Combinatorial bases for multilinear parts of free algebras with two compatible brackets

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Outline

- Introduction
- Statement and examples of the theorems
- Further discussion

PART I:

Introduction

***Summary*:** We will briefly go over some related results on free algebras with one Lie bracket.

Lie algebra and Poisson algebra

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We recall a *Lie algebra* over R is an R-module V equipped with a bilinear binary operation $[\cdot,\cdot]$, called a *Lie bracket*, satisfying two properties: for any $x, y, z \in V$,

antisymmetry $[x, y] = -[y, x]$,

Jacobi identity $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$.

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A *Poisson algebra* over R is an R-module V equipped with two bilinear binary operations: a Lie bracket $[\cdot,\cdot]$ and an associative commutative multiplication such that the Lie bracket is a *derivation* of the commutative multiplication: that is, for any $x, y, z \in V$, we have

$$[x, yz] = [x, y]z + y[x, z].$$

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The *free Lie algebra* on X over R is the Lie algebra over R that is generated by all possible Lie bracketings of elements of X with no relations other than antisymmetries and Jacobi identities.

Let $\mathscr{L}ie(n)$ be the *multilinear part* of this free Lie algebra: i.e., the subspace consisting of all elements containing each x_i exactly once.

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Monomials in $\mathscr{P}(2) : [x_1, x_2], [x_2, x_1], x_1x_2, x_2x_1$.

A basis for $\mathscr{P}(2) : \{[x_1, x_2], x_1x_2\}$.

Dimension formulas and bases for $\mathscr{L}ie(n)$ and $\mathscr{P}(n)$

Example: A basis for $\mathscr{L}ie(3)$: $\{[x_1, [x_2, x_3]], [x_2, [x_1, x_3]]\}$.

$[x_3, [x_1, x_2]] = -[x_1, [x_2, x_3]] + [x_2, [x_1, x_3]]$ by the Jacobi identity.

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A basis for $\mathscr{L}ie(n)$:

$$\{[x_{\sigma(1)},[x_{\sigma(2)},[x_{\sigma(3)},[\cdots,[x_{\sigma(n-1)},x_n]\cdots]]]] \mid \sigma \in S_{n-1}\}.$$

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Suppose $B(X)$ is a basis for $\mathscr{L}ie(n)$ on the alphabet X. Then the following is a basis for $\mathscr{P}(n)$:

$\{b_1 b_2 \cdots b_k \mid \cup_{i=1} X_i$ is a partition of X with $\max(X_1) < \cdots < \max(X_k)$, and $b_i \in B(X_i)\}$

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$\mathscr{P}(n)$ has dimension $n!$.

PART II:

Statement and examples of the theorems

Summary: In this part, we introduce free algebras with two compatible brackets. After defining the problem and giving a little background, we state our main theorems together with examples in the cases when $n = 3$.

Free algebras with two compatible Lie brackets

We consider a free algebra on X with two Lie brackets $[\cdot,\cdot]$ and $\langle\cdot,\cdot\rangle$, which are *compatible*: that is, any linear combination of them is a Lie bracket.

If we write out this condition explicitly, the compatibility gives one condition, which we call the *mixed Jacobi identity*, in addition to the antisymmetry and Jacoby identity for each of the two brackets. For any x, y, z,

(S1) $[x,y] + [y,x] = 0,$

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Dimension formulas for $\mathscr{L}ie_2(n)$ and $\mathscr{P}_2(n)$

Theorem 1.

$$\dim(\mathscr{L}ie_2(n)) = n^{n-1}, \quad (2)$$

$$\dim(\mathscr{P}_2(n)) = (n+1)^{n-1}. \quad (3)$$

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- Dotsenko and Khoroshkin independently prove Theorem 1 using the theory of operads. They also obtain character formulas for the representation of the symmetric groups and SL_2 in $\mathscr{L}ie_2(n)$ and $\mathscr{P}_2(n)$.
- It turns out (2) and (3) are equivalent to each other. Therefore, it is enough to show one of them. We will focus on (2).

Two-colored rooted trees

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For any edge $\{i, j\}$ in a rooted tree, if i is closer to the root than j, then we call i the *parent* of j and j a *child* of i. Furthermore, if i is the parent of j, we call the edge $\{i, j\}$ an *increasing edge* if $i < j$ and a *decreasing edge* if $i > j$.

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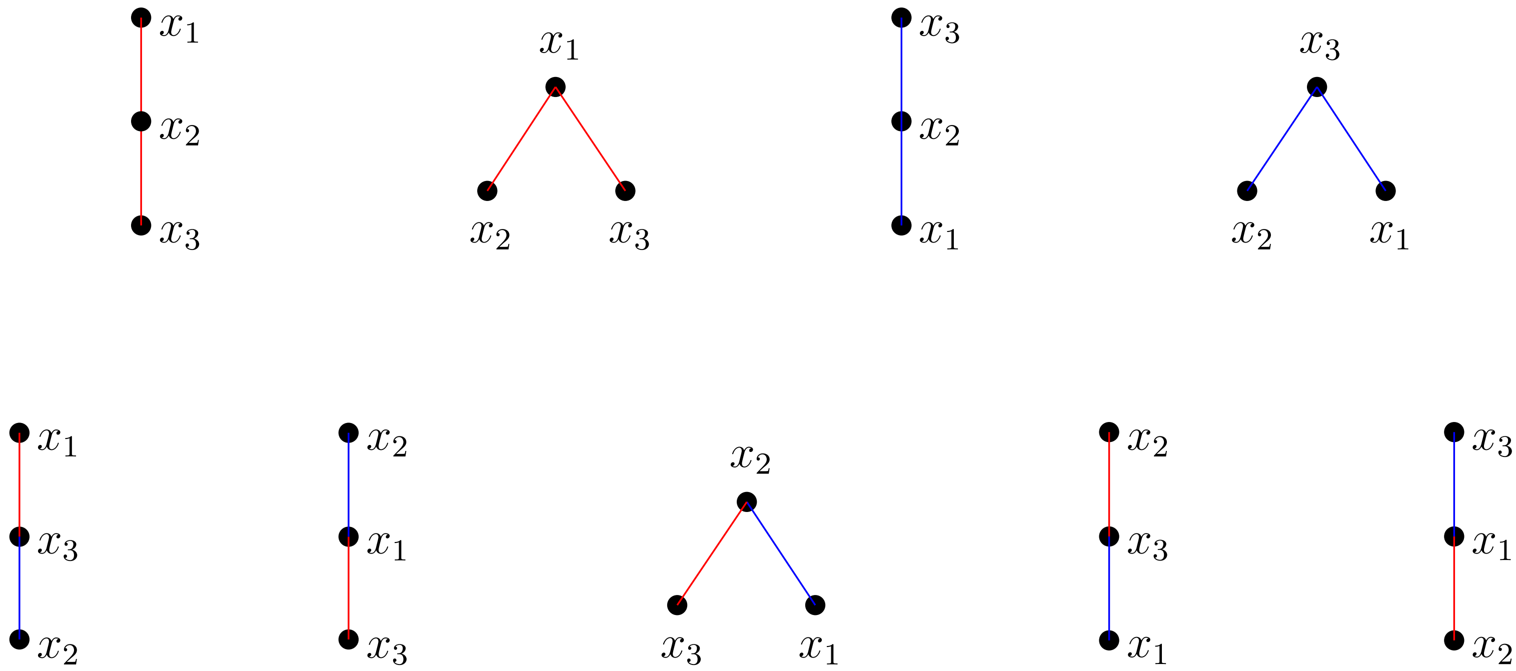
We define a color map c : Given any rooted tree $T \in \mathcal{R}_n$, we color all of the increasing edges by red and all of the decreasing edges by blue.

Let $\overline{\mathcal{G}}_n := c(\mathcal{R}_n)$ be set of all *two-colored rooted trees* on X.

Examples of two-colored rooted trees



A basis for $\mathscr{L}ie_2(n)$ constructed from $\overline{\mathcal{G}}_n$

For any two colored rooted tree G in $\overline{\mathcal{G}}_n$ with root r, we define a monomial $b_G \in M_n$ recursively as follows:

(i) If $G = r$, let $b_G := r$.

(ii) If $G \neq r$, let $c_1 < \cdots < c_k$ be the vertices connected to r, and $G_1, \ldots, G_k$ be the corresponding subtrees.

- If $r < c_k$, i.e., there are red edges adjacent to r, choose the smallest c_i such that $\{r, c_i\}$ is a red edge. Let $b_G := [b_{G \setminus G_i}, b_{G_i}]$.
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We define $\mathcal{B}_n(X)$ to be the set of all monomials obtained from $\overline{\mathcal{G}}_n$:

$$\mathcal{B}_n(X) := \{b_G \mid G \in \overline{\mathcal{G}}_n\}.$$

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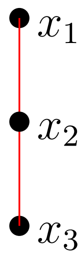
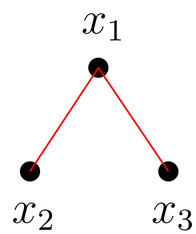
Theorem 5. *$\mathcal{B}_n(X)$ is a basis for $\mathscr{L}ie_2(n)$.*

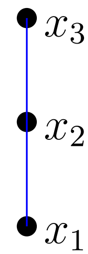
Examples of the construction of $\mathcal{B}_n(X)$



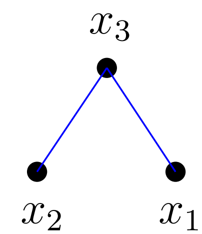
$[x_1, [x_2, x_3]]$



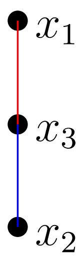
$[[x_1, x_3], x_2]$



$\langle\langle x_1, x_2\rangle, x_3\rangle$



$\langle x_2, \langle x_1, x_3\rangle\rangle$



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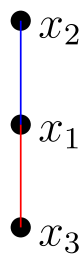

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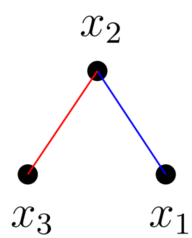

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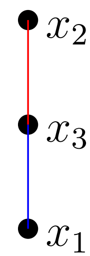
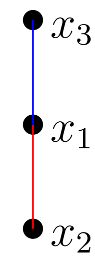


$[x_2, \langle x_1, x_3\rangle]$



$\langle [x_1, x_2], x_3\rangle$

The 9 monomials above form a basis for $\mathscr{L}ie_2(3)$.

Idea of the proof

i. We first show that $\mathcal{B}_n(X)$ spans $\mathscr{L}ie_2(n)$ by giving an explicit algorithm to express each monomial in $\mathscr{L}ie_2(n)$ as linear combinations of elements in $\mathcal{B}_n(X)$.

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ii. We give two methods to prove the independence of $\mathcal{B}_n(X)$. The first method is purely algebraic.

iii. The second method is using the idea of pairing: We define a complementary space $\mathscr{E}il_2(n)$ to $\mathscr{L}ie_2(n)$ by using the combinatorial objects *oriented two-colored graphs*, give a pairing between $\mathscr{L}ie_2(n)$ and $\mathscr{E}il_2(n)$, and show that the pairing is perfect.

More bases for $\mathscr{L}ie_2(n)$ constructed from $\overline{\mathcal{G}}_n$

When we constructed $\mathcal{B}_n(X)$, we used an ordering to decide which edge connected to the root should be removed first. It turns out the ordering is not necessary.

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Definition 6. For any two-colored rooted tree G in $\overline{\mathcal{G}}_n$ with root r, we run an algorithm $rand$ on G as follows:

(i) If $G = r$, output $rand(G) := r$.

(ii) If $G \neq r$, let $c_1 < \cdots < c_k$ be the vertices connected to r, and $G_1, \dots, G_k$ be the corresponding subtrees. Randomly choose i from $\{1, 2, \dots, k\}$.

- Output $rand(G) := [rand(G \setminus G_i), rand(G_i)]$, if c_i is red.
- Output $rand(G) := \langle rand(G_i), rand(G \setminus G_i)\rangle$, if c_i is blue.

More bases for $\mathscr{L}ie_2(n)$ constructed from $\overline{\mathcal{G}}_n$

When we constructed $\mathcal{B}_n(X)$, we used an ordering to decide which edge connected to the root should be removed first. It turns out the ordering is not necessary.

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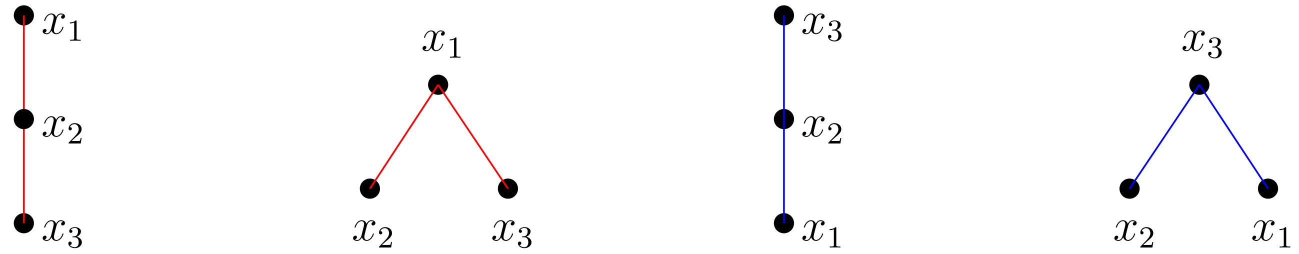
Theorem 7. $rand_n(X) := \{rand(G) \mid G \in \overline{\mathcal{G}}_n\}$ *is a basis for* $\mathscr{L}ie_2(n)$.

Examples of $rand_n(X)$



$[x_1, [x_2, x_3]]$ $\quad$ $[[x_1, x_3], x_2]$ $\quad$ $\langle\langle x_1, x_2\rangle, x_3\rangle$ $\quad$ $\langle x_2, \langle x_1, x_3\rangle\rangle$

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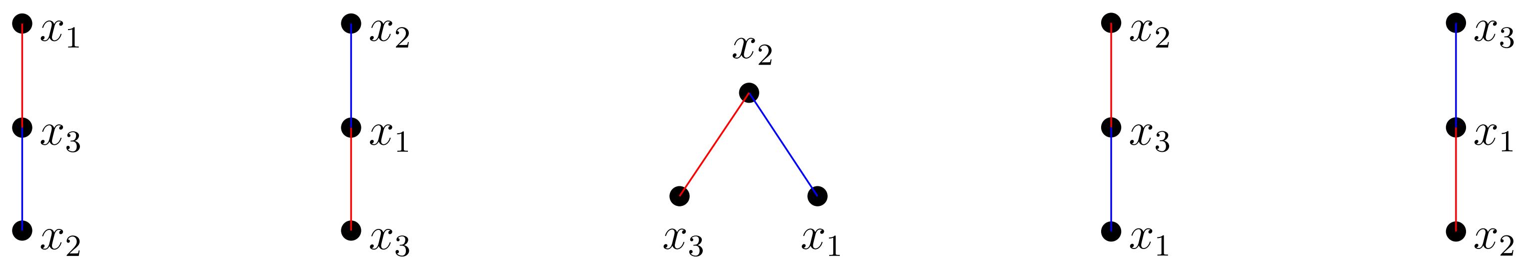

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For each two-colored rooted tree above, we pick one monomial under it. Then the 9 picked monomials together form a basis for $\mathscr{L}ie_2(3)$.

PART III:

Further discussion

***Summary*: We give further combinatorial results on rooted trees, and then ask several natural questions.**

$\mathscr{L}ie_2(n, i)$: submodules of $\mathscr{L}ie_2(n)$

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Definition 8. For any $i = 0, 1, \ldots, n-1$, we define $\mathscr{L}ie_2(n, i)$ to be the submodule of $\mathscr{L}ie_2(n)$ that is generated by all the monomials in $\mathscr{L}ie_2(n)$ with exactly i $[\cdot, \cdot]$'s (and $n-1-i$ $\langle \cdot, \cdot \rangle$'s).

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Proposition 9. *The set*

$$\mathcal{B}_{n,i}(X) := \{b_G \mid G \in \overline{\mathcal{G}}_n \text{ has } i \text{ red/increasing edges}\}$$

or

$$Rand_{n,i}(X) := \{rand(G) \mid G \in \overline{\mathcal{G}}_n \text{ has } i \text{ red/increasing edges}\}$$

is a basis for $\mathscr{L}ie_2(n,i)$.

$$\dim(\mathscr{L}ie_2(n,i)) = \#\{\text{rooted trees on } n \text{ vertices with } i \text{ increasing edges}\}.$$

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Corollary 10.

$$\dim(\mathscr{L}ie(n)) = \#\{\text{increasing rooted trees on } n \text{ vertices}\} = (n-1)!.$$

Number of rooted trees with i increasing edges

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$a(n, i) :=$ the number of rooted trees on n vertices with i increasing edges.

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By the exponential generating function for the SL_2-characters for $\mathscr{L}ie_2(n)$ with SL_2 action obtained by Dotsenko and Khoroshkin, we get the generating function for $a(n,i)$.

Corollary 11.

$$\sum_{i=0}^{n-1} a(n,i)x^i = \prod_{k=1}^{n-1}(kx + (n-k)). \tag{12}$$

Hence, the number of rooted trees on n vertices with i increasing edges is given by

$$a(n,i) = \sum_{K:\text{ a } i\text{-subset of } [n-1]} \prod_{k \in K} k \prod_{k' \in [n-1] \setminus K} (n-k'). \tag{13}$$

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Question: Can one find a combinatorial proof for formulas (12) and (13)?

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What are the right combinatorial objects for $\mathscr{L}ie_k(n)$, if it can be defined?