l}).$$

The operator $\mathcal{A}_0(k), k > 0$, is the generator of a strongly mixing, reversible Markov process and, as a result of (3.8), satisfies the spectral gap property

$$-(\mathcal{A}_0(k)\mathcal{P}(k)F, \mathcal{P}(k)F)_{L^2} \geq c_1 \|\mathcal{P}(k)F\|^2_{L^2}, \quad F \in L_0^2. \tag{3.9}$$

We now show that (3.4) follows from (3.7) and (3.6). We have from (3.7) that

$$\mathcal{E}_{\mathcal{A}}(F,F) = \int_0^{K_0} l^{2\beta}\,(d\mathcal{A}_0(l)F,F)_{L^2}, \tag{3.10}$$

where $d\mathcal{A}_0(l)$ denotes the differential of $\mathcal{A}_0(l)$. An integration by parts gives

$$\begin{aligned}\mathcal{E}_{\mathcal{A}}(F,F) &= \left[l^{2\beta}(\mathcal{A}_0(l)F,F)_{L^2}\right]_{l=0}^{K_0} - 2\beta\int_0^{K_0} l^{2\beta-1}(\mathcal{A}_0(l)F,F)_{L^2}\,dl\\ &= -2\beta\int_0^{K_0} l^{2\beta-1}(\mathcal{A}_0(l)F,F)_{L^2}\,dl.\end{aligned}$$

Here we have used the fact that the support of the cutoff function a is contained in $[0, K_0]$. Similar to (3.6), we have

$$-(\mathcal{A}_0(l)F,F)_{L^2} \geq -(\mathcal{A}_0(l)\mathcal{P}(l)F,\mathcal{P}(l)F)_{L^2}$$

for all $F \in L_0^2$. Since $\mathcal{P}(l)F \in L_0^2$ for all $F \in L_0^2$, (3.4) follows from (3.9).

3.3. Example: The Ornstein–Uhlenbeck velocity.

We now describe an important example of Gaussian velocity fields in greater detail.

The Fourier modes of the velocity field are the stationary solution of the infinite dimensional SDEs:

$$d\hat{\mathbf{V}}^c_{d\mathbf{k}}(t) = -|\mathbf{k}|^{2\beta}\hat{\mathbf{V}}^c_{d\mathbf{k}}(t)dt + |\mathbf{k}|^{\beta}\sqrt{2\overline{\mathcal{E}}(|\mathbf{k}|)\Gamma(\mathbf{k})}\,W^c_{d\mathbf{k}}(dt) \tag{3.11}$$

and

$$d\hat{\mathbf{V}}^s_{d\mathbf{k}}(t) = -|\mathbf{k}|^{2\beta}\hat{\mathbf{V}}^s_{d\mathbf{k}}(t)dt + |\mathbf{k}|^{\beta}\sqrt{2\overline{\mathcal{E}}(|\mathbf{k}|)\Gamma(\mathbf{k})}\,W^s_{d\mathbf{k}}(dt), \tag{3.12}$$

where $W^c_{d\mathbf{k}}(dt)$, $W^s_{d\mathbf{k}}(dt)$ are two independent R^d valued space-time white noise fields. The generator of the process $(\hat{\mathbf{V}}^c_{d\mathbf{k}}(t), \hat{\mathbf{V}}^s_{d\mathbf{k}}(t))$ is of the form $|\mathbf{k}|^{2\beta}\widehat{\mathcal{A}}_0(d\mathbf{k})$ with

$$\widehat{\mathcal{A}}_0(d\mathbf{k}) := \overline{\mathcal{E}}(|\mathbf{k}|)(\nabla_{\hat{f}^c_{d\mathbf{k}}}\cdot\Gamma(\mathbf{k})\nabla_{\hat{f}^c_{d\mathbf{k}}} + \nabla_{\hat{f}^s_{d\mathbf{k}}}\cdot\Gamma(\mathbf{k})\nabla_{\hat{f}^s_{d\mathbf{k}}}) - \hat{f}^c_{d\mathbf{k}}\cdot\nabla_{\hat{f}^c_{d\mathbf{k}}} - \hat{f}^s_{d\mathbf{k}}\cdot\nabla_{\hat{f}^s_{d\mathbf{k}}},$$

where $\nabla_{\hat{f}^c_{d\mathbf{k}}}$ and $\nabla_{\hat{f}^s_{d\mathbf{k}}}$ denote the differentiations with respect to $\hat{f}^c_{d\mathbf{k}}$ and $\hat{f}^s_{d\mathbf{k}}$, respectively. Conditions (3.7) and (3.8) can be easily verified. Further, note that the space H_1 is invariant under $\mathcal{A}$, and the action of $\mathcal{A}$ on H_1 is given by

$$\begin{aligned}&\mathcal{A}\textstyle\int(\phi_1(\mathbf{k})\cdot f^c_{d\mathbf{k}} + \phi_2(\mathbf{k})\cdot f^s_{d\mathbf{k}})\\ &= -\textstyle\int |\mathbf{k}|^{2\beta}(\phi_1(\mathbf{k})\cdot f^c_{d\mathbf{k}} + \phi_2(\mathbf{k})\cdot f^s_{d\mathbf{k}}) \quad \forall\phi_1,\phi_2\in\mathcal{S}(R^d;R^d).\end{aligned} \tag{3.13}$$

Here we make a useful observation that the generator $\mathcal{A}$ commutes with the orthogonal projection Π_1 onto H_1 because H_1 is an invariant subspace of the self-adjoint operator $\mathcal{A}$, and so (3.5) is automatically satisfied.

The initial condition $(\hat{\mathbf{V}}^c_{d\mathbf{k}}(0), \hat{\mathbf{V}}^s_{d\mathbf{k}}(0))$, $\mathbf{k}\in R^d$, is independent of the white noise fields and is distributed according to the Gaussian invariant measure μ, formally written as

$$\mu[A] = \int\cdots\int_{f\in A}\prod_{\mathbf{k}\in R^d}[(2\pi)^d\overline{\mathcal{E}}(|\mathbf{k}|)|d\mathbf{k}|]^{-1}\exp\left\{-\frac{|\hat{f}^c_{d\mathbf{k}}|^2 + |\hat{f}^s_{d\mathbf{k}}|^2}{2\overline{\mathcal{E}}(|\mathbf{k}|)|d\mathbf{k}|}\right\}d\hat{f}^c_{d\mathbf{k}}\,d\hat{f}^s_{d\mathbf{k}} \qquad \forall A\in\mathcal{B}(\mathbb{X}). \tag{3.14}$$

The continuum product can be defined as the weak limit of Gaussian periodic measures given by

$$\mu_n[A] = \int_{\mathcal{T}^{-1}(A)} \cdots \int \prod_{\mathbf{i}:0<|\mathbf{k_i}|\le 2^n} [(2\pi\overline{\mathcal{E}})^d(|\mathbf{k_i}|)|\delta\mathbf{k}|]^{-1} \exp\left\{-\frac{|\hat{f}^c_{\mathbf{k_i}}|^2 + |\hat{f}^s_{\mathbf{k_i}}|^2}{2\overline{\mathcal{E}}(|\mathbf{k_i}|)|\delta\mathbf{k}|}\right\} d\hat{f}^c_{\mathbf{k_i}}\, d\hat{f}^s_{\mathbf{k_i}}$$

for all $A \in \mathcal{B}(\mathbb{X})$, where $\mathcal{B}(\mathbb{X})$ is the Borel σ-algebra of $\mathbb{X}$, $\mathbf{k_i} := \mathbf{i}2^{-n}$, $\mathbf{i} \in \mathbb{Z}^d$, $|\delta\mathbf{k}| := 2^{-nd}$, and the map $\mathcal{T}$ is given by

$$\mathcal{T}\left((\hat{f}^c_{\mathbf{k_i}}\, \hat{f}^s_{\mathbf{k_i}})_{\mathbf{i}\in\mathbb{Z}^d}\right)(\mathbf{x}) := \sum_{\mathbf{i}:0<|\mathbf{k_i}|\le 2^n} [\hat{f}^c_{\mathbf{k_i}} \cos(\mathbf{x}\cdot\mathbf{k_i}) + \hat{f}^s_{\mathbf{k_i}} \sin(\mathbf{x}\cdot\mathbf{k_i})]$$

for all $(\hat{f}^c_{\mathbf{k_i}}\, \hat{f}^s_{\mathbf{k_i}})$, with $0 < |\mathbf{k_i}| \le 2^n$ and $\hat{f}^c_{\mathbf{k_i}} \cdot \mathbf{i} = \hat{f}^s_{\mathbf{k_i}} \cdot \mathbf{i} = 0$ (see [11] for details). The orthogonal projection $\mathcal{P}(l_0)$ can be written as

$$\begin{aligned}\mathcal{P}(l_0)F = \int_{|\mathbf{k}|\le l_0} & F(f^c_{d\mathbf{k}}, f^s_{d\mathbf{k}}; \mathbf{k} \in R^d) \\ & \cdot \prod_{|\mathbf{k}|\le l_0} [2\pi\overline{\mathcal{E}}(|\mathbf{k}|)|d\mathbf{k}|]^{-d} \exp\left\{-\frac{|\hat{f}^c_{d\mathbf{k}}|^2 + |\hat{f}^s_{d\mathbf{k}}|^2}{\overline{\mathcal{E}}(|\mathbf{k}|)|d\mathbf{k}|}\right\} d\hat{f}^c_{d\mathbf{k}} d\hat{f}^s_{d\mathbf{k}}.\end{aligned} \tag{3.15}$$

For non-Gaussian examples, one may consider, instead of the white noise driving field in (3.11)–(3.12), the differential of Poisson random fields as the driving field to construct Poisson shot noise velocity fields (cf. [5]).

4. The Lagrangian velocity process.

4.1. The generator and the correctors.

Set

$$\eta(t) := \tau_{\mathbf{x}(t)}\mathbf{V}(t), \quad t \ge 0.$$

It is well known (see, e.g., [15]) that this process is Markovian, stationary, and ergodic with respect to the invariant measure μ. The generator of the process is given by

$$\mathcal{L}F = D_0\,\Delta F + \mathcal{A}F + \tilde{\mathbf{V}} \cdot \nabla F \quad \forall F \in \mathcal{C}_{\mathcal{A}} := D(\mathcal{A}) \cap W^{2,\infty}, \tag{4.1}$$

where $\tilde{\mathbf{V}}$ is defined in (3.3), $\nabla := (D_1, \ldots, D_d)$, $\Delta := D_1^2 + \cdots + D_d^2$, and $D(\mathcal{A})$ is the domain of $\mathcal{A}$. On $\mathcal{C}_{\mathcal{A}} \times \mathcal{C}_{\mathcal{A}}$ we define a nonnegative definite bilinear form

$$(f, g)_+ := \mathcal{E}_{\mathcal{A}}(f, g) + D_0 \int \nabla f \cdot \nabla g\, d\mu. \tag{4.2}$$

The form is closable, and we denote by $\mathcal{H}_+$ the completion of $\mathcal{C}_{\mathcal{A},0} := \mathcal{C}_{\mathcal{A}} \cap L^2_0$ under the norm $\|\cdot\|_+ := (\cdot,\cdot)_+^{1/2}$. It is easy to see that $\mathcal{H}_+ = W^{1,2} \cap D(\mathcal{E}_{\mathcal{A}})$ (cf. [9]), the latter being the domain of $\mathcal{E}_{\mathcal{A}}$. The scalar product $(\cdot,\cdot)_+$ over $\mathcal{H}_+$ is the Dirichlet form associated with the Markovian process $\xi_t := \tau_{\mathbf{w}(t)}\mathbf{V}(t)$, $t \ge 0$, with $\mathbf{w}(t)$, $t \ge 0$, a standard d-dimensional Brownian motion independent of $\mathbf{V}(t)$, $t \ge 0$.

Consider the resolvent equation on $\mathcal{H}_+$:

$$-\mathcal{L}\chi_{\lambda,i} + \lambda\chi_{\lambda,i} = \tilde{V}_i, \quad i = 1, \ldots, d. \tag{4.3}$$

It can be shown that (4.3) has a unique weak solution $\chi_{\lambda,i} \in \mathcal{H}_+$ in the sense that

$$\mathcal{E}_{\mathcal{A}}(\chi_{\lambda,i}, \Phi) + D_0 \int \nabla\chi_{\lambda,i} \cdot \nabla\Phi\, d\mu + \int \tilde{\mathbf{V}} \cdot \nabla\chi_{\lambda,i}\Phi\, d\mu + \lambda \int \chi_{\lambda,i}\Phi\, d\mu = \int \tilde{V}_i\Phi d\mu \tag{4.4}$$

for all $\Phi \in L^\infty \cap \mathcal{H}_+$ (see [11]).

4.2. The energy estimate and the energy identity. To study the long time limit, we need to pass to the limit $\lambda \to 0$. To this end, we consider the right side of (4.4) as a linear functional

$$G_i(\Phi) := \int \tilde{V}_i \, \Phi \, d\mu = \int \tilde{V}_i \, \Pi_1 \Phi \, d\mu = \int_{R^d} \overline{\mathcal{E}}(|\mathbf{k}|) \left(\delta_{i,\cdot} - \frac{k_i \mathbf{k}}{|\mathbf{k}|^2} \right) \cdot \phi_1(\mathbf{k}) \, d\mathbf{k},$$

with $\Pi_1 \Phi = \int_{R^d} (\phi_1(\mathbf{k}) f^c_{d\mathbf{k}} + \phi_2(\mathbf{k}) f^s_{d\mathbf{k}}) \in H_1$ for some $\phi_1, \phi_2 \in \mathcal{S}(R^d; R^d)$. By the Cauchy–Schwarz inequality we have

$$\begin{aligned} \left| \int \tilde{V}_i \, \Phi \, d\mu \right| &\le \sqrt{\int_{R^d} \overline{\mathcal{E}}(|\mathbf{k}|) |\mathbf{k}|^{-2\beta} \, d\mathbf{k}} \sqrt{\int_{R^d} |\mathbf{k}|^{2\beta} \overline{\mathcal{E}}(|\mathbf{k}|) |\phi_1(\mathbf{k})|^2 \, d\mathbf{k}} \\ &\le C' \mathcal{E}_{\mathcal{A}}(\Pi_1 \Phi, \Pi_1 \Phi)^{1/2} \\ &\le C_2 C' \mathcal{E}_{\mathcal{A}}(\Phi, \Phi)^{1/2} \\ &\le C' \|\Phi\|_+, \end{aligned} \tag{4.5}$$

with C_2 given in (3.5) and

$$C' = \sqrt{\int_{R^d} \overline{\mathcal{E}}(|\mathbf{k}|) |\mathbf{k}|^{-2\beta} \, d\mathbf{k}} < \infty \quad \text{for } \alpha + \beta < 1.$$

Thus $G_i \in \mathcal{H}_-$, the dual space to $\mathcal{H}_+$. Using the test function $\xi_n(\chi_\lambda)$ with $\xi_n(r) := -n^\gamma \vee (r \wedge n^\gamma)$ in (4.4) and passing to the limit as $n \to \infty$, we have

$$\|\chi_{\lambda,i}\|_+^2 + \lambda \|\chi_{\lambda,i}\|_{L^2}^2 = G_i(\chi_{\lambda,i}) \le \|G_i\|_- \|\chi_{\lambda,i}\|_+ \tag{4.6}$$

(cf. [11]), from which the energy estimate follows:

$$\|\chi_{*,i}\|_+^2 \le G_i(\chi_{*,i}). \tag{4.7}$$

Moreover, $\chi_{\lambda,i}$, $\lambda > 0$, is weakly compact in $\mathcal{H}_+$. Let $\chi_{*,i}$ be an $\mathcal{H}_+$-weak limit of $\chi_{\lambda,i}$ as $\lambda \downarrow 0$. It satisfies the following equation:

$$\begin{aligned} \mathcal{E}_{\mathcal{A}}(\chi_{*,i}, \Phi) + D_0 \int \nabla \chi_{*,i} \cdot \nabla \Phi \, d\mu + \int \tilde{\mathbf{V}} \cdot \nabla \chi_{*,i} \Phi \, d\mu \\ = G_i(\Phi) \quad \forall \Phi \in L^\infty \cap \mathcal{H}_+. \end{aligned} \tag{4.8}$$

Now we prove the crucial fact:

$$\mathcal{E}_{\mathcal{A}}(\chi_{*,i}, \chi_{*,i}) + D_0 \int \nabla \chi_{*,i} \cdot \nabla \chi_{*,i} \, d\mu = G_i(\chi_{*,i}).$$

The key point of the proof is to show that the convection term drops out in the limit. Here we repeat the argument in [11].

THEOREM 1 (energy identity). *Suppose $\alpha + \beta < 1$. Then*

$$\|\chi_{*,i}\|_+^2 = G_i(\chi_{*,i}). \tag{4.9}$$

Proof. Property (3.4) implies that $\mathcal{P}(l)\chi_{*,i} \in L_0^2 \cap \mathcal{H}_+$ for any $l > 0$. Since $U^{\mathbf{x}}, \mathbf{x} \in R^d$, and $\mathcal{P}(l), l > 0$, commute, we have

$$\mathcal{P}(l)(\tilde{\mathbf{V}} \cdot \nabla \chi_{*,i}) = \mathcal{P}(l)(\tilde{\mathbf{V}}_{\le l} \cdot \nabla \chi_{*,i}) + \tilde{\mathbf{V}}_{>l} \cdot \nabla \mathcal{P}(l) \chi_{*,i},$$

where $\tilde{\mathbf{V}}_{\leq l}(f) := \int_{|\mathbf{k}|\leq l} f^c_{d\mathbf{k}}$ and $\tilde{\mathbf{V}}_{>l}(f) := \int_{|\mathbf{k}|>l} f^c_{d\mathbf{k}}$. Hence $\mathbf{V}(f) = \mathbf{V}_{\leq l}(f) + \mathbf{V}_{>l}(f)$.

Consider the test function $\Phi := \xi_n(\mathcal{P}(l)\chi_{*,i})$ in (4.8) with ξ_n as before. Notice that

$$\int \tilde{\mathbf{V}}_{>l} \cdot \nabla \mathcal{P}(l)\chi_{*,i}\, \xi_n(\mathcal{P}(l)\chi_{*,i})\, d\mu = \int \nabla \cdot (\tilde{\mathbf{V}}_{>l} \Xi_n(\mathcal{P}(l)\chi_{*,i}))\, d\mu = 0,$$

with Ξ_n the primitive of ξ_n satisfying $\Xi_n(0) = 0$. Passing to the limit as $n \to \infty$, in (4.8) we get (cf. [11])

$$\begin{aligned} \mathcal{E}_{\mathcal{A}}(\chi_{*,i}, \mathcal{P}(l)\chi_{*,i}) + D_0 \textstyle\int \nabla\chi_{*,i} \cdot \nabla \mathcal{P}(l)\chi_{*,i}\, d\mu + \int \mathbf{V}_{\leq l} \cdot \nabla \chi_{*,i} \mathcal{P}(l)\chi_{*,i}\, d\mu \\ = G_i(\mathcal{P}(l)\chi_{*,i}) \quad \forall \Phi \in L^\infty \cap \mathcal{H}_+. \end{aligned} \tag{4.10}$$

Note that

$$\begin{aligned} \int \left|\tilde{\mathbf{V}}_{\leq l} \cdot \nabla\chi_{*,i}\, \mathcal{P}(l)\chi_{*,i}\right| d\mu &\leq \left(\int \left|\tilde{\mathbf{V}}_{\leq l}\right|^2 |\mathcal{P}(l)\chi_{*,i}|^2\, d\mu\right)^{1/2} \|\nabla\chi_{*,i}\|_{L^2_d} \\ &= \|\tilde{\mathbf{V}}_{\leq l}\|_{L^2_d}\, \|\mathcal{P}(l)\chi_{*,i}\|_{L^2}\, \|\nabla\chi_{*,i}\|_{L^2_d}. \end{aligned} \tag{4.11}$$

Using (3.2) and (3.4), we find $\|\tilde{\mathbf{V}}_{\leq l}\|_{L^2_d} \|\mathcal{P}(l)\chi_{*,i}\|_{L^2} \sim o(1) l^{1-\alpha-\beta}$ as $l \to 0$. Passing to the limit as $l \downarrow 0$, in (4.10) we conclude (4.9). □

It is easy to see that the $\mathcal{H}_+$-weak solution of (4.8) satisfying the energy identity (4.9) is unique up to a constant.

From the energy identity, we immediately have the following corollary.

COROLLARY 1. *For any $i = 1, \ldots, d$ we have*

$$\lim_{\lambda\downarrow 0} \lambda \|\chi_{\lambda,i}\|^2_{L^2} = 0. \tag{4.12}$$

In addition, $\chi_{\lambda,i}$ converges $\mathcal{H}_+$ strongly as $\lambda \downarrow 0$.

5. Convergence of finite dimensional distributions. The remainder $R_{\varepsilon,i}$ (2.1) satisfies

$$\mathbf{ME}|R_{\varepsilon,i}(t)| \leq \varepsilon^3 \int_0^{t/\varepsilon^2} \mathbf{ME}|\chi_{\varepsilon^2,i}(\eta(s))|\, ds + 2\varepsilon\|\chi_{\varepsilon^2,i}\|_{L^2} \leq \varepsilon \|\chi_{\varepsilon^2,i}\|_{L^2}(t+2). \tag{5.1}$$

The right-hand side of (5.1) tends to 0 as $\varepsilon \downarrow 0$ by Corollary 1. A standard calculation shows that, for any $\varepsilon_1, \varepsilon_2 > 0$,

$$\begin{aligned} \mathbf{ME}|N_i^{\varepsilon_1}(t) - N_i^{\varepsilon_2}(t)|^2 &= -2t(\mathcal{L}(\chi_{\varepsilon_1^2,i} - \chi_{\varepsilon_2^2,i}), \chi_{\varepsilon_1^2,i} - \chi_{\varepsilon_2^2,i})_{L^2} \\ &= 2t\|\chi_{\varepsilon_1^2,i} - \chi_{\varepsilon_2^2,i}\|^2_+ \end{aligned} \tag{5.2}$$

(cf. [11]). From Corollary 1, (5.2), and Kolmogorov's inequality for martingales, we know immediately that for an arbitrary $\varrho > 0$ there exists $\varepsilon_0 > 0$ such that, for $0 < \varepsilon < \varepsilon_0$,

$$\mathbf{ME} \sup_{0\leq t\leq T} \left[\varepsilon N_i^{\varepsilon}\left(\frac{t}{\varepsilon^2}\right) - \varepsilon N_i^{\varepsilon_0}\left(\frac{t}{\varepsilon^2}\right)\right]^2 \leq C\|\chi_{\varepsilon^2,i} - \chi_{\varepsilon_0^2,i}\|^2_+ T < \varrho T \quad \forall T \geq 0. \tag{5.3}$$

Here $C > 0$ is some constant.

Stationarity and ergodicity of $\eta(\cdot)$ implies that $N^{\varepsilon_0}(\cdot)$ is a martingale with stationary and ergodic increments (see [9, Lemma 2]). The classical martingale central limit theorem of Billingsley [4, Theorem 23.1] implies that finite dimensional distributions of $\varepsilon N^{\varepsilon_0}(t/\varepsilon^2)$, $t \geq 0$, tend weakly as $\varepsilon \downarrow 0$ to those of a Brownian motion whose covariance equals $2\mathbf{D}(\varepsilon_0) := [2D_{i,j}(\varepsilon_0)]$ with

$$D_{i,j}(\varepsilon_0) = (\chi_{\varepsilon_0^2,i}, \chi_{\varepsilon_0^2,j})_+ + D_0\delta_{i,j}.$$

Passing to the limit $\varepsilon_0 \downarrow 0$, we conclude convergence of the finite dimensional distributions of $\mathbf{x}_\varepsilon(t)$, $t \geq 0$, as $\varepsilon \downarrow 0$, to the respective finite dimensional distributions of the Wiener measure with the covariance matrix given by $\mathbf{D} = [D_{i,j}]$, where

$$D_{i,j} := (\chi_{*,i}, \chi_{*,j})_+ + D_0\delta_{i,j} \geq D_0\delta_{i,j}. \tag{5.4}$$

6. Vanishing molecular diffusion limit. In this section, we shall write the dependence of the correctors and the effective diffusivity on D_0 explicitly.

By the same argument, the uniform estimate (4.6) is valid independent of λ and D_0. Passing to the limit $\lambda \to 0$, we have

$$\|\chi_*^{(p)}(D_0)\|_+ \leq C \quad \text{for some constant } C > 0, \tag{6.1}$$

which, together with (5.4), implies that the limiting diffusivity is bounded as D_0 tends to zero.

For the lower bound, we turn to (4.8), which with the test function $\tilde{V}_i$ can be written, after some elementary approximation arguments, in the form

$$\mathcal{E}_{\mathcal{A}}(\chi_{*,i})(D_0), \tilde{V}_i) + D_0 \int \nabla\chi_{*,i}(D_0) \cdot \nabla\tilde{V}_i \, d\mu + \int \tilde{\mathbf{V}} \cdot \nabla\chi_{*,i}(D_0)\tilde{V}_i \, d\mu = G_i(\tilde{V}_i). \tag{6.2}$$

We want to show that, if the infimum of $\mathbf{D}$ is zero as D_0 tends to zero, then the entire left side of (6.2) drops out in the limit while the right side equals $\|V_i\|_{L^2}^2 > 0$, leading to a contradiction.

Let us assume the infimum of $\mathbf{D}$ as $D_0 \to 0$ is zero and take an infimum-achieving subsequence of $\chi_{*,i}(D_0)$, $i = 1, \ldots, d$. For that sequence we shall have

$$\sum_{i=1}^{d} \mathcal{E}_{\mathcal{A}}(\chi_{*,i}(D_0), \chi_{*,i}(D_0)) \to 0. \tag{6.3}$$

By (6.1), the Cauchy inequality, and the assumption, both the first and second terms on the left side of (6.2) vanish.

Let us denote $W_i := \tilde{\mathbf{V}} \cdot \nabla\tilde{V}_i$ and

$$\|W_i\|_{H_-} := \sup_{\mathcal{E}_{\mathcal{A}}(F,F)=1} \int W_i \, F \, d\mu.$$

Then

$$\begin{aligned} \|W_i\|_{H_-}^2 &= \int_0^{+\infty} (R^t \, W_i, W_i)_{L^2} \, dt \\ &= \int_0^{+\infty} \mathbf{E}\left[\tilde{\mathbf{V}}(t) \cdot \nabla\tilde{V}_i(t) \, \tilde{\mathbf{V}}(0) \cdot \nabla\tilde{V}_i(0)\right]. \end{aligned} \tag{6.4}$$

The right side of (6.4) can be explicitly calculated using the Feynman diagrams, and the result is

$$\sum_{i=1}^{d} \|W_i\|_{H_-}^2 = d \int_{R^d}\int_{R^d} \frac{1}{|\mathbf{k}|^{2\beta} + |\mathbf{k}'|^{2\beta}} \times \frac{\mathcal{E}(|\mathbf{k}|)\mathcal{E}(|\mathbf{k}'|)}{|\mathbf{k}|^{d-1}|\mathbf{k}'|^{d-1}} \left\{ |\mathbf{k}'|^2 - \frac{(\mathbf{k}\cdot\mathbf{k}')^2}{|\mathbf{k}|^2} \right\} d\mathbf{k}\, d\mathbf{k}' < +\infty$$

for $\alpha + \beta < 1$. The third term on the left-hand side of (6.2) can therefore be bounded by

$$\|W_i\|_{H_-} \mathcal{E}_{\mathcal{A}}^{1/2}(\chi_{*,i})(D_0), \chi_{*,i}(D_0)), \qquad i = 1, \ldots, d,$$

which vanishes by (6.3).

7. Heuristic criterion for diffusive limit. In this section, we explain why the condition (1.8) is a plausible criterion for the diffusive limit for Markovian flows and discuss its possible extensions and limitations.

First, consider Markovian flows with a symmetric, positive generator $\mathcal{A}$. We have the identity

$$\begin{aligned} \int_0^\infty \mathbf{ME}\left[V_i(t, \sqrt{2D_0}\mathbf{w}(t))\, V_j(0, \mathbf{0})\right] dt \\ = \left((D_0\Delta + \mathcal{A})^{-1}\widetilde{V}_i, \widetilde{V}_j\right)_{L^2}, \quad i, j = 1, \ldots, d, \end{aligned} \tag{7.1}$$

if the right side can be suitably defined. This suggests the definition of reduced λ-correctors $\bar{\chi}_{\lambda,i}$ as solutions of the *reduced* resolvent equation

$$(D_0\Delta + \mathcal{A})\bar{\chi}_{\lambda,i} + \lambda\bar{\chi}_{\lambda,i} = -\widetilde{V}_i, \quad i = 1, \ldots, d. \tag{7.2}$$

Note that the convection term $\widetilde{\mathbf{V}} \cdot \nabla$ is dropped out of (7.2). The condition that the right side of (7.1) is finite is equivalent to $\widetilde{V}_i \in \mathcal{H}_-$, the dual of $\mathcal{H}_+$.

As before, if $\widetilde{V}_i \in \mathcal{H}_-$, the energy estimate

$$\|\bar{\chi}_{\lambda,i}\|_+^2 \le (\widetilde{V}_i, \bar{\chi}_{\lambda,i})_{L^2}$$

can be established in general by a standard argument. Passing to the limit with $\lambda \to 0$, we have an $\mathcal{H}_+$-weak solution $\bar{\chi}_{*,i}$ of (7.2). We want to show that

$$\lim_{\lambda\to 0} \lambda \|\bar{\chi}_{\lambda,i}\|_{L^2}^2 = 0, \tag{7.3}$$

which is equivalent to the energy identity for the limiting solution $\bar{\chi}_{*,i}$

$$\|\bar{\chi}_{*,i}\|_+^2 = G_i(\bar{\chi}_{*,i}).$$

Since there is no convection term $\widetilde{\mathbf{V}} \cdot \nabla$ in (7.2), the energy identity is true in general by the truncation argument. Note that $G_i(\bar{\chi}_{*,i})$ equals the right side of (7.1). Now, since the operator $\widetilde{\mathbf{V}} \cdot \nabla$ is antisymmetric and of lower order than the symmetric part $D_0\Delta + \mathcal{A}$, we can expect to bound $\|\chi_{\lambda,i}\|_{L^2}$ by $\|\bar{\chi}_{\lambda,i}\|_{L^2}$, where $\chi_{\lambda,i}$ is the "true" corrector, i.e.,

$$(D_0\Delta + \mathcal{A} + \widetilde{\mathbf{V}} \cdot \nabla)\chi_{\lambda,i} + \lambda\chi_{\lambda,i} = -\widetilde{V}_i, \quad i = 1, \ldots, d.$$

Therefore, we expect to see

$$\lim_{\lambda \to 0} \lambda \|\chi_{\lambda,i}\|_{L^2}^2 = 0$$

and, hence, the convergence to the diffusive limit under the condition $\widetilde{V}_i \in H_-$.

In previous sections, we did not prove the diffusive limit theorem in such a generality. Because the formulation of (1.8) does not rely on the Markov property of the velocity, it is tempting to apply it to general (Markovian or non-Markovian) flows whose time correlation functions $\rho(t, \mathbf{k})$ decay *exponentially* in time for each $\mathbf{k}$. For flows with nondecaying but oscillating time correlation functions, condition (1.8) may be far from optimal (see [10] for examples).

8. Nondiffusive limit. In this section we present a nondiffusive scaling argument under the conditions $\alpha + \beta > 1$, $\alpha + 2\beta < 2$. We set $\mathbf{x}^\varepsilon(t) := \varepsilon \mathbf{x}(t/\varepsilon^{2\delta})$, where

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{V}(t, \mathbf{x}(t)), \quad \mathbf{x}(0) = \mathbf{0},$$

and the parameter $\delta < 1$ is yet to be determined. Here we drop the molecular diffusion for the simplicity of the argument. Molecular diffusion is irrelevant because, as we shall see below, the scaling is superdiffusive (i.e., $\delta < 1$). Let $\mathbf{V}^\varepsilon(t, \mathbf{x})$ be the Gaussian velocity with the power spectrum

$$\mathcal{E}_\varepsilon(k) = a(\varepsilon k) k^{1-2\alpha} \tag{8.1}$$

instead of (1.4). Then it follows from the spectral representation that $\mathbf{V}^\varepsilon$ is related to $\mathbf{V}$ via

$$\mathbf{V}\left(\frac{t}{\varepsilon^{2\delta}}, \frac{\mathbf{x}}{\varepsilon}\right) = \varepsilon^{1-\alpha} \mathbf{V}^\varepsilon \left(\frac{t}{\varepsilon^{2(\delta-\beta)}}, \mathbf{x}\right).$$

With a unique pair of parameters $\delta = \beta/(\alpha + 2\beta - 1)$, $\eta_\varepsilon = \varepsilon^{2-(\alpha+2\beta)}$, the equation of motion can be written as

$$\frac{d\mathbf{x}^\varepsilon(t)}{dt} = \frac{1}{\eta_\varepsilon^{2\delta-1}} \mathbf{V}^\varepsilon \left(\frac{t}{\eta_\varepsilon^{2\delta}}, \mathbf{x}^\varepsilon(t)\right), \quad \mathbf{x}^\varepsilon(0) = \mathbf{0}. \tag{8.2}$$

Since $\alpha + 2\beta < 2$, η_ε must tend to zero, and (8.2) is in the form of the FBM limit theorem established in [8] under the conditions $\alpha + \beta > 1$ and $\alpha < 1$. The only difference is that the velocity $\mathbf{V}^\varepsilon$ has increasingly smaller scales as ε tends to zero, while the FBM limit theorem of [8] deals with velocity fields with a fixed ultraviolet cutoff. However, as the following physical argument shows, the high frequency components of the velocity fields are negligible under $\alpha + \beta > 1, \alpha < 1$, and $\alpha + 2\beta < 2$. Therefore, the FBM limit should be valid. The small scale velocity has the magnitude

$$\sqrt{\int_{l \le k} \mathcal{E}(l) dl} \sim k^{1-\alpha}, \quad k \gg 1,$$

and the correlation time of the order $k^{-2\beta}$. Then particles carried by small scale velocity travel a distance less than or equal to the sum of $tk^{2\beta}$ of roughly uncorrelated random increments of magnitude $k^{1-\alpha-2\beta}$. Thus, by the central limit theorem on the time scale $t \sim \eta_\varepsilon^{-2\delta}$, the displacement induced by a high wave number k is of the order of magnitude less than or equal to $\sqrt{\eta_\varepsilon^{-2\delta} k^{2\beta}} k^{1-\alpha-2\beta}$, which equals $\eta_\varepsilon^{-\delta} k^{1-\alpha-\beta}$ and is always smaller than η_ε^{-1} (the scale of observation) since $\alpha+\beta>1$ and $\delta<1$. The two conditions are equivalent to each other: $\alpha+\beta>1$ if and only if $\delta<1$. Thus, for $\alpha+\beta>1$, the transport effect of high wave numbers is negligible compared to that of low wave numbers. Transport is essentially determined by the wave numbers of order $\eta_\varepsilon^{\delta/\beta}$, which is right on the edge of Eulerian correlation; therefore, we could derive the scaling exponent based solely on those wave numbers.

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