Pure-cycle Hurwitz factorizations and multi-noded rooted trees

Fu Liu

University of California, Davis

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This is joint work with Rosena R.X. Du.

PART I:

Definitions and Backgrounds

Hurwitz's problem

Definition 1. Given integers d and r, and r partitions $\lambda^1, \dots, \lambda^r \vdash d$, a *Hurwitz factorization* of type $(d, r, (\lambda^1, \dots, \lambda^r))$ is an r-tuple $(\sigma_1, \dots, \sigma_r)$ satisfying the following conditions:

(i) $\sigma_i \in \mathfrak{S}_d$ has cycle type (or is in the conjugacy class) λ^i, for every i;

(ii) $\sigma_1 \cdots \sigma_r = 1$;

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Definition 2. The *Hurwitz number* $h(d, r, (\lambda^1, \dots, \lambda^r))$ is the number of Hurwitz factorizations of type $(d, r, (\lambda^1, \dots, \lambda^r))$ divided by $d!$.

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Question: What is the Hurwitz number $h(d, r, (\lambda^1, \dots, \lambda^r))$?

This question originally arises from geometry: Hurwitz number counts the number of degree-d covers of the projective line with r branch points where the monodromy over the ith branch point has cycle type λ^i.

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We consider instead the *pure-cycle* case. This means each λ^i has the form $(e_i, 1, \dots, 1)$, for some $e_i \geq 2$, or equivalently, each σ_i is an e_i cycle. In this case, we use the notation $h(d, r, (e_1, \dots, e_r))$ for the Hurwitz number.

We also focus on the *genus*-0 case, which simply means that

$$2d - 2 = \sum_{i=1}^{r} (e_i - 1).$$

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Example 3. Let $d = 5$, $r = 4$, $(e_1, e_2, e_3, e_4) = (2, 2, 3, 5)$. One can check that

$$((2\ 3), (4\ 5), (1\ 3\ 5), (5\ 4\ 3\ 2\ 1))$$

is a genus-0 pure-cycle Hurwitz factorization.
(Genus-0: $2d - 2 = 8 = \sum_{i=1}^{4} (e_i - 1) = 1 + 1 + 2 + 4$.)

Previous results on the pure-cycle case

Lemma 4 (L-Osserman)**.** *In the genus-*0 *pure-cycle case, when* $r = 3$,

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Theorem 5 (L-Osserman)**.** *In the genus-*0 *pure-cycle case, when* $r = 4$,

$$h(d, 4, (e_1, e_2, e_3, e_4)) = \min\{e_i(d + 1 - e_i)\}$$

Hurwitz factorizations with a d-cycle

We study a special case of genus-0 pure-cycle Hurwitz factorizations: when one of the e_i is d. W.L.O.G, we assume $e_r = d$.

Then the "genus-0" condition becomes:

$$2d - 2 = \sum_{i=1}^{r} (e_i - 1) \quad \Rightarrow \quad \sum_{i=1}^{r-1} (e_i - 1) = d - 1.$$

Since σ_r is a d-cycle, $\langle \sigma_1, \ldots, \sigma_r \rangle$ is automatically transitive in $\mathfrak{S}_d$.

Moreover,

$$\sigma_1 \ldots \sigma_r = 1 \quad \Leftrightarrow \quad \sigma_1 \ldots \sigma_{r-1} = \sigma_r^{-1}.$$

Factorizations of a d-cycle

Definition 6. Assume $d, r \geq 1, e_1, \dots, e_{r-1} \geq 2$ are integers satisfying $\sum_{i=1}^{r-1}(e_i - 1) = d - 1$. Fix a d-cycle $\tau \in \mathfrak{S}_d$, We say $(\sigma_1, \dots, \sigma_{r-1})$ is a *factorization of τ of type $(e_1, \dots, e_{r-1})$* if the followings are satisfied:

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It is clear that

$$h(d, r; (e_1, \dots, e_{r-1}, e_r = d)) = \frac{1}{d} \mathrm{fac}(d, r, (e_1, \dots, e_{r-1}))$$

The Formula

Goulden and Jackson considered more general factorizations of a d-cycle, where they allow σ_i to be any cycle type, that is, σ_i does not have to be a cycle. They gave a formula for the factorization number in this situation. Specializing their formula to the pure-cycle case gives the following formula.

Theorem 9. *Suppose* $\sum_{i=1}^{r-1}(e_i - 1) = d - 1$. *Then*

$$\mathrm{fac}(d, r, (e_1, \ldots, e_{r-1})) = d^{r-2}$$

Hence,

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An equivalent *symmetrized* version of Theorem 9 was proved by Springer and Irving separately: e.g., when $(e_1, e_2, e_3) = (2, 2, 3)$, we only allow factorizations where the first and second cycles have length 2 and the third cycle has length 3. They included all factorizations with one 3-cycle and two 2-cycles.

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We construct a class of combinatorial objects that are counted by d^{r-2}, and then describe a bijection between factorizations and these objects.

PART II:

Multi-noded Rooted Trees

Definition of Multi-noded Rooted Trees

Definition 10. Suppose $f_0, f_1, \ldots, f_n$ are positive integers and $S = \{s_1, \ldots, s_n\}$. We say G is a *multi-noded rooted tree* on $S \cup \{0\}$ *of vertex data* $(f_0, f_1, \ldots, f_n)$ if we have the followings:

(i) The vertex set of G is $S \cup \{0\}$.

(ii) For each vertex s_i, it includes f_i ordered nodes (by convention, $s_0 := 0$).

(iii) Considering only vertices and edges, G is a rooted tree with root 0, but in addition each edge is connected to a particular node of the parent vertex.

We denote by $\mathcal{M}R_S(f_0, f_1, \ldots, f_n)$ the set of multi-noded rooted trees.

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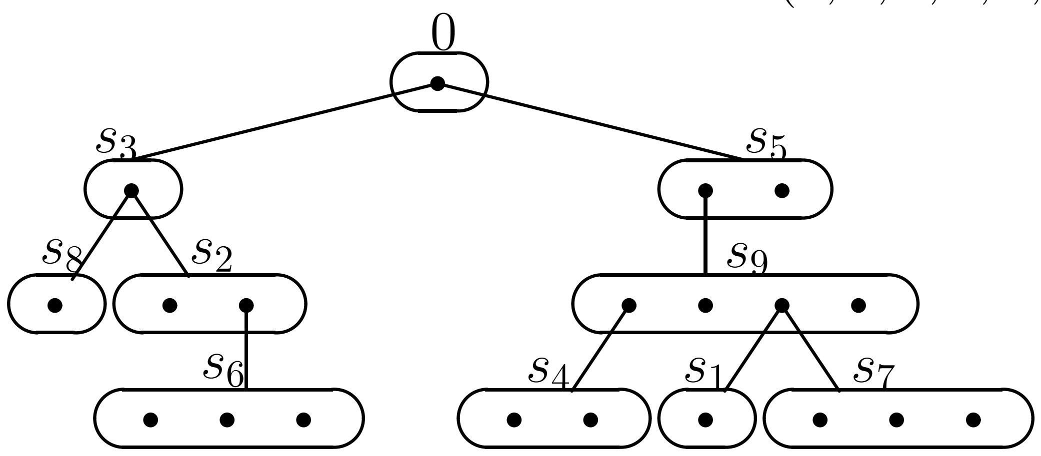

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Theorem 12. $|\mathcal{M}R_S(f_0, f_1, \dots, f_n)| = f_0 \left(\sum_{i=0}^n f_i\right)^{n-1}.$

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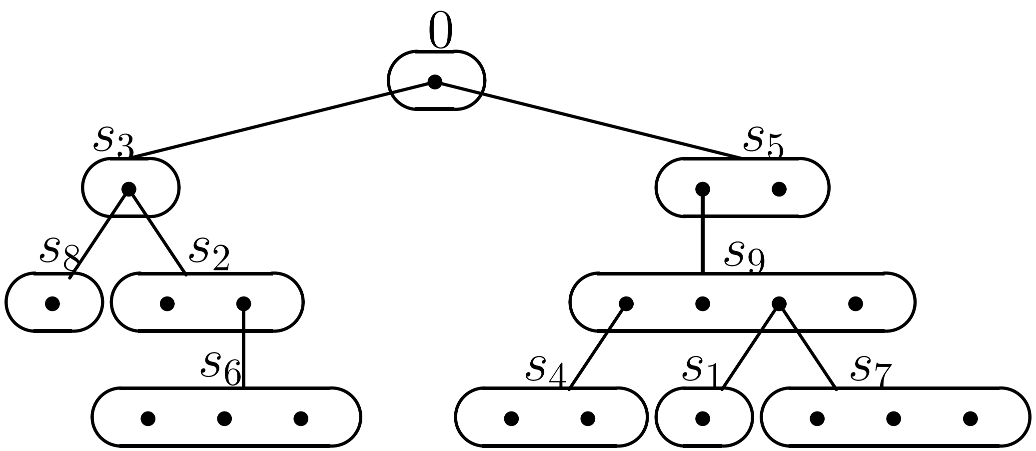

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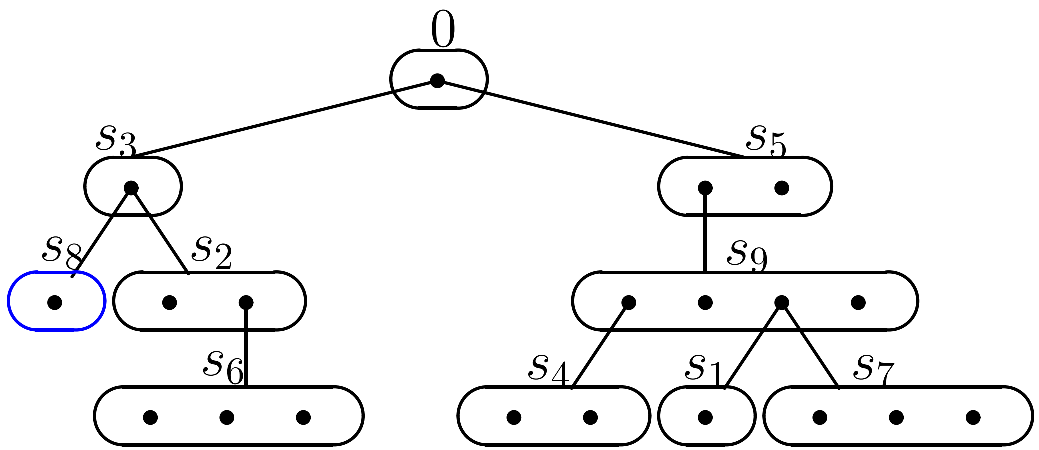

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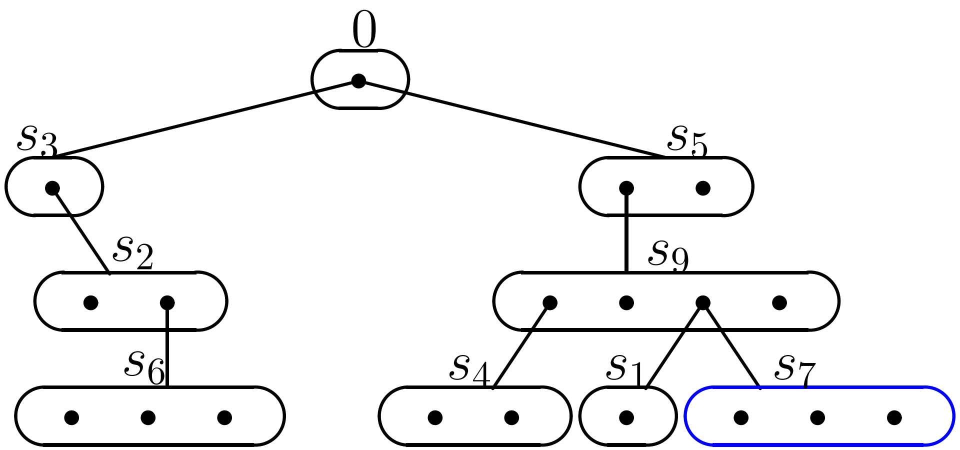

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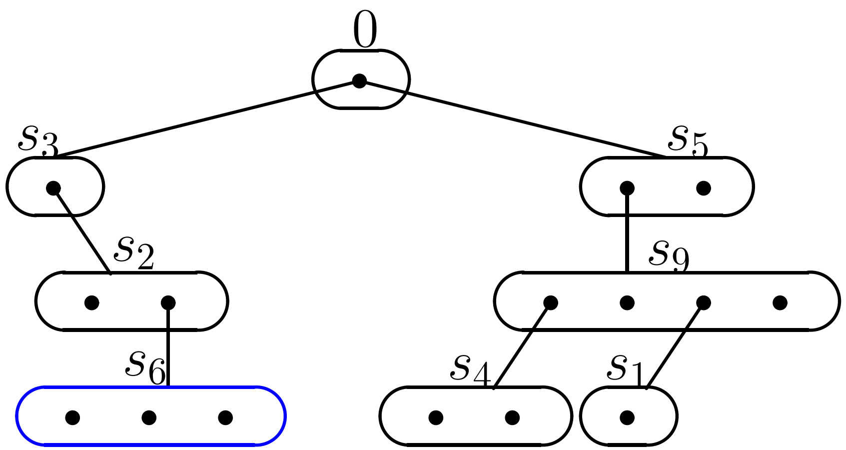

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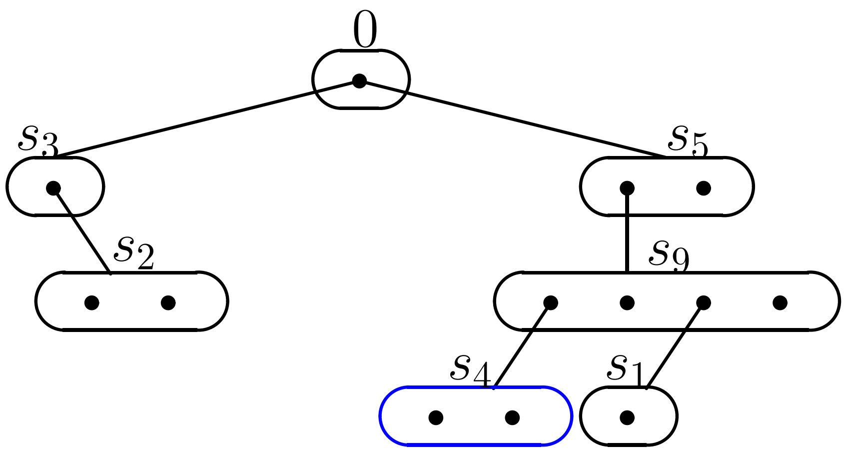

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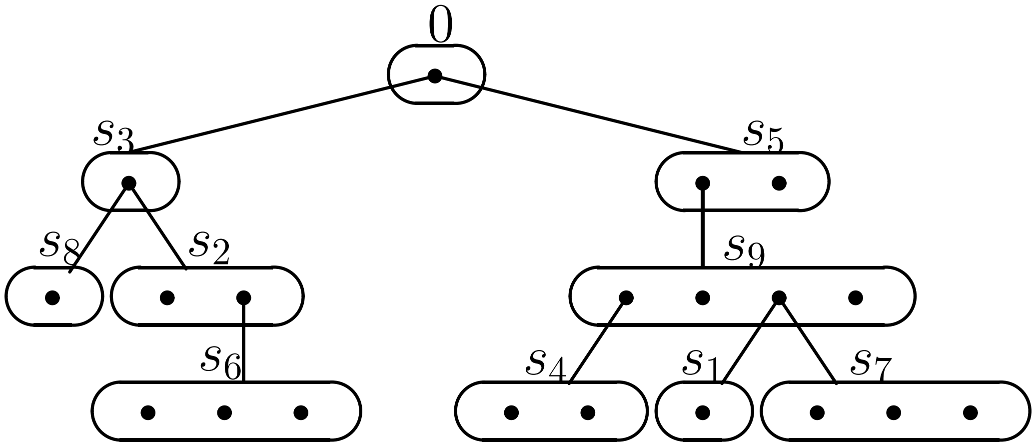

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PART III:

Bijection between Factorizations and Multi-noded Rooted Trees

Factorization Graphs

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A factorization of $\tau = (1\ 2\ \cdots\ 20)$ of type $(2, 3, 2, 3, 3, 4, 4, 2, 5)$:

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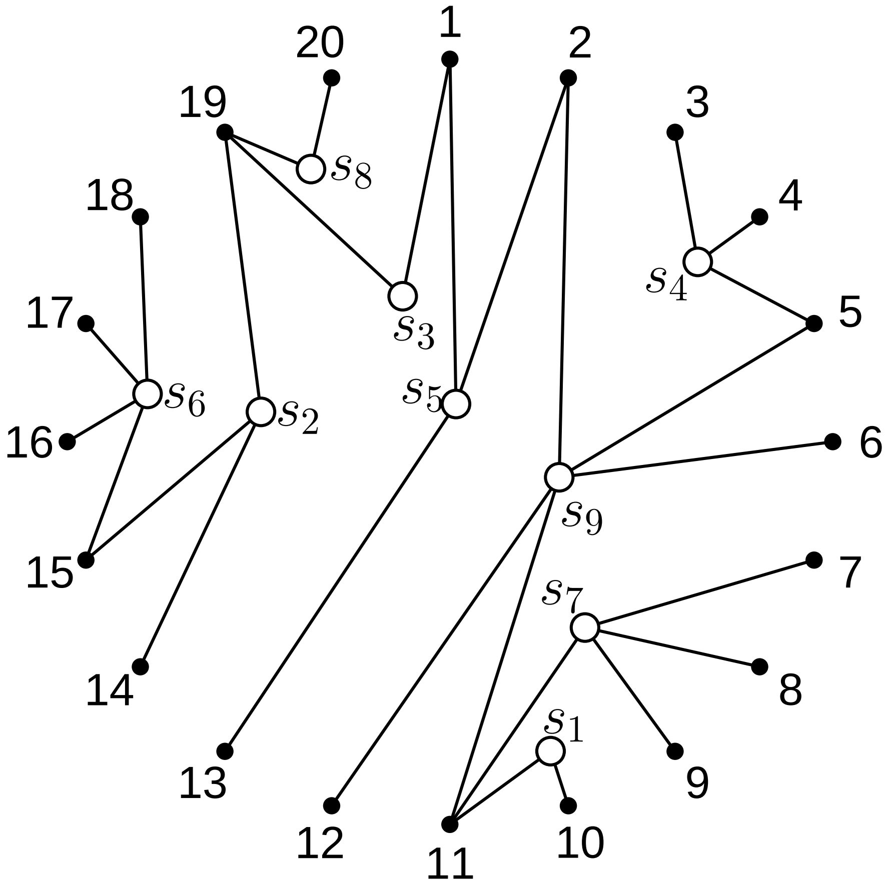

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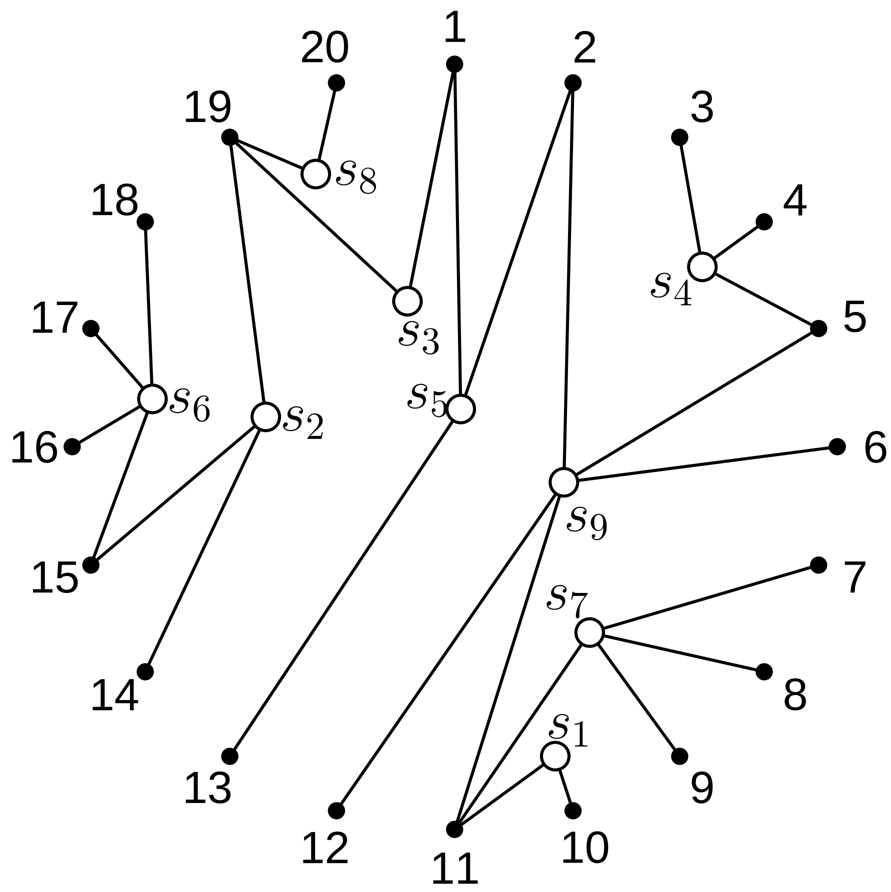

Facts:

1. G is a bipartite graph on $S \cup [d]$.

2. Any vertex s_i has degree e_i.

Characterization of factorization graphs

Proposition 14. *Suppose* $\sum_{j=1}^{r-1}(e_j - 1) = d - 1$, *and* G *is a bipartite graph on* $S \cup [d]$ *such that vertex* s_i *has degree* e_i.

Then G *is a factorization graph associated to a factorization of* τ *of type* $(e_1, \dots, e_{r-1})$ *if and only if* G *satisfies the following conditions:*

i. G *is a tree.*

ii. For each $[d]$*-vertex* ν *of* G, *suppose* $\{s_{j_1} < s_{j_2} < \cdots < s_{j_t}\}$ *are the vertices adjacent to* ν *in* G. *We get* t *subtrees after removing* ν *and all its incident edges. Then*

(a) The $[d]$*-vertices of the* t *subtrees partition* $[d] \setminus \{\nu\}$ *into contiguous pieces.*

(b) If we order the pieces in counterclockwise order on τ *starting from* ν, *then the* m*-th piece is exactly the subtree that contains vertex* s_{j_m} *for any* $1 \leq m \leq t$.

Factorization Graphs to Labeled Multi-noded Rooted Trees

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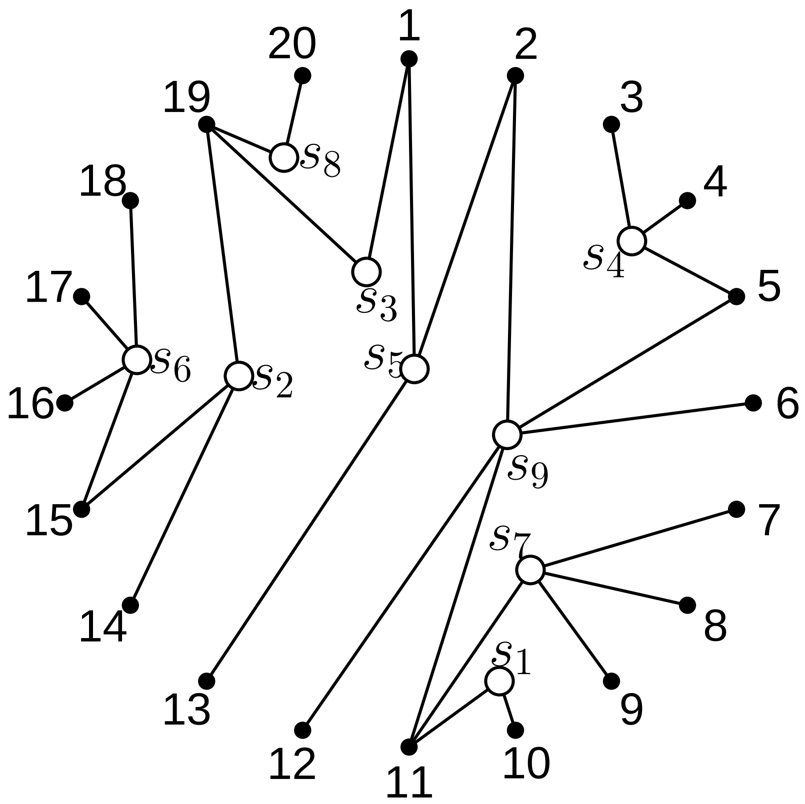

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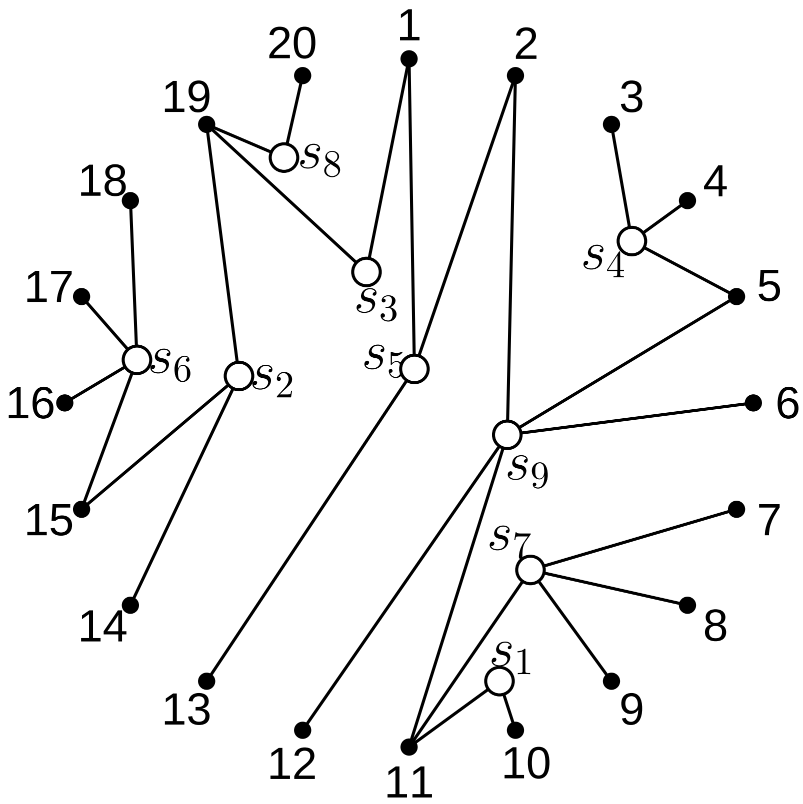

A labelled multi-noded rooted tree of vertex data $(1, 1, 2, 1, 2, 2, 3, 3, 1, 4)$