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Lagrangian dynamics for a passive tracer in a class of Gaussian Markovian flows

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Abstract

We formulate a stochastic differential equation describing the Lagrangian environment process of a passive tracer in Ornstein–Uhlenbeck velocity fields. We subsequently prove a local existence and uniqueness result when the velocity field is regular. When the Ornstein–Uhlenbeck velocity field is only spatially Hölder continuous we construct and identify the probability law for the Lagranging process under a condition on the time correlation function and the Hölder exponent.

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1. Introduction

A major problem in fluid turbulence is the transport of a passive tracer whose displacement $\boldsymbol{x}(\cdot)$ in a d-dimensional flow V satisfies the ordinary differential equation

$$\mathrm{d}\boldsymbol{x}(t)/\mathrm{d}t = V(t,\boldsymbol{x}(t)), \quad \boldsymbol{x}(0)=\mathbf{0}. \tag{1.1}$$

When the velocity field V is temporally and spatially continuous, Peano's theorem guarantees the existence of solutions of Eq. (1.1) and when V is also spatially Lipschitz, the solution is unique. If V is not spatially Lipschitz, then the equation typically has many trajectories for each given initial point $\boldsymbol{x}(0)$.

The inertial-scale (see below) turbulent velocity fluctuation is usually modelled by an irregular time–space stationary random field. The correlation functions $R=[R_{ij}]$ of the stationary process V naturally has the spectral representation

$$R(t,\mathbf{x}) := \mathbb{E}[V(t,\mathbf{x})\otimes V(0,\mathbf{0})] = \int_{\mathbb{R}^d} \cos(\mathbf{k}\cdot\mathbf{x}) r(t,\mathbf{k}) \frac{\mathscr{E}(\mathbf{k})}{|\mathbf{k}|^{d-1}}\,\mathrm{d}\mathbf{k}, \tag{1.2}$$

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where r is the time correlation function, $\mathscr{E}$ the matrix-valued spectral (shell) density and $\mathbb{E}$ denotes the average w.r.t. the ensemble of the velocity field. According to Kolmogorov's similarity hypothesis, an incompressible turbulent velocity in dimension $d=3$ has the following power-law spectral density:

$$\mathscr{E}(\mathbf{k}) = c_0|\mathbf{k}|^{-5/3}(\mathbf{I} - \mathbf{k}\otimes\mathbf{k}/|\mathbf{k}|^2), \quad \ell_0^{-1} < |\mathbf{k}| < \ell_K^{-1} \tag{1.3}$$

in the so-called inertial range of scales (ℓ_K, ℓ_0). Here ℓ_0 is commonly called the integral scale, ℓ_K the Kolmogorov dissipation scale and c_0 the Kolmogorov constant. Eq. (1.3) is called Kolmogorov's 5/3-law. In the large Reynolds number limit $Re \gg 1$, the dissipation length $\ell_K \asymp \ell_0 Re^{-3/4}$ tends to zero while the integral length ℓ_0 stays fixed. It is often assumed (see e.g. Carmona, 1996) that, in the first approximation, V is Gaussian. Then, in the large Reynolds number limit, the field with the spectrum described by Eq. (1.3) has a Hölder continuous modification with exponent less than 1/3.

The question is how to make sense of the trajectories of the passive scalar in turbulent velocities. Our approach is based on the notion of the *Lagrangian environment process*, which roughly speaking describes the medium from the point of view of the moving particle and is defined formally as

$$\eta(t,\cdot) := V(t, \boldsymbol{x}(t) + \cdot), \quad t \geqslant 0.$$

The particle trajectory is then recovered as an additive functional of η:

$$\boldsymbol{x}(t) = \int_0^t \eta(s, \mathbf{0})\,\mathrm{d}s, \quad t \geqslant 0. \tag{1.4}$$

The representation of the trajectory in the form given by (1.4) is quite useful in the study of long time behavior of the particle displacement (see Kozlov, 1985; Papanicolaou and Varadhan, 1982) and was found particularly suitable for the case of transport in incompressible Markovian flows, see Fannjiang and Komorowski (1999). We will not, however, be restricted to divergence-free velocity fields, so in the present article the spectral density $\mathscr{E}$ is a general positive definite symmetric real $d\times d$ matrix satisfying $\mathscr{E}(-\mathbf{k}) = \mathscr{E}(\mathbf{k})$. As we will see in Section 2.2, the Eulerian velocity process V can be thought of as a stationary infinite dimensional Ornstein–Uhlenbeck process in an appropriate Hilbert space and described by the stationary solution of an infinite dimensional linear stochastic differential equation (SDE)

$$\mathrm{d}V(t) = -AV(t)\,\mathrm{d}t + B\,\mathrm{d}W(t), \tag{1.5}$$

where A, B are linear operators related to the spectral density $\mathscr{E}$ and the time correlation function r, W is a cylindrical Wiener process (see Section 2.2) and the law of $V(0)$ is a stationary measure for (1.5). We show that for regular velocity V, see (2.2), the Lagrangian process η satisfies a nonlinear SDE

$$\mathrm{d}\eta(t) = (-A\eta(t) + \eta(t,\mathbf{0})\cdot\nabla_{\mathbf{x}}\eta(t))\,\mathrm{d}t + B\,\mathrm{d}\tilde{W}(t) \tag{1.6}$$

for a suitable initial condition and the cylindrical Wiener process $\tilde{W}$ obtained from W by an appropriate spatial translation (see Theorem 1 for a precise statement).

One may therefore ask first whether the equation

$$\mathrm{d}\eta_f(t) = (-A\eta_f(t) + \eta_f(t,\mathbf{0}) \cdot \nabla_{\mathbf{x}}\eta_f(t))\,\mathrm{d}t + B\,\mathrm{d}W(t), \quad \eta_f(0) = f \tag{1.7}$$

with a given cylindrical Wiener process W and an initial condition f possesses a solution. For regular f and the velocity field the answer to this question is affirmative. We prove the local existence and uniqueness of strong solutions in an appropriate Hilbert space in Theorems 2 and 3. We also show that for almost sure initial condition in the support of the stationary measure for (1.5) the existence of solution can, in fact, be guaranteed globally (see Corollary 1).

In Section 6 we turn to irregular fields with the power-law type spectral density

$$|\mathscr{E}(\mathbf{k})| \sim |\mathbf{k}|^{1-2\alpha}, \quad 1 < \alpha < 2, \text{ for } |\mathbf{k}| \gg 1$$

and the power-law correlation time

$$r(t,\mathbf{k}) = c_1 \exp\{-c_1|\mathbf{k}|^{2\beta}t\}, \quad \beta > 0. \tag{1.8}$$

Without loss of generality we set $c_1 = 1$ in the analysis. These Ornstein–Uhlenbeck fields are spatially Hölder continuous with exponent $H < \alpha - 1$. The Kolmogorov spectrum (1.3) corresponds to $\alpha = 4/3$. Instead of the sample-wise construction we obtain the probability law of the Lagrangian process η by studying Eq. (1.7) via a limiting procedure: consider a sequence of velocity fields V_K, $K \geqslant 1$ obtained by truncation of the spectrum of V for large wave numbers i.e.

$$R_K(t,\mathbf{x}) := \mathbb{E}[V_K(t,\mathbf{x}) \otimes V_K(0,\mathbf{0})] = \int_{\mathbb{R}^d} \cos(\mathbf{k}\cdot\mathbf{x})\mathrm{e}^{-|\mathbf{k}|^{2\beta}t} \frac{\mathbf{1}_{[0,K]}(|\mathbf{k}|)\mathscr{E}(\mathbf{k})}{|\mathbf{k}|^{d-1}}\,\mathrm{d}\mathbf{k}. \tag{1.9}$$

This truncation corresponds to replacing in (1.5) the diffusion operator B and the initial invariant measure μ by some other operator B_K and measure μ_K, respectively. Such fields V_K are smooth, in fact analytic, therefore we can define their corresponding Lagrangian processes η_K, $K \geqslant 1$ satisfying (1.6). The key observation is that the law of η_K on $C([0,+\infty); C(\mathbb{R}^d;\mathbb{R}^d))$ coincides with that of the solution $\bar{\eta}_K$ of the modified equation (1.7) with B_K in place of B. With the Wiener process W fixed it is possible to prove almost sure convergence of $\bar{\eta}_K$, as $K \uparrow \infty$ when $\alpha+\beta > 2$ and $\alpha+3\beta > 3$. Thus, we can conclude that the laws of η_K, $K \geqslant 1$ on $C([0,+\infty); C(\mathbb{R}^d;\mathbb{R}^d))$ converge weakly to a certain law, which then is naturally called the probability law of the Lagrangian environment process for the irregular velocity field V, see Theorem 4 of Section 6 for more details.

In the case of temporally white-noise, spatially Hölderian velocity fields the Lagrangian semigroup has been constructed in Le Jan and Raimond (1999) in the presence of molecular diffusion (see also E and Vanden Eijnden, 2000 and Gawedzki and Vergassola, 2000). Roughly speaking, white-noise-in-time corresponds to $\beta = 0$ and $c_1 \to \infty$ in (1.8).

2. Preliminaries

2.1. Functional spaces and operators

For brevity we set $\mathscr{S}_d := \mathscr{S}(\mathbb{R}^d, \mathbb{R}^d)$. Given $\rho > 0$ and a positive integer m, we define $\vartheta_\rho(\mathbf{x}) = (1 + |\mathbf{x}|^2)^{-\rho}$, $\mathbf{x} \in \mathbb{R}^d$, and $\mathbb{W}^{p,m}_\rho$ as the completion of $\mathscr{S}_d$ with respect to the norm

$$\|\psi\|^p_{\mathbb{W}^{p,m}_\rho} := \int_{\mathbb{R}^d} (|\psi(\mathbf{x})|^p + |\nabla_{\mathbf{x}}\psi(\mathbf{x})|^p + \cdots + |\nabla^m_{\mathbf{x}}\psi(\mathbf{x})|^p)\vartheta_\rho(\mathbf{x})\,\mathrm{d}\mathbf{x}.$$

For $p = 2$, we write $\mathbb{H}^m_\rho := \mathbb{W}^{2,m}_\rho$, or $\mathbb{L}^2_\rho$ depending on whether $m \neq 0$ or $m = 0$. We shall omit writing the subscript ρ in case when $\rho = 0$. Using Sobolev embedding, see e.g. (Gilbarg and Trudinger, 1983, Theorem 7.10, p. 154), it can easily be observed that for $m > d/2$ there exists a constant $C > 0$ such that $|\psi(\mathbf{0})| \leqslant C\|\psi\|_{\mathbb{H}^m_\rho}$, $\psi \in \mathbb{H}^m_\rho$. Notice also that $J_{\rho,p} : \mathbb{W}^{p,m}_\rho \to \mathbb{W}^{p,m}$ given by $J_{\rho,p}\psi(\mathbf{x}) := \psi(\mathbf{x})\vartheta_{\rho/p}(\mathbf{x})$ is a linear isomorphism satisfying $J_{\rho,p}(\mathscr{S}_d) = \mathscr{S}_d$, and that in the special case $m = 0$, $p = 2$ $J_{\rho,2} : \mathbb{L}^2_\rho \to \mathbb{L}^2$ is unitary. In the sequel we shall consider spaces $C([0,T]; \mathbb{H}^m_\rho)$, $T > 0$ consisting of continuous functions $f : [0,T] \to \mathbb{H}^m_\rho$ equipped with the norm $\|f\|_{\mathbb{H}^m_\rho,T} := \sup_{0 \leqslant t \leqslant T} \|f(t)\|_{\mathbb{H}^m_\rho}$.

The following proposition is an immediate consequence of the Rellich–Kondrashov compact embedding theorem (cf. e.g. Theorem 7.22, p. 163 of Gilbarg and Trudinger, 1983).

Proposition 1. *For any $m' > m$ and $\rho' < \rho$ the embedding $\mathbb{H}^{m'}_{\rho'} \subseteq \mathbb{H}^m_\rho$ is compact.*

Let

$$\mathscr{H}_\rho := \bigcap_{m \geqslant 1} \mathbb{H}^m_\rho, \quad \mathscr{C}_\rho := \bigcap_{p,m \geqslant 1} \mathbb{W}^{p,m}_\rho.$$

Obviously $\mathscr{H}_\rho$ and $\mathscr{C}_\rho$ are dense in any $\mathbb{H}^m_\rho$.

Let $S(d)$ denotes the space of all real, symmetric, positive definite $d \times d$ matrices. Given $\beta \geqslant 0$ and $\mathscr{E} : \mathbb{R}^d \to S(d)$ we set

$$\widehat{B_\beta\psi}(\mathbf{k}) = \sqrt{2|\mathbf{k}|^{2\beta}\bar{\mathscr{E}}(\mathbf{k})}\hat{\psi}(\mathbf{k}) \quad \text{and} \quad \widehat{S_\beta(t)}\psi(\mathbf{k}) = \mathrm{e}^{-|\mathbf{k}|^{2\beta}t}\hat{\psi}(\mathbf{k}), \quad \psi \in \mathscr{S}_d, \tag{2.1}$$

with $\bar{\mathscr{E}}(\mathbf{k}) := \mathscr{E}(\mathbf{k})|\mathbf{k}|^{1-d}$. Throughout Sections 2 and 5 we assume that

$$\int_{\mathbb{R}^d} (1 + |\mathbf{k}|^2)^m \operatorname{Tr} \bar{\mathscr{E}}(\mathbf{k})\,\mathrm{d}\mathbf{k} < \infty \quad \text{for any } m \in \mathbb{N}. \tag{2.2}$$

In what follows $\|\cdot\|_{L_{(HS)}(\mathbb{L}^2,\mathbb{H}^m_\rho)}$ denotes the respective Hilbert–Schmidt operator norm, see Da Prato and Zabczyk (1992). Let $\gamma(\beta) = \beta$ if $\beta \notin \mathbb{Z}$ and $+\infty$ otherwise. Both here and throughout the remainder of the paper we assume that $\rho \in (d/2, d/2 + \gamma(\beta))$.

Proposition 2.

(i) *B_β extends to a Hilbert–Schmidt operator $B_\beta : \mathbb{L}^2 \to \mathbb{H}^m_\rho$.*

(ii) *For any $t \geqslant 0$ $S_\beta(t)$ extends to a bounded linear operator $S_\beta(t) : \mathbb{W}^{p,m}_\rho \to \mathbb{W}^{p,m}_\rho$. In addition, $S_\beta(\cdot)$ form a C_0-semigroup on $\mathbb{W}^{p,m}_\rho$ for any $p \geqslant 1$, $m \geqslant 0$. In what* *follows we denote by* $-A_\beta : D(A_\beta) \to \mathbb{H}^m_\rho$ *the generator of the semigroup* $S_\beta(\cdot)$ *on* $\mathbb{H}^m_\rho$.

(iii) $\mathscr{C}_\rho$, $\mathscr{H}_\rho$ *and* $\mathscr{S}_d$ *are cores of any* A_β, $\beta \geqslant 0$. *Moreover*, $\widehat{A_\beta \psi}(\mathbf{k}) = |\mathbf{k}|^{2\beta}\hat{\psi}(\mathbf{k})$ *when* $\psi \in \mathscr{S}_d$, *and* $\mathbb{H}^{m+[2\beta]+1}_\rho \subseteq D(A_\beta)$, $\forall m \geqslant 0$.

(iv) *Suppose that* $\beta, \beta' \geqslant 0$, $t > 0$ *and* $\rho \in (d/2, d/2 + \gamma(\beta) \wedge \gamma(\beta'))$. *Then for any* $f \in \mathbb{H}^m_\rho$ *we have* $S_\beta(t) f \in D(A_{\beta'})$. *Moreover, there exists C possibly depending on* β, β', d *but independent of t such that*

$$\|A_{\beta'} S_\beta(t) B_\beta\|_{L_{(HS)}(\mathbb{L}^2, \mathbb{H}^m_\rho)} \leqslant C \tag{2.3}$$

and

$$\|A_{\beta'} S_\beta(t) f\|_{\mathbb{H}^m_\rho} \leqslant C t^{-\beta'/\beta} \|f\|_{\mathbb{H}^m_\rho}. \tag{2.4}$$

Due to rather technical nature of the proposition we postpone its proof till Appendix A. In what follows we shall suppress the index β of the operators when there is no danger of confusion.

Let $\mathbf{v} \in \mathbb{R}^d$. We define

$$R_\mathbf{v}(t) f(\mathbf{x}) := f(\mathbf{x} + t\mathbf{v}), \quad t \in \mathbb{R}, \ \ f \in \mathbb{H}^m_\rho, \ \ \mathbf{x} \in \mathbb{R}^d.$$

Then $R_\mathbf{v}$ forms a C_0-group of operators on $\mathbb{H}^m_\rho$.

Proposition 3. $\|R_\mathbf{v}(t) f\|_{\mathbb{H}^m_\rho} \leqslant (1 + |\mathbf{v}t|)^\rho \|f\|_{\mathbb{H}^m_\rho}$ *for all* $f \in \mathbb{H}^m_\rho$, $t \in \mathbb{R}$, $\mathbf{v} \in \mathbb{R}^d$.

Proof. The conclusion of the proposition can be reached by a direct calculation with the help of change of variables and an elementary inequality

$$\frac{1}{1 + |\mathbf{x} - \mathbf{v}t|^2} \leqslant \frac{(1 + |\mathbf{v}t|)^2}{1 + |\mathbf{x}|^2} \quad \text{for all } t \in \mathbb{R}, \ \ \mathbf{x}, \mathbf{v} \in \mathbb{R}^d. \qquad \square$$

Let

$$C_\mathbf{v}(t) := R_\mathbf{v}(t) S(t), \ \ t \geqslant 0. \tag{2.5}$$

The following result holds.

Proposition 4.

(i) $[S(t), R_\mathbf{v}(s)] := S(t) R_\mathbf{v}(s) - R_\mathbf{v}(s) S(t) = 0$ *for all* $t, s \geqslant 0$.

(ii) $C_\mathbf{v}(\cdot)$ *is a* C_0*-semigroup of operators on* $\mathbb{H}^m_\rho$ *for any* $\mathbf{v} \in \mathbb{R}^d$.

(iii) $\sup_{0 \leqslant t \leqslant T} \|C_\mathbf{v}(t) B\|_{L_{(HS)}(\mathbb{L}^2, \mathbb{H}^m_\rho)} < \infty$ *for all* $T > 0$, $\mathbf{v} \in \mathbb{R}^d$.

Proof. Notice that $\widehat{R_\mathbf{v}(s)\psi}(\mathbf{k}) = \mathrm{e}^{s\mathrm{i}\mathbf{k}\cdot\mathbf{v}} \hat{\psi}(\mathbf{k})$ for any $\psi \in \mathscr{S}_d$, hence (i) holds trivially in light of Proposition 2. Part (ii) is a consequence of (i). Part (iii) follows from the following computation:

$$\begin{aligned} &\sup_{0 \leqslant t \leqslant T} \|C_\mathbf{v}(t) B\|_{L_{(HS)}(\mathbb{L}^2, \mathbb{H}^m_\rho)} \\ &\quad \leqslant \sup_{0 \leqslant t \leqslant T} \|R_\mathbf{v}(t)\|_{L(\mathbb{H}^m_\rho)} \sup_{0 \leqslant t \leqslant T} \|S(t) B\|_{L_{(HS)}(\mathbb{L}^2, \mathbb{H}^m_\rho)} < \infty. \qquad \square \end{aligned}$$

Denote by $D_{\mathbf{v}}$ the generator of $C_{\mathbf{v}}(\cdot)$. A direct computation shows that

$$D_{\mathbf{v}} f = -Af + \mathbf{v}\cdot\nabla f, \quad f \in \mathscr{C}_\rho. \tag{2.6}$$

Additionally, $\mathscr{C}_\rho$ is a core of $D_{\mathbf{v}}$. This follows from the fact that $\mathscr{C}_\rho$ is invariant both under $R_{\mathbf{v}}(\cdot)$ and $S(\cdot)$, thus it is also invariant under $C_{\mathbf{v}}(\cdot)$.

2.2. The Ornstein–Uhlenbeck process

Let $(\Omega, \mathscr{V}, \mathbb{P})$ be a probability space. By $W(t) = \{W_j(t);\ j = 1,\dots,d\}$, $t \geqslant 0$ we denote a cylindrical Wiener process on $\mathbb{L}^2$ over the given probability space, that is an $\mathscr{S}_d' := \mathscr{S}'(\mathbb{R}^d;\mathbb{R}^d)$-valued Gaussian process satisfying

(A.1) for every $\psi \in \mathscr{S}_d$, $\{\langle W(t),\psi\rangle\}_{t\in[0,\infty)}$ is a one dimensional Wiener process. Here $\langle\cdot,\cdot\rangle$ denotes the pairing between $\mathscr{S}_d'$ and $\mathscr{S}_d$ spaces.

(A.2) for all $\psi, \varphi \in \mathscr{S}_d$ and $t, s \in [0,\infty)$ one has

$$\mathbb{E}[\langle W(t),\psi\rangle\langle W(s),\varphi\rangle] = t\wedge s \sum_{l=1}^{d}\int_{\mathbb{R}^d}\psi_l(\mathbf{x})\cdot\varphi_l(\mathbf{x})\,\mathrm{d}\mathbf{x}.$$

We shall denote by $(\mathscr{W}_t)_{t\geqslant 0}$ the natural filtration corresponding to $W(\cdot)$. By virtue of Proposition 2(iv) and the argument used in the proof of Proposition 4.15, p. 104 of Da Prato and Zabczyk (1992) we conclude that for any $f \in \mathbb{H}_\rho^m$ there exists a unique $\mathbb{H}_\rho^m$-valued strong solution V of (1.5) satisfying $V(0) = f$, that is $V(t) \in D(A)$ for $t > 0$ and

$$V(t) = f - \int_0^t AV(s)\,\mathrm{d}s + BW(t), \quad t \geqslant 0.$$

We denote this solution by V_f. From Theorem 5.4 of Da Prato and Zabczyk (1992) we have $V_f(t) = S(t)f + W_A(t)$, $t \geqslant 0$, with

$$W_A(t) := \int_0^t S(t-s)B\,\mathrm{d}W(s).$$

When $m > d/2$ we can define a Gaussian random field $W_A(t,\mathbf{x}) := W_A(t)(\mathbf{x})$, $(t,\mathbf{x}) \in [0,+\infty)\times\mathbb{R}^d$ over $(\Omega,\mathscr{V},\mathbb{P})$ whose co-variance matrix is given by

$$\begin{aligned} R(s,t,\mathbf{x}-\mathbf{y}) &:= \mathbb{E}[W_A(t,\mathbf{x})\otimes W_A(s,\mathbf{y})] \\ &= \int_{\mathbb{R}^d}\cos(\mathbf{k}\cdot(\mathbf{x}-\mathbf{y}))\mathrm{e}^{-|\mathbf{k}|^{2\beta}|t-s|}[1-\mathrm{e}^{-2|\mathbf{k}|^{2\beta}(s\wedge t)}]\bar{\mathscr{E}}(\mathbf{k})\,\mathrm{d}\mathbf{k} \end{aligned}$$

for any $(t,\mathbf{x}),(s,\mathbf{y}) \in [0,+\infty)\times\mathbb{R}^d$. Standard regularity results for Gaussian processes (see corollary to Theorem 3.4.1 of Adler, 1981) allow us to conclude that there exists a modification of $W_A(t,\mathbf{x})$, $(t,\mathbf{x}) \in [0,+\infty)\times\mathbb{R}^d$ that is jointly continuous and, thanks to (2.2), C^∞ in $\mathbf{x}$ for a fixed t.

Let $T, \kappa > 0$ be arbitrary. Define the random field

$$Y(t,\mathbf{x}) := W_A(t,\mathbf{x})(1+|\mathbf{x}|)^{-\kappa}, \quad (t,\mathbf{x}) \in \mathbb{R}_T, \tag{2.7}$$

where $\mathbb{R}_T := [0,T]\times\mathbb{R}^d$.

By a d-ball we mean a ball in $\mathbb{R}_T$ in the pseudo-metric

$$\mathrm{d}(t,\mathbf{x};s,\mathbf{y}) := [\mathbb{E}|Y(t,\mathbf{x}) - Y(s,\mathbf{y})|^2]^{1/2}, \quad (t,\mathbf{x}),(s,\mathbf{y}) \in \mathbb{R}_T.$$

The entropy number $N(\varepsilon)$, $\varepsilon > 0$ of $\mathbb{R}_T$ with respect to the field $Y(\cdot,\cdot)$ is the smallest number d-balls of radius at most $\varepsilon > 0$ needed to cover $\mathbb{R}_T$.

It can be seen that $N(\varepsilon) \leqslant K\varepsilon^{-k}$ for some constants $K, k > 0$ independent of $\varepsilon > 0$. Indeed, suppose that $\varepsilon > 0$ is chosen arbitrarily and set $B_R := [\mathbf{x} \in \mathbb{R}^d\colon |\mathbf{x}| \leqslant R]$. Note that, since $\kappa > 0$, one can choose a sufficiently large $R > 0$ to cover the outside of $[0,T] \times B_R$ with a single d-ball of radius $\varepsilon > 0$. Since the co-variance matrix of the field $Y(\cdot,\cdot)$ is Hölder regular the same is true about the function $D\colon \mathbb{R}_T \times \mathbb{R}_T \ni (t,\mathbf{x},s,\mathbf{y}) \mapsto \mathrm{d}(t,\mathbf{x};s,\mathbf{y}) \in [0,\infty)$. Therefore, in order to cover the remaining set $[0,T] \times B_R$ one needs to use at most $[\varepsilon^{-k}]$ d-balls, for some $k > 0$ adjusted accordingly to the Hölder exponent of the function D.

Thus, by virtue of Theorem 6.9.2, p. 161 of Adler (1990) there exist constants $c_1, c_2 > 0$ such that

$$\mathbb{P}\left(\sup_{(t,\mathbf{x}) \in \mathbb{R}_T} |Y(t,\mathbf{x})| > u\right) \leqslant c_1 \exp\{-c_2 u^2/(2\sigma^2)\} \quad \text{for all } u > 0,$$

where

$$\sigma^2 := \sup_{(t,\mathbf{x}) \in \mathbb{R}_T} \mathbb{E}|Y(t,\mathbf{x})|^2.$$

Therefore, for any $T > 0$ there exists a random constant C_T such that $\mathbb{P}$-a.s. $0 < C_T < \infty$ and

$$\sup_{0 \leqslant t \leqslant T} |W_A(t,\mathbf{x})| \leqslant C_T(1 + |\mathbf{x}|)^\kappa, \quad \forall\, \mathbf{x} \in \mathbb{R}^d.$$

2.3. *Invariant measure*

From (2.2), cf. the proof of Proposition 2, we conclude that

$$\int_0^\infty \|S(t)B\|^2_{L_{(HS)}(\mathbb{L}^2,\mathbb{H}^m_\rho)}\,\mathrm{d}t \leqslant C\int_{\mathbb{R}^d} (1 + |\mathbf{k}'|^2)^m \operatorname{Tr} \bar{\mathscr{E}}(\mathbf{k}')\,\mathrm{d}\mathbf{k}' < \infty.$$

Let μ be a Gaussian measure on $\mathbb{H}^m_\rho$ that is of zero mean and with the covariance given by QQ^*, where

$$Q := \int_0^\infty S(t)B\,\mathrm{d}t \in L_{(HS)}(\mathbb{L}^2,\ \mathbb{H}^m_\rho).$$

By Theorem 11.7, p. 308 of Da Prato and Zabczyk (1992) the measure is stationary for Eq. (1.5).

$G(\mathbf{x}; f) := f(\mathbf{x})$, $f \in \mathbb{H}^m_\rho$, $\mathbf{x} \in \mathbb{R}^d$ is a Gaussian, homogeneous random field over the probability space $(\mu, \mathbb{H}^m_\rho, \mathscr{B}(\mathbb{H}^m_\rho))$ with the co-variance matrix

$$R_G(\mathbf{x}) := \int f(\mathbf{x}) \otimes f(\mathbf{0})\mu(\mathrm{d}f) = \int_{\mathbb{R}^d} \cos(\mathbf{x}\cdot\mathbf{k})\bar{\mathscr{E}}(\mathbf{k})\,\mathrm{d}\mathbf{k}.$$

Proposition 5. $\mu(\mathscr{C}_\rho) = 1$.

Proof. With no loss of generality we can assume that $m > d/2$. Since R_G is C^∞, by virtue of (2.2), the realizations of the field are C^∞ μ-a.s. The conclusion of the proposition follows from Theorem 3.1, p. 168 of Qualls and Watanabe (1973) that implies

$$|f(\mathbf{x})| + \cdots + |\nabla_{\mathbf{x}}^m f(\mathbf{x})| \leqslant C(1 + |\log \mathbf{x}|)^{1/2}$$

and, in consequence, $\|f\|_{\mathbb{W}_\rho^{p,m}} < \infty$, $\forall m,\ p \geqslant 1$ for μ-a.s. f. Recall that $\rho \in (d/2, d/2 + \gamma(\beta))$. □

Note that $V(t) := V_f(t)$, $t \geqslant 0$ and $f \in \mathbb{H}_\rho^m$ is a stationary solution of (1.5) over the probability space $(\Omega \times \mathbb{H}_\rho^m, \mathscr{V} \otimes \mathscr{B}(\mathbb{H}_\rho^m), \mathbb{P} \otimes \mu)$, that is adapted with respect to the filtration $\mathscr{V}_t := \mathscr{W}_t \otimes \mathscr{B}(\mathbb{H}_\rho^m)$, $t \geqslant 0$. For $m > d/2$ we can define a stationary random field $V(t, \mathbf{x}) := V(t)(\mathbf{x})$ with the co-variance matrix R given by (1.2).

3. S.D.E. for the Lagrangian process

Suppose that $m > d/2 + 1$ so that $\mathbb{H}_\rho^m \subseteq C^1(\mathbb{R}^d, \mathbb{R}^d)$. We can define a d-dimensional random field $V_f(t, \mathbf{x}) := V_f(t)(\mathbf{x})$, $(t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^d$. Its mean and co-variance matrix equal correspondingly $F(t, \mathbf{x}) := S(t) f(\mathbf{x})$ and

$$R_f(t, s, \mathbf{x} - \mathbf{y}) = \int_{R^d} \cos[(\mathbf{x} - \mathbf{y}) \cdot \mathbf{k}]\, \mathrm{e}^{-|\mathbf{k}|^{2\beta}|t-s|}[1 - \mathrm{e}^{-|\mathbf{k}|^{2\beta}(t \wedge s)}]\bar{\mathscr{E}}(\mathbf{k})\, \mathrm{d}\mathbf{k}.$$

Let $f \in \mathscr{C}_\rho$. Then $f \in \mathbb{W}_\rho^{p,m}$ for arbitrary $p \geqslant 1$, $m > d/2 + 1$. In particular, when $p > 2\rho > d$ one can find $C_f > 0$ such that $|f(\mathbf{x})| \leqslant C_f(1 + |\mathbf{x}|^2)^{\rho/p}$ and in consequence the ordinary differential equation (1.1) with V_f standing in place of V on its right-hand side has a unique global solution $\boldsymbol{x}_f(\cdot)$. Let us set $\eta_f(t, \cdot) := V_f(t, \boldsymbol{x}_f(t) + \cdot)$, $t \geqslant 0$.

Theorem 1. *Let $f \in \mathscr{C}_\rho$. Then there exists an $(\mathscr{W}_t)$-adapted cylindrical Wiener process W_f on $\mathbb{L}^2$, defined on $(\Omega, \mathscr{V}, \mathbb{P})$ such that η_f is a strong solution (in the sense of* Da Prato and Zabczyk (1992, p. 118)) *of*

$$\begin{cases} \mathrm{d}\eta_f(t) = (-A\eta_f(t) + \eta_f(t, \mathbf{0}) \cdot \nabla_{\mathbf{x}} \eta_f(t))\, \mathrm{d}t + B\, \mathrm{d}W_f(t), & t \geqslant 0 \\ \eta_f(0) = f. \end{cases}$$

Proof. Let $\psi \in \mathscr{S}_d$. Since V_f is a strong solution to (1.5), using Itô's formula we obtain

$$\begin{aligned} \langle \eta_f(t), \psi \rangle = \int_{\mathbb{R}^d} V_f(t, \mathbf{x}) \cdot \psi(\mathbf{x} - \boldsymbol{x}_f(t))\, \mathrm{d}\mathbf{x} &= \int_{\mathbb{R}^d} V_f(0, \mathbf{x}) \cdot \psi(\mathbf{x})\, \mathrm{d}\mathbf{x} \\ &+ \int_{\mathbb{R}^d} \int_0^t \Big[-AV_f(s)(\mathbf{x}) \cdot \psi(\mathbf{x} - \boldsymbol{x}_f(s)) \\ &+ V_f(s, \mathbf{x}) \cdot \frac{\mathrm{d}}{\mathrm{d}s} \psi(\mathbf{x} - \boldsymbol{x}_f(s)) \Big]\, \mathrm{d}s\, \mathrm{d}\mathbf{x} \\ &+ \int_0^t \langle B\, \mathrm{d}W(s), \psi(\cdot - \boldsymbol{x}_f(s)) \rangle. \end{aligned} \tag{3.1}$$

The first two terms on the utmost right-hand side of (3.1) can be rewritten as

$$\langle \eta_f(0), \psi \rangle + \int_0^t [\langle -A\eta_f(s), \psi \rangle - \langle \eta_f(s), \eta_f(s, \mathbf{0}) \cdot \nabla \psi \rangle] \, \mathrm{d}s$$

$$= \langle \eta_f(0), \psi \rangle + \int_0^t \langle -A\eta_f(s) + \eta_f(t, \mathbf{0}) \cdot \nabla \eta_f(s), \psi \rangle \, \mathrm{d}s.$$

The proof will be complete as soon as we show that

$$\langle W_f(t), \psi \rangle := \int_0^t \langle \mathrm{d}W(s), \psi(\cdot - \boldsymbol{x}_f(s)) \rangle, \quad t \geqslant 0, \ \ \psi \in \mathscr{S}_d,$$

defines a $(\mathscr{W}_t)$-adapted cylindrical Wiener process W_f on $\mathbb{L}^2$. This follows from the fact that for any ψ, $\langle W_f(t), \psi \rangle$, $t \geqslant 0$ is an $(\mathscr{W}_t)$-adapted square integrable martingale and for all ψ, φ, the quadratic variation $\langle \langle W_f(\cdot), \psi \rangle \langle W_f(\cdot), \varphi \rangle \rangle_t$, $t \geqslant 0$ of $\langle W_f(t), \psi \rangle$, $\langle W_f(t), \varphi \rangle$, $t \geqslant 0$ is equal to

$$\int_0^t \int_{\mathbb{R}^d} \psi(\mathbf{x} - \boldsymbol{x}_f(s)) \cdot \varphi(\mathbf{x} - \boldsymbol{x}_f(s)) \, \mathrm{d}\mathbf{x} \, ds = \int_0^t \int_{\mathbb{R}^d} \psi(\mathbf{x}) \cdot \varphi(\mathbf{x}) \, \mathrm{d}\mathbf{x} \, ds = t \langle \psi, \varphi \rangle. \quad \square$$

4. Existence of strong solutions

The main result of the present section is the following theorem on the existence of a solution to Eq. (1.7).

Theorem 2. *Suppose that $m > d/2$. Let W be a cylindrical Wiener process on $\mathbb{L}^2$, and $f \in \mathbb{H}_\rho^{m+[2\beta]+2}$. Then, there exist a stopping time $\tau_f > 0$ w.r.t. the standard filtration corresponding to the Wiener process, $\eta_f(t)$, $0 \leqslant t \leqslant \tau_f$ a non-anticipative process such that $\eta_f(t) \in D(A) \cap D(\nabla)$, $0 \leqslant t < \tau_f$ and*

$$\eta_f(t) = f + \int_0^t (-A\eta_f(s) + \eta_f(s, \mathbf{0}) \cdot \nabla \eta_f(s)) \, \mathrm{d}s + BW(t), \quad 0 \leqslant t < \tau_f. \tag{4.1}$$

Additionally, one can take $\tau_f = +\infty$, $\mathbb{P}$-a.s. for any $f \in \mathscr{C}_\rho$.

Take $f \in \mathbb{H}_\rho^m$. For a given positive integer $n \geqslant 1$ we take a partition of $[0, +\infty)$ given by points k/n. We construct an approximating sequence of solutions $\eta_f^{(n)}$ in the following way. First, we solve the linear equation obtained from (1.7) by putting the coefficient by the gradient term equal to $f(0)$ and thus obtain

$$\eta_f^{(n)}(t) := C_{f(0)}(t) f + \int_0^t C_{f(0)}(t-s) B \, \mathrm{d}W(s) \quad \text{for } 0 \leqslant t \leqslant 1/n.$$

Here $C_{\mathbf{v}}$ is given by (2.5). We can continue the construction of the approximate solution on a given partition interval feeding in for the coefficient by the gradient term the final value of the solution (taken at $\mathbf{x} = \mathbf{0}$) at the preceding partition interval and solving obtained that way linear equation with the initial condition given by the final value of the approximate solution in the previous interval. More precisely, the process $\eta_f^{(n)}(t)$ is defined for $t \in [k/n, (k+1)/n]$ by

$$\eta_f^{(n)}(t) = C_{v_k^{(n)}}\left(t - \frac{k}{n}\right)\left[\eta_f^{(n)}\left(\frac{k}{n}\right)\right] + \int_{k/n}^{t} C_{v_k^{(n)}}(t-s)B\,\mathrm{d}W(s),$$

$v_k^{(n)} := \eta_f^{(n)}(k/n, \mathbf{0})$. We show in Lemmas 2 and 3 that if $f \in \mathbb{H}_\rho^{m+[2\beta]+2}$, then there exists a stopping time $\tau_f > 0$ such that the sequence $\eta_f^{(n)}$ converges $\mathbb{P}$-a.s., as $n \uparrow \infty$, in the space $C([0,\tau_f]; \mathbb{H}_\rho^m)$.

First, note that as $BW(t) \in \mathscr{H}_\rho$, $t \geqslant 0$ we have

$$\eta_f^{(n)}(t) = C_{v_k^{(n)}}\left(t - \frac{k}{n}\right)\left[\eta_f^{(n)}\left(\frac{k}{n}\right)\right] + B\left[W(t) - W\left(\frac{k}{n}\right)\right]$$
$$+ \int_{k/n}^{t} C_{v_k^{(n)}}(t-s)(-A + v_k^{(n)}\cdot\nabla)B\left[W(s) - W\left(\frac{k}{n}\right)\right]\mathrm{d}s,$$

$t \in [k/n, (k+1)/n]$. We can write therefore that

$$F_{k+1}^{(n)} = C_{v_k^{(n)}}\left(\frac{1}{n}\right)F_k^{(n)} + R_k^{(n)}, \tag{4.2}$$

where $F_k^{(n)} := \eta_f^{(n)}(k/n) - W_k^{(n)}$, $W_k^{(n)} := BW(k/n)$ and

$$R_k^{(n)} := C_{v_k^{(n)}}\left(\frac{1}{n}\right)W_k^{(n)} - W_k^{(n)}$$
$$+ \int_{k/n}^{(k+1)/n} C_{v_k^{(n)}}\left(\frac{k+1}{n} - s\right)D_{v_k^{(n)}}B\left[W(s) - W\left(\frac{k}{n}\right)\right]\mathrm{d}s, \tag{4.3}$$

with $D_{\mathbf{v}}$ given by (2.6). Iterating (4.2) we get

$$F_{k+1}^{(n)} = C_{s_{k,0}^{(n)}}\left(\frac{1}{n}\right)f + \sum_{p=1}^{k+1} S\left(\frac{k+1-p}{n}\right)R_{s_{k,p}^{(n)}}\left(\frac{1}{n}\right)R_{p-1}^{(n)}, \tag{4.4}$$

with

$$s_{k,p}^{(n)} := \begin{cases} v_k^{(n)} + \cdots + v_p^{(n)}, & p \leqslant k \\ \mathbf{0}, & p > k. \end{cases}$$

Write $f_k^{(n)} := \|F_k^{(n)}\|_{\mathbb{H}_\rho^m}$, $r_k^{(n)} := \|R_k^{(n)}\|_{\mathbb{H}_\rho^m}$. We shall need the following fact.

Lemma 1. *Let $T > 0$, and let τ be any stopping time satisfying $\tau \in (0, T]$, $\mathbb{P}$-a.s. Define $a := \sup_{0\leqslant t\leqslant\tau}[\|ABW(t)\|_{\mathbb{H}_\rho^m} + \|\nabla BW(t)\|_{\mathbb{H}_\rho^m}]$ and $M := \sup_{0\leqslant t\leqslant\tau}\|BW(t)\|_{\mathbb{H}_\rho^m} + 1$. Then, there exists a deterministic constant $c > 0$ independent of n such that*

$$r_k^{(n)} \leqslant \frac{ca}{n}(1 + f_k^{(n)} + M)\left(1 + \frac{f_k^{(n)} + M}{n}\right)^{\rho} \quad \text{for } k = 0, \ldots, [n\tau] - 1, \quad \mathbb{P}\text{-a.s.} \tag{4.5}$$

Proof. We have

$$\begin{aligned}
&\left\| C_{v_k^{(n)}}\left(\frac{1}{n}\right) W_k^{(n)} - W_k^{(n)} \right\|_{\mathbb{H}_\rho^m} \\
&\quad \leqslant \left\| R_{v_k^{(n)}}\left(\frac{1}{n}\right) S\left(\frac{1}{n}\right) W_k^{(n)} - S\left(\frac{1}{n}\right) W_k^{(n)} \right\|_{\mathbb{H}_\rho^m} + \left\| S\left(\frac{1}{n}\right) W_k^{(n)} - W_k^{(n)} \right\|_{\mathbb{H}_\rho^m} \\
&\quad \leqslant \left\| \int_0^{1/n} S\left(\frac{1}{n}\right) R_{v_k^{(n)}}(s)(v_k^{(n)} \cdot \nabla W_k^{(n)})\, \mathrm{d}s \right\|_{\mathbb{H}_\rho^m} + \left\| \int_0^{1/n} S(s) A W_k^{(n)}\, \mathrm{d}s \right\|_{\mathbb{H}_\rho^m}.
\end{aligned} \tag{4.6}$$

The Sobolev inequality stating that $|F_k^{(n)}(\mathbf{0})| \leqslant c_1 f_k^{(n)}$ yields

$$|v_k^{(n)}| = |F_k^{(n)}(\mathbf{0}) + W_k^{(n)}(\mathbf{0})| \leqslant c_2(f_k^{(n)} + M). \tag{4.7}$$

Thus, from Proposition 3 we conclude that the utmost left-hand side of (4.6) is less than or equal to

$$\frac{c_3 a}{n}(1 + f_k^{(n)} + M)\left(1 + \frac{f_k^{(n)} + M}{n}\right)^\rho. \tag{4.8}$$

An analogous estimate can be carried out for the norm of the second term on the right-hand side of (4.3), which yields that it is less than or equal to

$$\frac{c_4 a}{n}\left(1 + f_k^{(n)} + M\right)\left(1 + \frac{f_k^{(n)} + M}{n}\right)^\rho. \tag{4.9}$$

Combining both (4.8) and (4.9) we conclude (4.5). □

Lemma 2. *There exists a stopping time* $\tau_f > 0$ *such that* $\sup_{n \geqslant 1} \|\eta_f^{(n)}\|_{\mathbb{H}_\rho^m, \tau_f} < \infty$, $\mathbb{P}$*-a.s.*

Proof. Given $M_0, a_0 > 0$ write $\tau_{M_0} := \min\{t \geqslant 0\colon \|BW(t)\|_{\mathbb{H}_\rho^m} \geqslant M_0 - 1\} \wedge 1$, $\sigma_{a_0} := \min\{t \geqslant 0 : \|ABW(t)\|_{\mathbb{H}_\rho^m} + \|\nabla BW(t)\|_{\mathbb{H}_\rho^m} \geqslant a_0\} \wedge 1$. From identity (4.4) we obtain

$$f_{k+1}^{(n)} \leqslant \left\| C_{s_{k,0}^{(n)}}\left(\frac{1}{n}\right) f \right\|_{\mathbb{H}_\rho^m} + \sum_{p=1}^{k+1} \left\| S\left(\frac{k+1-p}{n}\right) R_{s_{k,p}^{(n)}}\left(\frac{1}{n}\right) R_{p-1}^{(n)} \right\|_{\mathbb{H}_\rho^m}. \tag{4.10}$$

Denoting $C_1 := \sup_{0 \leqslant t \leqslant 1} \|S(t)\|$ and subsequently applying Proposition 3 we conclude that the left-hand side of (4.10) is less than or equal to

$$C_1 \left[\left(1 + \frac{1}{n}|s_{k,0}^{(n)}|\right)^\rho \|f\|_{\mathbb{H}_\rho^m} + \sum_{p=1}^{k+1} \left(1 + \frac{1}{n}|s_{k,p}^{(n)}|\right)^\rho r_{p-1}^{(n)} \right].$$

Obviously, one has

$$\left(1 + \frac{1}{n}\sum_{p=q}^{k}(f_p^{(n)} + M_0)\right)^\rho \leqslant \prod_{p=q}^{k}\left(1 + \frac{f_p^{(n)} + M_0}{n}\right)^\rho, \quad \forall q = 0, \ldots, k.$$

Hence, by Lemma 1 and identity (4.7), there is a constant C such that for all k and n satisfying $k/n \leqslant \tau_{M_0} \wedge \sigma_{a_0}$ one has $\mathbb{P}$-a.s.,

$$f_{k+1}^{(n)} \leqslant C\|f\|_{\mathbb{H}_\rho^m} \prod_{p=0}^{k}\left(1+\frac{f_p^{(n)}+M_0}{n}\right)^{\rho}$$

$$+\frac{Ca_0}{n}\sum_{p=0}^{k}(1+f_p^{(n)}+M_0)\prod_{q=p}^{k}\left(1+\frac{f_q^{(n)}+M_0}{n}\right)^{\rho}$$

and consequently

$$f_{k+1}^{(n)} \leqslant C\left(\|f\|_{\mathbb{H}_\rho^m}(1+a_0)+\frac{a_0(k+1)(1+M_0)}{n}+\frac{a_0}{n}\sum_{p=1}^{k} f_p^{(n)}\right)$$

$$\times\prod_{p=0}^{k}\left(1+\frac{f_p^{(n)}+M_0}{n}\right)^{\rho}.$$

Let us first choose K such that

$$K>(C+1)(1+\|f\|_{\mathbb{H}_\rho^m}+M_0)[1+\|f\|_{\mathbb{H}_\rho^m}(1+a_0)+a_0(M_0+1)]$$

and then $S>0$ sufficiently small so that

$$K > C(1+\|f\|_{\mathbb{H}_\rho^m}+M_0)\mathrm{e}^{\rho(K+M_0)S}[\|f\|_{\mathbb{H}_\rho^m}(1+a_0)+a_0(M_0+1)$$

$$+a_0(1+M_0+K)S].$$

Then $f_k^{(n)} \leqslant K$ for all k-s satisfying $k/n \leqslant \tau_f := \tau_{M_0} \wedge \sigma_{a_0} \wedge S$. $\square$

Lemma 3. *Suppose $f \in \mathbb{H}_{\rho'}^{m'}$ where $m' > m + [2\beta] + 2$, $d/2 < \rho' < \rho$. Then there is a stopping time $\tau_f > 0$, $\mathbb{P}$-a.s., such that the sequence $(\eta_f^{(n)})_{n\geqslant 1}$ converges, as $n \uparrow \infty$ in $C([0,\tau_f];\mathbb{H}_\rho^m)$, $\mathbb{P}$-a.s.*

Proof. Let $\tau_f^{(1)}$ and $\tau_f^{(2)}$ be the stopping times constructed in Lemma 2 for f and the pairs (m,ρ) and (m',ρ'). Thus taking $\tau_f := \tau_f^{(1)} \wedge \tau_f^{(2)}$, we obtain

$$R := \sup_{\substack{1\leqslant n \\ k\leqslant [n\tau_f]-1}} (f_k^{(n)} + \|F_k^{(n)}\|_{\mathbb{H}_{\rho'}^{m'}}) < \infty. \tag{4.11}$$

We will show that τ_f has the desired properties. Clearly $\mathbb{P}(\tau_f > 0) = 1$. To see the convergence of $(\eta_f^{(n)})$ on $[0,\tau_f]$ note that from (4.2), see also the proof of (4.4),

$$\|F_{k+1}^{(n)} - F_k^{(n)}\|_{\mathbb{H}_\rho^m} \leqslant \left\| C_{v_k^{(n)}}\left(\frac{1}{n}\right) F_k^{(n)} - F_k^{(n)} \right\|_{\mathbb{H}_\rho^m} + r_k^{(n)}$$

$$\leqslant \left\| \int_0^{1/n} S\left(\frac{1}{n}\right) R_{v_k^{(n)}}(s)(v_k^{(n)} \cdot \nabla F_k^{(n)})\,\mathrm{d}s \right\|_{\mathbb{H}_\rho^m}$$

$$+ \left\| \int_0^{1/n} S(s) A F_k^{(n)}\,\mathrm{d}s \right\|_{\mathbb{H}_\rho^m} + r_k^{(n)}.$$

Thus, using arguments from the proof of Lemma 1, one can easily obtain

$$\|F_{k+1}^{(n)} - F_k^{(n)}\|_{\mathbb{H}_\rho^m} \leqslant \frac{Ca_0}{n}(1 + f_k^{(n)} + M_0)\left(1 + \frac{f_k^{(n)} + M_0}{n}\right)^\rho$$

$$\times(\|F_k^{(n)}\|_{\mathbb{H}_{\rho'}^{m'}} + 1), \quad \mathbb{P}\text{-a.s.}$$

for $k = 0, \ldots, [n\tau_f] - 1$. Thus, by (4.11), $\eta_f^{(n)}(\cdot) - BW(\cdot)$, $n \geqslant 1$ are uniformly continuous in $C([0,\tau_f];\mathbb{H}_\rho^m)$, and by Lemma 2 it is also bounded in $C([0,\tau_f];\mathbb{H}_{\rho'}^{m'})$. From Proposition 1, the ball $B(R) := [\psi \in \mathbb{H}_{\rho'}^{m'} : \|\psi\|_{\mathbb{H}_{\rho'}^{m'}} \leqslant R]$ is pre-compact in $\mathbb{H}_\rho^m$. By virtue of the infinite dimensional version of Arzela–Ascoli lemma, see e.g. Dieudonné (1969, Section 7.5), $(\eta_f^{(n)}(\cdot))_{n\geqslant 1}$ is pre-compact in $C([0,\tau_f];\mathbb{H}_\rho^m)$, $\mathbb{P}$-a.s.

We show that the set of limiting points of $(\eta_f^{(n)}(\cdot))_{n\geqslant 1}$ is a singleton $\mathbb{P}$-a.s. Suppose that $\eta(\cdot)$ and $\bar{\eta}(\cdot) = v(\cdot) + \eta(\cdot)$ are two limiting points of $(\eta_f^{(n)}(\cdot))_{n\geqslant 1}$. They both satisfy (4.1), therefore

$$\begin{cases} \dfrac{\mathrm{d}v(t)}{\mathrm{d}t} = -Av(t) + g(t)\cdot\nabla_{\mathbf{x}} v(t) + h(t)\cdot\nabla_{\mathbf{x}}\eta(t), \\ v(0) = \mathbf{0}, \end{cases} \tag{4.12}$$

where $g(s) := \eta(s,\mathbf{0}) + v(s,\mathbf{0})$, $h(s) = v(s,\mathbf{0})$ are continuous $\mathbb{R}^d$-valued functions.

Claim. *Any solution $v(\cdot)$ of (4.12) satisfies*

$$v(t,\mathbf{x}) = \int_0^t v(s,\mathbf{0})\cdot\Phi\left(t,s,\mathbf{x} + \int_s^t \eta(\tau,\mathbf{0})\,\mathrm{d}\tau + \int_s^t v(\tau,\mathbf{0})\,\mathrm{d}\tau\right)\mathrm{d}s, \tag{4.13}$$

for all $(t,\mathbf{x}) \in [\mathbf{0},+\infty)\times\mathbb{R}^d$. Here $\Phi(t,s,\mathbf{x}) := [\nabla_{\mathbf{x}} S(t-s)\eta](s,\mathbf{x})$, $t \geqslant s \geqslant 0$, $\mathbf{x} \in \mathbb{R}^d$ is a continuous function of all its arguments.

Accepting this claim for a moment we show that $v(t) \equiv \mathbf{0}$. Indeed, thanks to the fact that $\eta(\cdot) \in C([0,\tau_f];\mathbb{H}_\rho^m)$ and $v(\cdot,\mathbf{0}), \eta(\cdot,\mathbf{0}) \in C([0,\tau_f];\mathbb{R}^d)$ we conclude from (4.13) that there exists a constant $C > 0$ such that

$$|v(t,\mathbf{0})| \leqslant C\int_0^t |v(s,\mathbf{0})|\,\mathrm{d}s, \quad \forall t \in [0,\tau_f]$$

and $v(\cdot,\mathbf{0}) \equiv \mathbf{0}$ by the classical Gronwall inequality. This, in turn, implies via (4.13) that $v(\cdot) \equiv \mathbf{0}$.

Proof of (4.13). Denoting the right-hand side of (4.13) by $\bar{v}(\cdot)$ we obtain, after a direct computation, that

$$\begin{cases} \dfrac{\mathrm{d}\bar{v}(t)}{\mathrm{d}t} = -A\bar{v}(t) + g(t)\cdot\nabla_{\mathbf{x}}\bar{v}(t) + h(t)\cdot\nabla_{\mathbf{x}}\eta(t), \\ \bar{v}(0) = \mathbf{0}, \end{cases} \tag{4.14}$$

with $g(\cdot), h(\cdot)$ as in (4.12). Denoting by $w(\cdot) := \bar{v}(\cdot) - v(\cdot)$ we conclude that

$$\begin{cases} \dfrac{\mathrm{d}w(t)}{\mathrm{d}t} = -Aw(t) + g(t)\cdot\nabla_{\mathbf{x}}w(t), \\ w(0) = \mathbf{0}. \end{cases} \tag{4.15}$$

In consequence

$$w(t) = \int_0^t g(s)\cdot\nabla_{\mathbf{x}}[S(t-s)w(s)]\,\mathrm{d}s. \tag{4.16}$$

We show that $w(\cdot)\equiv 0$. Let ψ be such that $\hat{\psi}\in C_0^\infty(\mathbb{R}^d\setminus\{\mathbf{0}\};\mathbb{R}^d)$. From (4.16) we know that the Fourier transform $\hat{w}(t)$ of $w(t,\cdot)$ is a distribution satisfying

$$\hat{w}(t)(\hat{\psi}) = \int_0^t \hat{w}(s)(g(s,\cdot))\,\mathrm{d}s, \tag{4.17}$$

where $g(s,\mathbf{k}) := \mathrm{i}\mathrm{e}^{-|\mathbf{k}|^{2\beta}(t-s)}(g(s)\cdot\mathbf{k})\hat{\psi}(\mathbf{k})$. Iterating we obtain that the left-hand side of (4.17) equals $\int_0^t \hat{w}(s)(g_N(s,\cdot))\,\mathrm{d}s$, with

$$g_N(s,\mathbf{k}) := \mathrm{i}^{N+1}\int\cdots\int_{t\geqslant s_1\geqslant\cdots\geqslant s_N\geqslant s} \mathrm{e}^{-|\mathbf{k}|^{2\beta}(t-s)}(g(s)\cdot\mathbf{k})(g(s_1)\cdot\mathbf{k})\cdots(g(s_N)\cdot\mathbf{k}) \times\hat{\psi}(\mathbf{k})\,\mathrm{d}s_1\cdots\mathrm{d}s_N$$

for an arbitrary integer $N\geqslant 1$. A direct calculation shows that

$$\sup_{0\leqslant s\leqslant t,\ \mathbf{k}\in\mathbb{R}^d} |\nabla_{\mathbf{k}}^m g_N(s,\mathbf{k})| \leqslant \frac{C^N}{N!}$$

with the constant C independent of N. Hence $g_N(s,\cdot)$ tends to 0 in the space $\mathscr{D}(\mathbb{R}^d;\mathbb{R}^d)$, as $N\uparrow\infty$, and therefore $\hat{w}(t)(\hat{\psi}) = 0$ for all $\hat{\psi}\in C_0^\infty(\mathbb{R}^d\setminus\{\mathbf{0}\};\mathbb{R}^d)$. When β is an integer (4.17) extends by the foregoing argument to any ψ such that $\hat{\psi}\in C_0^\infty(\mathbb{R}^d;\mathbb{R}^d)$. That implies $w(t,\cdot)\equiv\mathbf{0}$.

Suppose that β is a non-integer. The above argument shows $\operatorname{supp}\hat{w}(t,\cdot) = \{\mathbf{0}\}$ and in consequence, see e.g. Hörmander (1983, Theorem 2.3.4, pp. 46–47), we have $w(t,\mathbf{x}) = \sum_p c_p(t)\mathbf{x}^p$, where the summation extends over a certain finite set of multi-indices $p = (p_1,\ldots,p_d)\in\mathbb{Z}^d$. Suppose now that $p_* = (p_{*1},\ldots,p_{*d})$ is the multi-index with the largest norm $|p_*| := \sum p_{*i}$. We apply $\partial_{x_1}^{p_{*1}}\cdots\partial_{x_d}^{p_{*d}}$ to both sides of (4.14) and obtain that

$$p_*!c_{p_*}(t) = \int_0^t g(s)\cdot S(t-s)[\nabla_{\mathbf{x}}\partial_{x_1}^{p_{*1}}\cdots\partial_{x_d}^{p_{*d}}w(s)] = 0,$$

so that $c_p(t)\equiv\mathbf{0}$ for all p and $t\geqslant 0$. □

Proof of Theorem 2. Let $\eta_f(t)$, $0 \leqslant t \leqslant \tau_f$ be the limit of the sequence $(\eta_f^{(n)})$. It is a non-anticipative stochastic process with respect to the natural filtration corresponding to the Wiener process $W(\cdot)$. In view of Proposition 2(iii) we have $\eta_f(t) \in D(A) \cap D(\nabla)$, $0 \leqslant t \leqslant \tau_f$. The fact that $\eta_f(t)$, $0 \leqslant t \leqslant \tau_f$ satisfies (4.1) is obvious.

We prove now the global existence for any $f \in \mathscr{C}_\rho$. Let τ_∞ be a possible blow-up time of a solution. It is obviously a stopping time. Let us set

$$\boldsymbol{x}_f(t) := \int_0^t \eta_f(s, \mathbf{0})\,\mathrm{d}s \quad \text{and} \quad V_f(t, \mathbf{x}) := \eta_f(t, \mathbf{x} - \boldsymbol{x}_f(t)), \quad 0 \leqslant t < \tau_\infty.$$

$\boldsymbol{x}_f(t)$, $0 \leqslant t < \tau_\infty$ solves the initial value problem

$$\mathrm{d}\boldsymbol{x}_f(t)/\mathrm{d}t = V_f(t, \boldsymbol{x}_f(t)) \quad \text{and} \quad \boldsymbol{x}_f(0) = \mathbf{0}.$$

Using the Itô formula, exactly in the same fashion as we did to derive (1.6), we check that

$$V_f(t) = f - \int_0^t A V_f(s)\,\mathrm{d}s + B\tilde{W}_f(t), \quad 0 \leqslant t < \tau_\infty$$

with $\tilde{W}_f(t)$, $t \geqslant 0$ some cylindrical Brownian motion on $\mathbb{L}^2$. Notice also that by virtue of Proposition 2(iii) and the arguments made at the end of Section 2.2, $\sup_{0 \leqslant t \leqslant T} |V_f(t, \mathbf{x})|$ has sub-linear growth in $\mathbf{x}$, $\mathbb{P}$-a.s. for any $T > 0$. In consequence

$$C(\omega) := \sup_{0 \leqslant t < \tau_\infty} |\boldsymbol{x}_f(t)| < \infty.$$

Thus, by Proposition 3,

$$\begin{aligned}\sup_{0 \leqslant t < \tau_\infty} \|\eta_f(t)\|_{\mathbb{H}^m_\rho} &= \sup_{0 \leqslant t < \tau_\infty} \|V_f(t, \boldsymbol{x}_f(t) + \cdot)\|_{\mathbb{H}^m_\rho} \\ &\leqslant (1 + C(\omega))^\rho \sup_{0 \leqslant t < \tau_\infty} \|V_f(t)\|_{\mathbb{H}^m_\rho} < \infty,\end{aligned}$$

which contradicts the fact that τ_∞ is the blow-up time. □

5. Uniqueness

In this section we prove the following.

Theorem 3. *Suppose that $m > d/2$. Then, for any $f \in \mathbb{H}^m_\rho$, Eq.* (1.7) *possesses at most one* (*maximal in time*) *strong solution.*

In the first step of the proof we will show that any solution of (1.7) satisfies a certain integral equation that we call its *mild formulation*. This is the content of the following.

Lemma 4. *Suppose that $\tau > 0$ is a stopping time, $\eta(t) \in \mathbb{H}^m_\rho$, $t \in [0, \tau)$ is a local solution of* (1.7) *with the given cylindrical Brownian motion W on $\mathbb{L}^2$ and m is as* *in Theorem* 3. *Then*,

$$\eta(t,\mathbf{x}) = S(t)f\left(\mathbf{x} + \int_0^t \eta(s,\mathbf{0})\,\mathrm{d}s\right) + \bar{W}_A\left(t, \mathbf{x} + \int_0^t \eta(s,\mathbf{0})\,\mathrm{d}s\right), \quad t \geqslant 0, \qquad (5.1)$$

where $\bar{W}_A(t,\mathbf{x})$, $(t,\mathbf{x}) \in [0,+\infty)\times\mathbb{R}^d$ *is a Gaussian field given by*

$$\bar{W}_A(t) := \int_0^t S(t-s)B\,\mathrm{d}\bar{W}(s)$$

and $\bar{W}(t)$, $t \geqslant 0$ *is a cylindrical Wiener process on* $\mathbb{L}^2$ *given by*

$$\langle \bar{W}(t), \psi\rangle = \int_0^t \left\langle \mathrm{d}W(s), \psi\left(\cdot + \int_0^s \eta(u,\mathbf{0})\,\mathrm{d}u\right)\right\rangle, \quad t \geqslant 0, \ \psi \in \mathscr{S}_d.$$

Proof. Note that for any predictable Itô's integrable process ξ one has

$$\int_0^t \langle \mathrm{d}\bar{W}(s), \xi(s)\rangle = \int_0^t \left\langle \mathrm{d}W(s), \xi\left(s, \cdot + \int_0^s \eta(u,\mathbf{0})\,\mathrm{d}u\right)\right\rangle, \quad t \geqslant 0.$$

Thus, using the arguments from the proof of Theorem 1, one can show that the right-hand side $\bar{\eta}$ of (5.1) satisfies

$$\mathrm{d}\bar{\eta}(t) = (-A\bar{\eta}(t) + \eta(t,\mathbf{0})\cdot\nabla_{\mathbf{x}}\bar{\eta}(t))\,\mathrm{d}t + B\,\mathrm{d}W(t).$$

Hence $w(t) := \bar{\eta}(t) - \eta(t)$ satisfies (4.15), with $g(t) := \eta(t,\mathbf{0})$. We can use the argument presented there to demonstrate that $w(\cdot) \equiv \mathbf{0}$. □

Proof of Theorem 3. Taking $\mathbf{x} = 0$ in (5.1) we obtain

$$\eta(t,\mathbf{0}) = F\left(t, \int_0^t \eta(s,\mathbf{0})\,\mathrm{d}s\right) + B\bar{W}(t,\mathbf{0}) + \int_0^t G\left(t, s, \int_s^t \eta(u,\mathbf{0})\,\mathrm{d}u\right)\mathrm{d}s, \qquad (5.2)$$

with $F(t,\mathbf{x}) := S(t)f(\mathbf{x})$ a certain Lipschitz function in $\mathbf{x}$ variable and $G(t,s,\mathbf{x}) := AS(t-s)B\bar{W}(s,\mathbf{x})$—a Gaussian random field with continuous realizations that are C^∞ smooth in $\mathbf{x}$, $\mathbb{P}$-a.s. (thanks to (2.2)). Eq. (5.2) and thus also (5.1) admits a unique local solution. Hence, the desired conclusion follows from Lemma 4. □

Directly from the above result and the global existence part of Theorem 2 we conclude the following.

Corollary 1. *Suppose that* m *is as in the statement of Theorem* 3. *Then, for any* $f \in \mathscr{C}_\rho$ *there exists a unique strong global solution* η_f of (1.7) *satisfying* $\eta_f(0) = f$.

6. Lagrangian process for irregular fields

In this section we suppose that a Gaussian random field $V(t,\mathbf{x})$, $(t,\mathbf{x}) \in \mathbb{R}\times\mathbb{R}^d$ whose co-variance matrix is given by (1.2) satisfies

(C) $k_0 := \mathrm{dist}(\mathbf{0}, \mathrm{supp}\mathscr{E}) > 0$ and there exists $C > 0$ such that $\mathrm{Tr}\,\mathscr{E}(\mathbf{k}) \leqslant C|\mathbf{k}|^{1-2\alpha}$ for all $|\mathbf{k}| \geqslant k_0$ and $\alpha \in (1,2)$.

The class of fields satisfying (C) includes fields which are Hölder but not necessarily Lipschitz in the spatial variable. As we have pointed out in the introduction it may be impossible therefore to solve uniquely the equation (1.1). This difficulty prevents us from defining the Lagrangian dynamics directly as it was done in Komorowski (2000). Below we construct such a dynamics as the limit of Lagrangian processes $\eta_K(t) := V_K(t, \boldsymbol{x}_K(t) + \cdot)$, $t \geqslant 0$, as $K \uparrow \infty$. Each η_K corresponds to V_K whose spectrum is given by (1.9). Smoothness of V_K guarantees the existence and uniqueness of solutions of

$$\mathrm{d}\boldsymbol{x}_K(t)/\mathrm{d}t = V_K(t, \boldsymbol{x}_K(t)), \qquad \boldsymbol{x}_K(0) = \mathbf{0}. \tag{6.1}$$

By $\eta_K(t) := V_K(t, \boldsymbol{x}_K(t) + \cdot)$, $t \geqslant 0$ we denote the corresponding Lagrangian processes and by Q_K their respective laws on $\mathscr{M} := C([0, +\infty); \mathscr{X})$. Throughout this section we shall denote $\mathscr{X} := C(\mathbb{R}^d; \mathbb{R}^d)$. Suppose now that $V_K(\cdot)$ is the stationary solution of the Ornstein–Uhlenbeck equation (1.5) with A defined by (2.1). B_K is also defined with the help of (2.1) but with $\mathscr{E}_K(\mathbf{k}) := \mathbf{1}_{[0,K]}(|\mathbf{k}|)\mathscr{E}(\mathbf{k})$, $\mathbf{k} \in \mathbb{R}^d$ used in place of $\mathscr{E}(\cdot)$. Let μ_K be the law of $V_K(0)$ on $\mathscr{X}$. It is Gaussian with the co-variance matrix

$$\int f(\mathbf{x}) \otimes f(\mathbf{y}) \mu_K(\mathrm{d}f) = \Gamma_K(\mathbf{x} - \mathbf{y}),$$

where

$$\Gamma_K(\mathbf{x}) := \int_{\mathbb{R}^d} \cos(\mathbf{x} \cdot \mathbf{k}) \frac{\mathscr{E}_K(\mathbf{k})}{|\mathbf{k}|^{d-1}} \, \mathrm{d}\mathbf{k}.$$

As we already know Q_K is identical with the law on $\mathscr{M}$ of the solution $\bar{\eta}_K(\cdot)$ of Eq. (4.1) with operators A, B_K described above. According to Lemma 4, $\bar{\eta}_K$ satisfies

$$\bar{\eta}_K(t, \mathbf{x}) = S(t) f\left(\mathbf{x} + \int_0^t \bar{\eta}_K(s, \mathbf{0}) \, \mathrm{d}s\right) + \bar{W}_{A,K}\left(t, \mathbf{x} + \int_0^t \bar{\eta}_K(s, \mathbf{0}) \, \mathrm{d}s\right), \tag{6.2}$$

where

$$\bar{W}_{A,K}(t) = \int_0^t S(t-s) B_K \, \mathrm{d}\bar{W}(s)$$

and

$$\langle \bar{W}(t), \psi \rangle = \int_0^t \left\langle \mathrm{d}W(s), \psi\left(\cdot + \int_0^s \bar{\eta}_K(u, \mathbf{0}) \, \mathrm{d}u\right)\right\rangle, \qquad t \geqslant 0, \psi \in \mathscr{S}_d.$$

Define $U(t, \mathbf{x}) := W(t, \mathbf{x} - \int_0^t \bar{\eta}_K(u, \mathbf{0}) \, \mathrm{d}u)$. Then, the integration by parts formula yields

$$\begin{aligned} &\int_0^t \langle \mathrm{d}U(s), \psi \rangle \\ &\quad = \int_0^t \left\langle \mathrm{d}W(s), \psi\left(\cdot + \int_0^s \bar{\eta}_K(u, \mathbf{0}) \, \mathrm{d}u\right)\right\rangle \\ &\qquad + \int_0^t \left\langle W(s), \bar{\eta}_K(s, \mathbf{0}) \cdot \nabla_{\mathbf{x}} \psi\left(\cdot + \int_0^s \bar{\eta}_K(u, \mathbf{0}) \, \mathrm{d}u\right)\right\rangle \mathrm{d}s, \quad t \geqslant 0, \quad \psi \in \mathscr{S}_d. \end{aligned} \tag{6.3}$$

Using (6.3) we can rewrite (6.2) in the form

$$\bar{\eta}_K(t,\mathbf{x}) = S(t)f\left(\mathbf{x} + \int_0^t \bar{\eta}_K(u,\mathbf{0})\,\mathrm{d}u\right) + U_{A,K}\left(t,\mathbf{x} + \int_0^t \bar{\eta}_K(u,\mathbf{0})\,\mathrm{d}u\right)$$

$$+ \int_0^t \bar{\eta}(s,\mathbf{0})\cdot\nabla_{\mathbf{x}} S(t-s)B_K W\left(s,\mathbf{x} + \int_s^t \bar{\eta}_K(u,\mathbf{0})\,\mathrm{d}u\right)\,\mathrm{d}s, \tag{6.4}$$

where

$$U_{A,K}(t) = \int_0^t S(t-s)B_K\,\mathrm{d}U(s)$$

$$= S(t)B_K U(t) + \int_0^t AS(t-s)[B_K U(t) - B_K U(s)]\,\mathrm{d}s.$$

Denoting

$$F(t,\mathbf{x}) := S(t)f(\mathbf{x}) \tag{6.5}$$

a Gaussian random field over $(\mathscr{X},\mathscr{B}(\mathscr{X}),\mu_K)$, $G_K(t,\mathbf{x}) := S(t)B_K W(t,\mathbf{x})$ and $H_K(t,s,\mathbf{x}) := AS(t-s)B_K W(t,\mathbf{x})$, $I_K(t,s,\mathbf{x}) := AS(t-s)B_K W(s,\mathbf{x})$, $J_K(t,s,\mathbf{x}) := \nabla_{\mathbf{x}} S(t-s)B_K W(s,\mathbf{x})$—Gaussian fields over $(\Omega,\mathscr{V},\mathbb{P})$ we can rewrite (6.4) in the form

$$\bar{\eta}_K(t,\mathbf{x}) = F\left(t,\mathbf{x} + \int_0^t \bar{\eta}_K(s,\mathbf{0})\,\mathrm{d}s\right) + G_K(t,\mathbf{x})$$

$$+ \int_0^t \left[H_K(t,s,\mathbf{x}) - I_K\left(t,s,\mathbf{x} + \int_s^t \bar{\eta}_K(u,\mathbf{0})\,\mathrm{d}u\right)\right]\,\mathrm{d}s$$

$$+ \int_0^t \bar{\eta}_K(s,\mathbf{0})\cdot J_K\left(t,s,\mathbf{x} + \int_s^t \bar{\eta}_K(u,\mathbf{0})\,\mathrm{d}u\right)\,\mathrm{d}s. \tag{6.6}$$

A direct calculation shows that

$$\int |F(t,\mathbf{x})|^2\,\mathrm{d}\mu_K + \mathbb{E}\,|G_K(t,\mathbf{x})|^2$$

$$= \int_{\mathbb{R}^d} \mathrm{e}^{-2|\mathbf{k}|^{2\beta}t}(1 + |\mathbf{k}|^{2\beta}t)\frac{\mathrm{Tr}\,\mathscr{E}_K(\mathbf{k})}{|\mathbf{k}|^{d-1}}\,\mathrm{d}\mathbf{k} \leqslant C, \quad \forall K \geqslant 1$$

and

$$\mathbb{E}[|H_K(t,s,\mathbf{x})|^2 + |I_K(t,s,\mathbf{x}+(t-s)\mathbf{y})|^2 + |J_K(t,s,\mathbf{x}+(t-s)\mathbf{y})|^2] \leqslant C(t-s)^{-r}$$

for some $r>0$ and all $\mathbf{x},\mathbf{y}\in\mathbb{R}^d$. Skorohod's embedding theorem (see e.g. Theorem 2.4, p. 33 of Da Prato and Zabczyk (1992)) implies that there exist random fields

$$\mathscr{Z}_K(t,s,\mathbf{x},\mathbf{y}) := (\hat{F}_K(t,\mathbf{x}),\hat{G}_K(t,\mathbf{x}),\hat{H}_K(t,s,\mathbf{x}),\hat{I}_K(t,s,\mathbf{x}+(t-s)\mathbf{y}),$$

$$J_K(t,s,\mathbf{x}+(t-s)\mathbf{y})),$$

$$(t,s,\mathbf{x},\mathbf{y}) \in D := [(t,s,\mathbf{x},\mathbf{y}) : t > s \geqslant 0, \mathbf{x},\mathbf{y}\in\mathbb{R}^d]$$

defined over certain probability space, that with no loss of generality will be assumed to coincide with $(\Omega,\mathscr{V},\mathbb{P})$, whose laws coincide with those of respective (F,G_K,H_K,I_K,J_K)

over $(\mathscr{X}\times\Omega,\mathscr{B}(\mathscr{X})\otimes\mathscr{V},\mu_K\otimes\mathbb{P})$. These fields converge uniformly on compact subsets of D, as $K\uparrow\infty$, to a certain Gaussian field on D,

$$\mathscr{Z}(t,s,\mathbf{x},\mathbf{y})$$
$$:=(\hat F(t,\mathbf{x}),\hat G(t,\mathbf{x}),\hat H(t,s,\mathbf{x}),\hat I(t,s,\mathbf{x}+(t-s)\mathbf{y}),\hat J(t,s,\mathbf{x}+(t-s)\mathbf{y})),$$

whose law coincides with that of

$$(F(t,\mathbf{x}),G(t,\mathbf{x}),H(t,s,\mathbf{x}),I(t,s,\mathbf{x}+(t-s)\mathbf{y}),J(t,s,\mathbf{x}+(t-s)\mathbf{y})),$$

F being given by (6.5), a Gaussian random field over $(\mathscr{X},\mathscr{B}(\mathscr{X}),\mu)$, $G(t,\mathbf{x}):=S(t)BW(t,\mathbf{x})$,

$$H(t,s,\mathbf{x}):=AS(t-s)BW(t,\mathbf{x}),\qquad I(t,s,\mathbf{x})=AS(t-s)BW(s,\mathbf{x})$$

and

$$J(t,s,\mathbf{x}):=\nabla_{\mathbf{x}}S(t-s)B_KW(s,\mathbf{x}).$$

In what follows we suppress writing the hat sign over the relevant random fields.

We consider the equation

$$\begin{aligned}\bar\eta(t,\mathbf{x})&=F\left(t,\mathbf{x}+\int_0^t\bar\eta(s,\mathbf{0})\,\mathrm{d}s\right)+G(t,\mathbf{x})\\&\quad+\int_0^t\left[H(t,s,\mathbf{x})-I\left(t,s,\mathbf{x}+\int_s^t\bar\eta(u,\mathbf{0})\,\mathrm{d}u\right)\right]\mathrm{d}s\\&\quad+\int_0^t\bar\eta(s,\mathbf{0})\cdot J\left(t,s,\mathbf{x}+\int_s^t\bar\eta(u,\mathbf{0})\,\mathrm{d}u\right)\mathrm{d}s.\end{aligned}\tag{6.7}$$

We say that $\bar\eta(\cdot,\cdot)$ is a solution of the equation if $\bar\eta(\cdot,\cdot;\omega)\in\mathscr{M}$, $\mathbb{P}$-a.s. and (6.7) holds for all $\mathbf{x}\in\mathbb{R}^d$.

Theorem 4. *Suppose $\alpha\in(1,2)$, $\beta\geqslant0$ are such that $\alpha+\beta>2$, $\alpha+3\beta>3$. In addition, assume that condition (C) holds. Then (6.7) has a unique global solution $\bar\eta$. Moreover, if $(\bar\eta_K)$ are the solutions of (6.6) corresponding to the approximating fields constructed via the Skorohod embedding theorem, then for all T, $R>0$,*

$$\lim_{K\uparrow\infty}\sup_{0\leqslant t\leqslant T,|\mathbf{x}|\leqslant R}|\bar\eta_K(t,\mathbf{x})-\bar\eta(t,\mathbf{x})|=0,\qquad\mathbb{P}\text{-}a.s.\tag{6.8}$$

From the above theorem we conclude immediately the following.

Corollary 2. *The measures Q_K weakly converge over $(\mathscr{M},\mathscr{B}(\mathscr{M}))$, as $K\uparrow\infty$.*

The proof of Theorem 4. We start with the following lemma.

Lemma 5. *Under the assumptions of Theorem 4 the integro-differential equation*

$$\begin{aligned}\boldsymbol{x}'(t)&=F(t,\boldsymbol{x}(t))+G(t,\mathbf{0})+\int_0^t[H(t,s,\mathbf{0})-I(t,s,\boldsymbol{x}(t)-\boldsymbol{x}(s))\\&\quad+\boldsymbol{x}'(s)\cdot J(t,s,\boldsymbol{x}(t)-\boldsymbol{x}(s))]\,\mathrm{d}s,\quad t>0,\quad \boldsymbol{x}(0)=\mathbf{0}\end{aligned}\tag{6.9}$$

has a unique global solution $\mathbb{P}$-a.s.

Proof. *Uniqueness*. Suppose that $\boldsymbol{x}$ and $\bar{\boldsymbol{x}}$ are two solution of (6.9) and $T>0$ is such that $\sup_{0\leqslant t\leqslant T}[|\boldsymbol{x}(t)|+|\bar{\boldsymbol{x}}(t)|]=R<\infty$. Hence,

$$\begin{aligned}\boldsymbol{x}'(t)-\bar{\boldsymbol{x}}'(t)&=F(t,\boldsymbol{x}(t))-F(t,\bar{\boldsymbol{x}}(t))\\&\quad+\int_0^t[I(t,s,\bar{\boldsymbol{x}}(t)-\bar{\boldsymbol{x}}(s))-I(t,s,\boldsymbol{x}(t)-\boldsymbol{x}(s))]\,\mathrm{d}s\\&\quad+\int_0^t[\boldsymbol{x}'(s)\cdot J(t,s,\boldsymbol{x}(t)-\boldsymbol{x}(s))-\bar{\boldsymbol{x}}'(s)\cdot J(t,s,\bar{\boldsymbol{x}}(t)-\bar{\boldsymbol{x}}(s))]\,\mathrm{d}s.\end{aligned}\tag{6.10}$$

Notice that

$$\mathbb{E}\,|\nabla_{\mathbf{x}}F(t,\mathbf{x})|^2\leqslant C\int_{k_0}^{+\infty}\mathrm{e}^{-2k^{2\beta}t}k^{3-2\alpha}\,\mathrm{d}k\tag{6.11}$$

for some generic constant independent of t. Substituting $\xi:=k\,t^{1/(2\beta)}$ we obtain that the right-hand side of (6.11) equals

$$C\,t^{(\alpha-2)/\beta}\int_{k_0t^{1/(2\beta)}}^{+\infty}\mathrm{e}^{-2\xi^{2\beta}}\xi^{3-2\alpha}\,\mathrm{d}\xi\leqslant C\,t^{(\alpha-2)/\beta}.$$

We have also

$$\mathbb{E}|\nabla_{\mathbf{x}}I(t,s,\mathbf{x})|^2\leqslant C\int_{k_0}^{+\infty}\mathrm{e}^{-2k^{2\beta}(t-s)}k^{3+6\beta-2\alpha}\,\mathrm{d}k\tag{6.12}$$

and, after a substitution

$$\xi:=k(t-s)^{1/(2\beta)}\tag{6.13}$$

we obtain that the right-hand side of (6.12) is less than or equal to $C(t-s)^{(\alpha-3\beta-2)/\beta}$.

Finally, we get

$$\mathbb{E}|J(t,s,\mathbf{x})|^2\leqslant C\int_{k_0}^{+\infty}\mathrm{e}^{-2k^{2\beta}(t-s)}k^{3+2\beta-2\alpha}\,\mathrm{d}k\tag{6.14}$$

and

$$\mathbb{E}|\nabla_{\mathbf{x}}J(t,s,\mathbf{x})|^2\leqslant C\int_{k_0}^{+\infty}\mathrm{e}^{-2k^{2\beta}(t-s)}k^{5+2\beta-2\alpha}\,\mathrm{d}k,\tag{6.15}$$

which, after substitution (6.13), yield that the right-hand sides of (6.14) and (6.15) are less than or equal to $C(t-s)^{(\alpha-\beta-2)/\beta}$ and $C(t-s)^{(\alpha-\beta-3)/\beta}$ correspondingly. Hence by virtue of Theorem 6.9.2, p. 161 of Adler (1990), for any $\gamma>1$ one can find a random variable $C_{R,\gamma}$ such that

$$\begin{aligned}&\sup_{|\mathbf{x}|\leqslant R}|\nabla_{\mathbf{x}}F(t,\mathbf{x})|\leqslant C_{R,\gamma}\,t^{\gamma(\alpha-2)/(2\beta)},\\&\sup_{|\mathbf{x}|\leqslant R}|\nabla_{\mathbf{x}}I(t,s,\mathbf{x})|\leqslant C_{R,\gamma}\,(t-s)^{\gamma(\alpha-3\beta-2)/(2\beta)},\\&\sup_{|\mathbf{x}|\leqslant R}|J(t,s,\mathbf{x})|\leqslant C_{R,\gamma}\,(t-s)^{\gamma(\alpha-\beta-2)/(2\beta)},\\&\sup_{|\mathbf{x}|\leqslant R}|\nabla_{\mathbf{x}}J(t,s,\mathbf{x})|\leqslant C_{R,\gamma}\,(t-s)^{\gamma(\alpha-\beta-3)/(2\beta)}.\end{aligned}\tag{6.16}$$

Note that since $\alpha<2$ all the right-hand sides of (6.16) become unbounded when $t\to s+$.

Denoting $v(t):=|\mathbf{x}'(t)-\bar{\mathbf{x}}'(t)|$ we obtain from (6.10) and (6.16),

$$v(t)\leqslant C_{R,\gamma}\left\{\int_0^t s^{\gamma(\alpha-2)/(2\beta)}v(s)\,\mathrm{d}s+\int_0^t (t-s)^{\gamma(\alpha-3\beta-2)/(2\beta)}\,\mathrm{d}s\int_s^t v(u)\,\mathrm{d}u\right.$$
$$\left.+\int_0^t (t-s)^{\gamma(\alpha-\beta-2)/(2\beta)}v(s)\,\mathrm{d}s+\int_0^t (t-s)^{\gamma(\alpha-\beta-3)/(2\beta)}\,\mathrm{d}s\int_s^t v(u)\,\mathrm{d}u\right\}. \tag{6.17}$$

Suppose that $\gamma>1$ is sufficiently small so that $\gamma(\alpha-3\beta-2)+2\beta<0$. Interchanging the order of integration in the second and fourth terms on the right-hand side of (6.17) we obtain that

$$v(t)\leqslant C_{R,\gamma}\left\{\int_0^t s^{\gamma(\alpha-2)/(2\beta)}v(s)\,\mathrm{d}s+\int_0^t (t-s)^{[\gamma(\alpha-3\beta-2)+2\beta]/(2\beta)}v(s)\,\mathrm{d}s\right.$$
$$\left.+\int_0^t (t-s)^{\gamma(\alpha-\beta-2)/(2\beta)}v(s)\,\mathrm{d}s+\int_0^t K(t,s)v(s)\,\mathrm{d}s.\right\}.$$

Here

$$K(t,s):=\begin{cases}(t-s)^{[\gamma(\alpha-\beta-3)+2\beta]/(2\beta)}, & \text{if } \gamma(\alpha-\beta-3)+2\beta<0\\ t^{[\gamma(\alpha-\beta-3)+2\beta]/(2\beta)}, & \text{if otherwise.}\end{cases}$$

We can apply Gronwall's inequality, provided the kernels in the above integrals are integrable. This happens when $\gamma>1$ is sufficiently small and $\alpha+\beta>2$, $\alpha+3\beta>3$. We obtain therefore $v(\cdot)\equiv 0$.

Global existence. Denote $\mathbf{x}_K(t;\omega):=\int_0^t\bar{\eta}_K(s,\mathbf{0};\omega)\,\mathrm{d}s$.

Lemma 6. *Under the assumptions of Theorem* 4, $\mathbf{x}_K$ *is the solution of* (6.1) *and satisfies the equation*

$$\mathbf{x}_K'(t)=\int_0^t \Phi_K(t,s,\mathbf{x}_K(t),\mathbf{x}_K(t)-\mathbf{x}_K(s),\mathbf{x}_K'(s))\,\mathrm{d}s,\quad \mathbf{x}_K(0)=\mathbf{0},$$
$$\Phi_K(t,s,\mathbf{x},\mathbf{y},\mathbf{z}):=\frac{1}{t}[F_K(t,\mathbf{x})+G_K(t,\mathbf{0})]+[H_K(t,s,\mathbf{0})$$
$$-I_K(t,s,\mathbf{y})+\mathbf{z}\cdot J_K(t,s,\mathbf{y})]\,\mathrm{d}s. \tag{6.18}$$

Moreover, *for any* $T>0$ *there exist constants* $C_1,C_2>0$, *possibly depending on* T *but not on* K, *such that*

$$\mathbb{P}\left(\sup_{0\leqslant t\leqslant T}[|\mathbf{x}_K(t)|+|\mathbf{x}_K'(t)|]>M\right)\leqslant C_1\mathrm{e}^{-C_2M^2},\quad \forall K,M>0. \tag{6.19}$$

Proof. Only (6.19) requires a proof. Let

$$A_{n,K}(\lambda):=\left[\sup_{(t,\mathbf{x})\in\mathbb{R}_T}\frac{|V_K(t,\mathbf{x})|}{|\mathbf{x}|+n}\leqslant\lambda\right],\quad n\geqslant 1,\lambda>0.$$

By virtue of Theorem 6.9.2, p. 161 of Adler (1990) there exist constants $c_1, c_2, \lambda_0 > 0$, cf. the argument following (2.7), independent of n, K and such that

$$\mathbb{P}(A^c_{n,K}(\lambda_0)) \leqslant c_1 \mathrm{e}^{-c_2 n^2}, \quad \forall n \geqslant 1.$$

For $\omega \in A_{n,K}$ we obtain

$$|\boldsymbol{x}'_K(t)| \leqslant \lambda_0(|\boldsymbol{x}_K(t)| + n), \quad t \in [0,T].$$

Hence, there exists a deterministic constant $c_3 > 0$ such that for

$$\sup_{0 \leqslant t \leqslant T} |\boldsymbol{x}_K(t)| \leqslant c_3 n, \quad \text{for } \omega \in A_{n,K}(\lambda_0).$$

From the definition of the event $A_{n,K}(\lambda_0)$ we conclude also that

$$\sup_{0 \leqslant t \leqslant T} |V_K(t, \boldsymbol{x}_K(t))| \leqslant \lambda_0 \left(\sup_{0 \leqslant t \leqslant T} |\boldsymbol{x}_K(t)| + n \right) \leqslant \lambda_0 (c_3 + 1) n \quad \text{for } \omega \in A_{n,K}(\lambda_0)$$

and (6.19) follows from (6.1). □

Let us fix $T > 0$ and let $\psi_M : \mathbb{R} \to [0,1]$ be a C^∞ smooth function satisfying $\psi_M(r) = 1$, if $r \leqslant M$ and $\psi_M(r) = 0$, if $r \geqslant M + 1$. We define

$$\frac{\mathrm{d}\boldsymbol{x}^{(M)}_K(t)}{\mathrm{d}t} = \psi_M \left(|\boldsymbol{x}^{(M)}_K(t)| + \left| \frac{\mathrm{d}\boldsymbol{x}^{(M)}_K(t)}{\mathrm{d}t} \right| \right)$$
$$\int_0^t \Phi_K \left(t, s, \boldsymbol{x}^{(M)}_K(t), \boldsymbol{x}^{(M)}_K(t) - \boldsymbol{x}^{(M)}_K(s), \frac{\mathrm{d}\boldsymbol{x}^{(M)}_K(s)}{\mathrm{d}s} \right) \mathrm{d}s, \quad \boldsymbol{x}^{(M)}_K(0) = 0. \tag{6.20}$$

The existence of a solution $\boldsymbol{x}^{(M)}_K(\cdot) \in C^1([0,T]; \mathbb{R}^d)$ of (6.20) follows from a standard application of Schauder's Fixed Point Theorem. Let

$$\tau_{K,M} := \min \left[t : |\boldsymbol{x}^{(M)}_K(t)| + \left| \frac{\mathrm{d}\boldsymbol{x}^{(M)}_K(t)}{\mathrm{d}t} \right| \geqslant M \right] \wedge T.$$

Obviously, $\boldsymbol{x}^{(M)}_K(t) = \boldsymbol{x}_K(t)$, $t \in [0, \tau_{K,M}]$, $\forall K, M \geqslant 1$, therefore, according to Lemma 6 there exist constants $c_1, c_2 > 0$ independent of $K \geqslant 1$, $h \in (0, T/2)$ such that

$$\mathbb{P}(\tau_{K,M} \leqslant T - h) \leqslant c_1 \mathrm{e}^{-c_2 M^2}, \quad M \geqslant 1. \tag{6.21}$$

Let $K \uparrow \infty$, we have then $\boldsymbol{x}^{(M)}_K(\cdot)$ is uniformly convergent to the unique solution $\boldsymbol{x}^{(M)}(\cdot)$ of

$$\frac{\mathrm{d}\boldsymbol{x}^{(M)}(t)}{\mathrm{d}t} = \psi_M \left(|\boldsymbol{x}^{(M)}(t)| + \left| \frac{\mathrm{d}\boldsymbol{x}^{(M)}(t)}{\mathrm{d}t} \right| \right)$$
$$\int_0^t \Phi \left(t, s, \boldsymbol{x}^{(M)}(t), \boldsymbol{x}^{(M)}(t) - \boldsymbol{x}^{(M)}(s), \frac{\mathrm{d}\boldsymbol{x}^{(M)}(s)}{\mathrm{d}s} \right) \mathrm{d}s, \quad \boldsymbol{x}^{(M)}(0) = \mathbf{0}. \tag{6.22}$$

Here Φ is defined by formula (6.18) with the replacement of F_K, G_K, H_K, I_K, J_K appearing there by corresponding fields F, G, H, I, J. The proof of uniqueness of solutions of Eq. (6.22) can be carried out via an argument identical with the one used in the uniqueness part of the proof of the lemma. Moreover $\tau_{K,M} \to \tau_M$, as $K \uparrow \infty, \mathbb{P}$-a.s., where

$$\tau_M := \min\left[t : |\boldsymbol{x}^{(M)}(t)| + \left|\frac{\mathrm{d}\boldsymbol{x}^{(M)}(t)}{\mathrm{d}t}\right| \geqslant M\right] \wedge T.$$

Note also that, thanks to (6.21), we have $\mathbb{P}(\tau_M > 0) = 1$ and $\boldsymbol{x}^{(M)}(\cdot)$ is uniformly convergent, as $M \uparrow +\infty$, $\mathbb{P}$-a.s. on any compact sub-interval of $[0, \tau_M)$ to the unique solution $\boldsymbol{x}(\cdot)$ of (6.9).

We prove that $\tau_M \to T$, as $M \uparrow \infty$. This result establishes the global existence of the solutions of (6.9). First note that thanks to (6.21) we have

$$\mathbb{P}(\tau_M \leqslant T - h) \leqslant c_1 \mathrm{e}^{-c_2 M^2}, \quad M \geqslant 1, h \in (0, T/2). \tag{6.23}$$

The already proven uniqueness part implies $\tau_{M+1} \geqslant \tau_M$, so $T_* := \lim_{M\uparrow\infty} \tau_M$ satisfies

$$\mathbb{P}(T_* \leqslant T - h) = 0, \quad \forall h \in (0, T/2),$$

which shows that $T_* = T$, $\mathbb{P}$-a.s. □

Returning to the proof of the theorem we notice that having $\boldsymbol{x}$ as in Lemma 5 we can define

$$\bar{\eta}(t, \mathbf{x}) := F(t, \mathbf{x} + \boldsymbol{x}(t)) + G(t, \mathbf{x}) + \int_0^t [H(t, s, \mathbf{x}) - I(t, s, \mathbf{x} + \boldsymbol{x}(t) - \boldsymbol{x}(s))] \, \mathrm{d}s.$$

It can be easily verified that $\bar{\eta} \in \mathcal{M}$, $\mathbb{P}$-a.s., and that (6.8) is a conclusion from Lemma 6. □

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Appendix A. Proof of Proposition 2

Recall our standing assumption that $\rho \in (d/2, d/2 + \gamma(\beta))$.

Proof of (i). We denote by $\mathscr{F} f$ or $\hat{f}$ the Fourier transform of a given function f. Let $\{e_k\}_{k\in\mathbb{N}}$ be an orthonormal basis of $\mathbb{L}^2$. Recall, see Section 2, that $J_{\rho,2}\psi(x) = \psi(x)\vartheta_{\rho/2}(x)$ is an isomorphism between $\mathbb{H}^m_\rho$ and $\mathbb{H}^m$. Hence, there is a constant c_1

such that

$$\|S_\beta(t)B_\beta\|^2_{L_{(HS)}(\mathbb{L}^2,\mathbb{H}^m_\rho)} = \sum_k \|S_\beta(t)B_\beta e_k\|^2_{\mathbb{H}^m_\rho} \leqslant c_1 \sum_k \|J_{\rho,2}S_\beta(t)B_\beta e_k\|^2_{\mathbb{H}^m}$$

$$\leqslant c_1 \sum_k \int_{\mathbb{R}^d} (1+|\mathbf{k}|^2)^m |\mathscr{F}(\vartheta_\rho^{1/2} S_\beta(t)B_\beta e_k)(\mathbf{k})|^2_{\mathbb{R}^d}\,\mathrm{d}\mathbf{k}.$$

Note that

$$\mathscr{F}(S_\beta(t)B_\beta e_k)(\mathbf{k}) = \mathrm{e}^{-|\mathbf{k}|^{2\beta}t}\sqrt{2|\mathbf{k}|^{2\beta}\bar{\mathscr{E}}(\mathbf{k})}\hat{e}_k(\mathbf{k}).$$

Thus,

$$\sum_k |\mathscr{F}(\vartheta_\rho^{1/2} S_\beta(t)B_\beta e_k)(\mathbf{k})|^2_{\mathbb{R}^d}$$

$$= \sum_k \left| \int_{\mathbb{R}^d} \widehat{\vartheta_\rho^{1/2}}(\mathbf{k}-\mathbf{k}')\mathrm{e}^{-|\mathbf{k}'|^{2\beta}t}\sqrt{2|\mathbf{k}'|^{2\beta}\bar{\mathscr{E}}(\mathbf{k}')}\hat{e}_k(\mathbf{k}')\,\mathrm{d}\mathbf{k}' \right|^2_{\mathbb{R}^d}$$

$$\leqslant 2\int_{\mathbb{R}^d} |\widehat{\vartheta_\rho^{1/2}}(\mathbf{k}-\mathbf{k}')|^2 \mathrm{e}^{-2|\mathbf{k}'|^{2\beta}t}|\mathbf{k}'|^{2\beta}\,\mathrm{Tr}\,\bar{\mathscr{E}}(\mathbf{k}')\,\mathrm{d}\mathbf{k}'.$$

Hence

$$\|S_\beta(t)B_\beta\|^2_{L_{(HS)}(\mathbb{L}^2,\mathbb{H}^m_\rho)}$$

$$\leqslant 2c_1 \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (1+|\mathbf{k}|^2)^m |\widehat{\vartheta_\rho^{1/2}}(\mathbf{k}-\mathbf{k}')|^2 \mathrm{e}^{-2|\mathbf{k}'|^{2\beta}t}|\mathbf{k}'|^{2\beta}\,\mathrm{Tr}\,\bar{\mathscr{E}}(\mathbf{k}')\,\mathrm{d}\mathbf{k}'\,\mathrm{d}\mathbf{k}$$

$$= 2c_1 \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (1+|\mathbf{k}+\mathbf{k}'|^2)^m |\widehat{\vartheta_\rho^{1/2}}(\mathbf{k})|^2 \mathrm{e}^{-2|\mathbf{k}'|^{2\beta}t}|\mathbf{k}'|^{2\beta}\,\mathrm{Tr}\,\bar{\mathscr{E}}(\mathbf{k}')\,\mathrm{d}\mathbf{k}'\,\mathrm{d}\mathbf{k}. \tag{A.1}$$

Since $1+|\mathbf{k}+\mathbf{k}'|^2 \leqslant (1+|\mathbf{k}|^2)(1+|\mathbf{k}'|^2)$ we conclude that the utmost left-hand side of (A.1) is less than or equal to

$$c_2 \int_{\mathbb{R}^d} (1+|\mathbf{k}|^2)^m |\widehat{\vartheta_\rho^{1/2}}(\mathbf{k})|^2\,\mathrm{d}\mathbf{k} \int_{\mathbb{R}^d} (1+|\mathbf{k}'|^2)^m \mathrm{e}^{-2|\mathbf{k}'|^{2\beta}t}|\mathbf{k}'|^{2\beta}\,\mathrm{Tr}\,\bar{\mathscr{E}}(\mathbf{k}')\,\mathrm{d}\mathbf{k}'$$

and, because

$$\int_{\mathbb{R}^d} (1+|\mathbf{k}|^2)^m |\widehat{\vartheta_\rho^{1/2}}(\mathbf{k})|^2\,\mathrm{d}\mathbf{k} < \infty,$$

we have

$$\|S_\beta(t)\,B_\beta\|^2_{L_{(HS)}(\mathbb{L}^2,\mathbb{H}^m_\rho)} \leqslant c\int_{\mathbb{R}^d} (1+|\mathbf{k}'|^2)^m \mathrm{e}^{-2|\mathbf{k}'|^{2\beta}t}|\mathbf{k}'|^{2\beta}\,\mathrm{Tr}\,\bar{\mathscr{E}}(\mathbf{k}')\,\mathrm{d}\mathbf{k}'. \tag{A.2}$$

In particular, taking $t=0$ we obtain the desired conclusion from condition (2.2). □

Proof of (ii). We will treat the case of a non-integer β. The case of integer β has been considered in Peszat and Zabczyk (1997). Let

$$p_\beta(\mathbf{x}) := \int_{\mathbb{R}^d} \mathrm{e}^{-|\mathbf{k}|^{2\beta}}\mathrm{e}^{\mathrm{i}\,\mathbf{k}\cdot\mathbf{x}}\,\mathrm{d}\mathbf{k}.$$

Then, see e.g. Blumenthal and Getoor (1960, Theorem 2.1, p. 263), there exists a constant $C>0$ such that

$$|p_\beta(\mathbf{x})| \leqslant \frac{C}{(1+|\mathbf{x}|^2)^{d/2+\beta}}.$$

We show boundedness of

$$S_\beta(t)f(\mathbf{x}) = \frac{1}{t^{d/(2\beta)}}\int_{\mathbb{R}^d} p_\beta\left(\frac{|\mathbf{x}-\mathbf{y}|}{t^{1/(2\beta)}}\right) f(\mathbf{y})\,\mathrm{d}\mathbf{y}$$

in $\mathbb{L}^p_\rho$ using an interpolation argument. First consider the operator on $\mathbb{L}^\infty$. Then

$$\|S_\beta(t)f\|_{\mathbb{L}^\infty} \leqslant \sup_{\mathbf{x}\in\mathbb{R}^d}\left[\frac{1}{t^{d/(2\beta)}}\int_{\mathbb{R}^d}\left|p_\beta\left(\frac{|\mathbf{x}-\mathbf{y}|}{t^{1/(2\beta)}}\right)\right|\,\mathrm{d}\mathbf{y}\right]\|f\|_{\mathbb{L}^\infty} = \|p_\beta\|_{L^1}\|f\|_{\mathbb{L}^\infty},$$

and the boundedness follows from $p_\beta \in L^1$. Next, consider the operator on $\mathbb{L}^1_\rho$. The boundedness of $S_\beta(t)$ on this space is equivalent with the same property of

$$T_{\beta,\rho}(t)f(\mathbf{x}) := \frac{1}{t^{d/(2\beta)}}\int_{\mathbb{R}^d}\left(\frac{1+|\mathbf{y}|^2}{1+|\mathbf{x}|^2}\right)^\rho p_\beta\left(\frac{|\mathbf{x}-\mathbf{y}|}{t^{1/(2\beta)}}\right) f(\mathbf{y})\,\mathrm{d}\mathbf{y}$$

on $\mathbb{L}^1$. Using an elementary inequality $(1+|\mathbf{y}|^2)^\rho \leqslant C[(1+|\mathbf{x}|^2)^\rho + |\mathbf{x}-\mathbf{y}|^{2\rho}]$ we conclude that

$$\|T_{\beta,\rho}(t)f\|_{\mathbb{L}^1} \leqslant I_1(f) + I_2(f),$$

where

$$I_1(f) = \frac{C}{t^{d/(2\beta)}}\int_{\mathbb{R}^d}\mathrm{d}\mathbf{x}\int_{\mathbb{R}^d}\left|p\left(\frac{|\mathbf{x}-\mathbf{y}|}{t^{1/(2\beta)}}\right) f(\mathbf{y})\right|\,\mathrm{d}\mathbf{y},$$

$$I_2(f) = \frac{C}{t^{d/(2\beta)}}\int_{\mathbb{R}^d}\frac{\mathrm{d}\mathbf{x}}{(1+|\mathbf{x}|^2)^\rho}\int_{\mathbb{R}^d}|\mathbf{x}-\mathbf{y}|^{2\rho}\left|p\left(\frac{\mathbf{x}-\mathbf{y}}{t^{1/(2\beta)}}\right) f(\mathbf{y})\right|\,\mathrm{d}\mathbf{y}.$$

Then $I_1(f) = C\,\|p_\beta\|_{L^1}\|f\|_{\mathbb{L}^1}$ while

$$I_2(f) \leqslant Ct^{(2\rho-d)/(2\beta)}\sup_{u>0}(u^{2\rho}|p_\beta(u)|)\int_{\mathbb{R}^d}\frac{\mathrm{d}\mathbf{x}}{(1+|\mathbf{x}|^2)^\rho}\|f\|_{\mathbb{L}^1}, \tag{A.3}$$

where the supremum appearing in (A.3) remains finite as long as $2\rho \leqslant d+2\beta$. In conclusion we obtain that $S_\beta(t):\mathbb{L}^p_\rho \to \mathbb{L}^p_\rho$ is a bounded operator when $1 \leqslant p \leqslant \infty$. This together with the continuity of the semigroup on $\mathscr{S}_d$ proves that S_β is a C_0-semigroup on any $\mathbb{L}^p_\rho$. In fact since $S_\beta(t)$ commutes with $\nabla_\mathbf{x}$ we have proved C_0-continuity of the semigroup on any $\mathbb{W}^{p,m}_\rho$. □

In the proof of the third part of the proposition we need two lemmas. Write

$$\widehat{K_{\beta,\lambda}\psi}(\mathbf{k}) := |\mathbf{k}|^{2\beta}(|\mathbf{k}|^2+\lambda)^{-\beta}\hat\psi(\mathbf{k}) \quad \text{for any } \lambda>0,\ \psi\in\mathscr{S}_d.$$

Lemma 7. *$K_{\beta,\lambda}$ extends to a bounded operator $K_{\beta,\lambda}:\mathbb{H}^m_\rho \to \mathbb{H}^m_\rho$ for all $\lambda>0$, $m\geqslant 0$.*

Proof. By Stein (1970, Lemma 3.2.2, p. 133) the Fourier transform $\mu_{2\beta} := \mathscr{F}(|\mathbf{k}|^{2\beta}(|\mathbf{k}|^2+\lambda)^{-\beta})$ is a signed measure with a finite total variation $\|\mu_{2\beta}\|_{TV}$. Hence the operator $K_{\beta,\lambda}:\mathbb{L}^\infty \to \mathbb{L}^\infty$ is bounded and

$$|K_{\beta,\lambda}f(\mathbf{x})| \leqslant \int_{\mathbb{R}^d}|f(\mathbf{x}-\mathbf{y})|\,|\mu_{2\beta}|(\mathrm{d}\mathbf{y}) \leqslant \|f\|_{\mathbb{L}^\infty}\|\mu_{2\beta}\|_{TV}, \quad \mathbf{x}\in\mathbb{R}^d.$$

Using a general interpolation argument, as in the proof of (ii), we note that in order to finish the proof of the lemma we only need to show that $K_{\beta,\lambda}$ is bounded on $\mathbb{L}^1_\rho$. This is equivalent with proving that

$$R_{\beta,\lambda}f(\mathbf{x}) := \int_{\mathbb{R}^d} \left(\frac{1+|\mathbf{x}-\mathbf{y}|^2}{1+|\mathbf{x}|^2}\right)^\rho f(\mathbf{x}-\mathbf{y})\mu_{2\beta}(\mathrm{d}\mathbf{y})$$

extends to a bounded operator in $\mathbb{L}^1$. It is known, see Blumenthal and Getoor (1960, p. 134), that

$$\left(\frac{|\mathbf{k}|^2}{|\mathbf{k}|^2+\lambda}\right)^\beta = 1 + \sum_{m=1}^{[d/2]} c_m(\lambda+|\mathbf{k}|^2)^{-m} + \hat{r}(\mathbf{k}),$$

where $\hat{r}(\cdot)\in L^1(\mathbb{R}^d)\cap C^\infty(\mathbb{R}^d)$ so its inverse Fourier transform r satisfies $r(\mathbf{x})\leqslant|\mathbf{x}|^{-n}$, $|\mathbf{x}|\geqslant 1$ for an arbitrary $n\geqslant 1$. Hence

$$\mu_{2\beta}(\mathrm{d}\mathbf{x}) = \delta_0(\mathrm{d}\mathbf{x}) + G(\mathbf{x})\,\mathrm{d}\mathbf{x} + r(\mathbf{x})\,\mathrm{d}\mathbf{x},$$

with

$$G(\mathbf{x}) := \sum_{m=1}^{[d/2]} c_m G_{2m}(\mathbf{x})$$

and

$$G_{2m}(\mathbf{x}) = 2^{d-2\beta}(\sqrt{\lambda}\pi)^{d-\beta}\frac{1}{\Gamma(\beta)}\int_0^{+\infty} \mathrm{e}^{-\pi|\mathbf{x}|^2/\delta}\mathrm{e}^{-\delta/(4\pi)}\delta^{(-d+2\beta)/2}\frac{\mathrm{d}\delta}{\delta}.$$

Therefore

$$\int_{\mathbb{R}^d}|R_{\beta,\lambda}f(\mathbf{x})|\,\mathrm{d}\mathbf{x} \leqslant \int_{\mathbb{R}^d}|f(\mathbf{x})|\,\mathrm{d}\mathbf{x} + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left(\frac{1+|\mathbf{y}|^2}{1+|\mathbf{x}|^2}\right)^\rho \times[|G(\mathbf{x}-\mathbf{y})+r(\mathbf{x}-\mathbf{y})|]|f(\mathbf{y})|\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}.$$

The second term on the right-hand side is less than or equal to

$$\|G+r\|_{\mathbb{L}^1}\|f\|_{\mathbb{L}^1} + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left(\frac{|\mathbf{x}-\mathbf{y}|^2}{1+|\mathbf{x}|^2}\right)^\rho [|G(\mathbf{x}-\mathbf{y})|+|r(\mathbf{x}-\mathbf{y})|]|f(\mathbf{y})|\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}. \tag{A.4}$$

(29) and (30) from p. 132 of Blumenthal and Getoor (1960) imply that $G_{2m}(\mathbf{x})|\mathbf{x}|^{2\rho}\leqslant C$, $\mathbf{x}\in\mathbb{R}^d$ for some constant $C>0$, provided $\rho>d/2$. Repeating estimates performed in the proof of (ii) we bound the second term of (A.4) by

$$C\sup_{\mathbf{x}\in\mathbb{R}^d}|\mathbf{x}|^{2\rho}(|G(\mathbf{x})|+|r(\mathbf{x})|)\int_{\mathbb{R}^d}\frac{\mathrm{d}\mathbf{x}}{(1+|\mathbf{x}|^2)^\rho}\|f\|_{\mathbb{L}^1}\leqslant C\|f\|_{\mathbb{L}^1},$$

which proves that $R_{\beta,\lambda}$ is a bounded operator on $\mathbb{L}^1$. □

Lemma 8. *Suppose that β, m are as in Lemma 7. Then there exists $C>0$ such that $\|A_\beta f\|_{\mathbb{H}^m_\rho}\leqslant C\|f\|_{\mathbb{H}^{m+[2\beta]+1}_\rho}$ for any $f\in\mathscr{S}_d$.*

Proof. Let us define

$$L_{\rho,\lambda}f := J_{\rho,2}G_{\lambda,1}J_{-\rho,2}f, \quad \text{with } \widehat{G_{\lambda,\beta}f}(\mathbf{k}) := (|\mathbf{k}|^2+\lambda)^\beta\hat{f}(\mathbf{k}), \quad f\in\mathscr{S}_d,\ \beta,\lambda\geqslant 0.$$

For a sufficiently large $\lambda > 0$ $L_{\rho,\lambda}$ is a positive definite elliptic operator with $-\Delta_{\mathbf{x}}$— the principal part and bounded C^{∞}-smooth coefficients. By a classical theory it extends uniquely to a generator (i.e. S_d is a respective core) $-L_{\rho,\lambda}: D(L_{\rho,\lambda}) \to \mathbb{L}^2$ of an analytic semigroup $U(\cdot): \mathbb{L}^2 \to \mathbb{L}^2$ such that

$$U(t)f = \mathrm{e}^{-\lambda t} J_{\rho,2} S_1(t) J_{-\rho,2} f, \quad \forall f \in \mathscr{S}_d, \ t \geqslant 0.$$

Using formulas (6.14), (6.9) of Pazy (1983) we obtain

$$f := J_{\rho,2} G_{\lambda,\beta} J_{-\rho,2} f, \quad \forall f \in \mathscr{S}_d.$$

It is known, see Theorem 6.10, p. 73 of Pazy (1983) that $\|L^{\beta}_{\rho,\lambda} f\|_{\mathbb{H}^m} \leqslant C\|f\|_{\mathbb{H}^{m+[2\beta]+1}}$. Hence, by Lemma 7,

$$\begin{aligned} \|A_\beta f\|_{\mathbb{H}^m_\rho} &= \|K_{\beta,\lambda} G_{\lambda,\beta} f\|_{\mathbb{H}^m_\rho} \leqslant C\|G_{\lambda,\beta} f\|_{\mathbb{H}^m_\rho} \leqslant C\|L^{\beta}_{\rho,\lambda} J_{\rho,2} f\|_{\mathbb{H}^m} \\ &\leqslant C\|J_{\rho,2} f\|_{\mathbb{H}^{m+[2\beta]+1}} \leqslant C\|f\|_{\mathbb{H}^{m+[2\beta]+1}_\rho} \end{aligned}$$

for all $f \in S_d$ and the conclusion of the lemma follows. □

Proof of (iii). From (ii) we conclude that $S_\beta(t)(\mathscr{C}_\rho) \subseteq \mathscr{C}_\rho$ and $S_\beta(t)(\mathscr{H}_\rho) \subseteq \mathscr{H}_\rho$ for all $t \geqslant 0$. Hence, in a consequence of Proposition 3.3 of Ethier and Kurtz (1986), $\mathscr{C}_\rho$ and $\mathscr{H}_\rho$ are cores of A_β. We prove now that $\mathscr{S}_d$ is also a core of A_β. Let us consider first the case when β is a positive integer. Then, obviously $S_\beta(t)(\mathscr{S}_d) \subseteq \mathscr{S}_d$, $t \geqslant 0$ and our claim follows from Proposition 3.3 of Ethier and Kurtz (1986). Suppose that $\beta \notin \mathbb{Z}$. Since $\mathscr{S}_d$ is dense in $\mathbb{H}^{m+[2\beta]+1}$ there exists a sequence $f_n \in \mathscr{S}_d$, $n \geqslant 1$ such that $f_n \to f$ in $\mathbb{H}^{m+[2\beta]+1}$. From Lemma 8 we conclude that $A_\beta f_n$, $n \geqslant 1$ is convergent in $\mathbb{H}^m_\rho$. Hence from closedness of A_β we conclude that $A_\beta f_n \to A_\beta f$, as $n \uparrow \infty$, in $\mathbb{H}^m_\rho$. Our claim holds thanks to the fact that $\mathscr{H}_\rho$ is a core of A_β.

We prove now that $\mathbb{H}^{m+[2\beta]+1}_\rho \subseteq D(A_\beta)$. Let $f \in \mathbb{H}^{m+[2\beta]+1}_\rho$. We can approximate f by elements $f_n \in \mathscr{S}_d$, $n \geqslant 1$ in that space. Thanks to Lemma 8, $(A_\beta f_n)$ converges in $\mathbb{H}^m_\rho$. Then $f \in D(A_\beta)$ and $A_\beta f_n \to A_\beta f$ in $\mathbb{H}^m_\rho$ as a consequence of closedness of the graph of A_β. □

Proof of (iv). We only consider the case of non-integer β, β'. Let

$$q_{\beta,\beta'}(|\mathbf{x}|, t) := \int_{\mathbb{R}^d} |\mathbf{k}|^{2\beta'} \mathrm{e}^{-|\mathbf{k}|^{2\beta} t} \mathrm{e}^{\mathrm{i}\,\mathbf{k}\cdot\mathbf{x}}\, \mathrm{d}\mathbf{k}.$$

Then

$$A_{\beta'} S_\beta(t) \psi(\mathbf{x}) = \int q_{\beta,\beta'}(|\mathbf{x}-\mathbf{y}|, t) \psi(\mathbf{y})\, \mathrm{d}\mathbf{y}$$

for any $\psi \in \mathscr{S}_d$. For any integer $M \geqslant 1$ one can choose N sufficiently large and $c_{m,n} \in \mathbb{R}$, $m,n = 0,\ldots,N$ such that $c_{0,0} = 0$ and

$$H(\mathbf{k}) := (|\mathbf{k}|^{2\beta'} + 1)\mathrm{e}^{-|\mathbf{k}|^{2\beta}} - \sum_{m,n=0}^{N} c_{m,n} \mathrm{e}^{-(m|\mathbf{k}|^{2\beta'} + n|\mathbf{k}|^{2\beta})}$$

is M-times differentiable and $|H(\mathbf{k})| \leqslant C\mathrm{e}^{-(1/2)|\mathbf{k}|^{2\gamma}}$, where $\gamma = \beta \wedge \beta'$. Hence

$$q_{\beta,\beta'}(|\mathbf{x}|, 1) \leqslant \frac{C}{(1 + |\mathbf{x}|^2)^{d/2+\gamma}}.$$

Repeating now the argument leading to estimate of the $\mathbb{H}_\rho^m$-norm of $S_\beta(t)f$, $f \in \mathbb{H}_\rho^m$, see the calculations made in the proof of (ii), we obtain the estimate (2.4). The inequality (2.3) follows from calculations identical with those made in the proof of (i). □

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