Math 108, Fall 2013.

## Some challenging logical puzzles

This is a collection of well-known puzzles related to mathematical logic and information. To solve these problems, you will need to use such devices as the *law of excluded middle* (every statement is either true or false), *proof by contradiction* (when you try to prove something, you may assume its converse, in addition to stated assumptions, and deduce a contradiction) and *mathematical induction* (you can prove that natural numbers have property **P** by checking that the number 1 has this property, and that if $n$ has property **P**, then $n+1$ must have it as well). We will learn these principles in the course.

1. A politician comes to a press conference, reads the following list of statements, then leaves.

#1. I have never taken a bribe.
#2. Exactly two statements on this list are false.
#3. Exactly three statements on this list are false.

Using the law of excluded middle, determine which statements are true. Then devise a similar list of 10 statements with a similar unique solution.

2. A certain island contains two tribes: Liars (every statement they utter is false) and Truthtellers (every statement they utter is true).

The classic puzzle finds you traveling on the island, and arriving at a fork in the road. The two roads you can take are labeled A and B, and exactly one of them leads to your destination. There is a native coming down the road, and you may ask him a single yes-no question that will determine which road is correct. What do you ask?

Recent immigration has created a third tribe, Normals (their statements can be true or false). A crime is committed and three suspects A, B, and C are brought in for questioning. It is known that exactly one of them committed the crime.

(a) It becomes known that one of the three is Normal, one Liar, one Truthteller. Determine which is which on the basis of the following three statements they give:

A: I am a Normal.
B: The above A's statement is correct.
C: I am not a Normal.

(b) In a different situation, further evidence surfaces that a Truthteller is the criminal, and that there is exactly one Truthteller among the three suspects. This time the statements are:

A: I am innocent.
B: The above A's statement is correct.
C: B is not a Normal.

Who is the criminal?

The author of these puzzle is the famous logician Raymond Smullyan. If you enjoy solving such problems, check out some of his books, such as *What Is the Name of This Book?: The Riddle of Dracula and Other Logical Puzzles*.)

3. This is the famous *unexpected hanging paradox*. A prisoner is sentenced to be hanged. "The hanging will take place at noon," adds the judge, "on one of the seven days, Monday through Sunday, of next week. But you will not know which day it is until you are so informed on the day of the hanging." This is how the prisoner reasons: "They can't hang me on Sunday, since in this case I will be able to predict this on Saturday afternoon. If they hang me on Saturday, I'll be able to predict this on Friday afternoon, so Saturday is ruled out too. So is Friday, by the same logic. So are Thursday, Wednesday, Tuesday and, finally, Monday. They can't hang me at all!" Yet the execution squad arrives on Thursday and catches the prisoner completely by surprise, vindicating the judge's word. Explain.

This story has generated hundreds of scientific papers, mostly in the field of philosophy. If interested, read a review in *American Mathematical Monthly* 105 (1998), 41–51.

4. A Zen master devises the following test as the final exam for his 50 disciples. In the evening, before they retire to their cells, he tells them each of their foreheads will be painted either black or white during the night. During the meditation on any of the following days, they will be able to see the others' foreheads, but not their own. There are no reflecting surfaces in the monastery, and the disciples have to observe a vow of silence during the exam. If a disciple is able to logically deduce the color of his forehead during a particular day, he passes the exam and can leave the monastery that night.

The master decides to paint all foreheads white. Next morning, at the beginning of meditation, the master also divulges the following apparently useless piece of information: "There is at least one forehead painted white." Or is it really useless? What if only one forehead were painted white? Or only two? How long will the exam go on?

Next year, the master has just two disciples, and as a final exam he tells one of them (in secret) "your number is 13," and the other "your number is 14." This time, if a disciple is able to deduce the other's number during a particular day, he passes the exam and can leave the monastery that night. The vow of silence is still in effect. Next morning, during the meditation, the master says: "I've given you two consecutive integers, each at least 1." How long will the exam go on this time?

5. A band of $n$ pirates has just robbed a ship on an open sea. The loot consists of 10 valuable gold plates. The pirates are ranked $P_1, P_2, \ldots, P_n$ in increasing rank. This is how they divide the spoils (the individual plates are indivisible). The highest ranked pirate proposes a division, which is then voted on by all $n$ of them. If at least $n/2$ approve, the division is accepted and the story ends. Otherwise, $P_n$ is thrown overboard, and the procedure is repeated with $n-1$ pirates. This is how each pirate ranks his preferences, in decreasing order: getting one or more plates (the more the better), seeing the spectacle of somebody being thrown overboard, getting nothing, being thrown overboard.

If $n = 2$, $P_2$ clearly takes all 10 plates, as his vote alone assures the plan is approved. If

$n = 3$, $P_3$ has to give $P_1$ a plate, hence assuring his vote ($P_1$ knows he gets nothing if $P_3$ is thrown overboard), so the split is (1,0,9) in this case. Determine the split for *every* $n \geq 4$.

6. This is the famous *prisoners and the light bulb* puzzle. The version here is an adaptation from the *Car Talk* radio show, and is a popular job interview problem.

A warden meets with $n$ new prisoners when they arrive. He tells them:

> "You may meet today and plan a strategy. But after today, you will be in isolated cells and will have no communication with one another. In the prison is a switch room, which contains a light switch, which can turn the single light bulb in the room on or off. I am not telling you its present position. After today, from time to time whenever I feel so inclined, I will select one prisoner at random and escort him to the switch room. This prisoner may, but is not obligated to, reverse the position of the switch. Then he'll be led back to his cell. No one else will enter the switch room until I lead the next prisoner there, and he'll be instructed to do the same thing. I'm going to choose prisoners at random. However, this is not a probability problem — I'm only assuring you that, given enough time, every one of you will eventually visit the switch room arbitrarily many times. At any time anyone of you may declare to me, 'We have all visited the switch room.' If it is true, then you will all be set free. If it is false, and somebody has not yet visited the switch room, you will be fed to the alligators."

What is the strategy the prisoners devise?

7. The Zen master has $n$ disciples, who are painted on their foreheads positive numbers $a_1, \ldots, a_n$. The master also publicly announces the numbers $s_1 < \cdots < s_k$, for some $k \leq n$, and tells the disciples that an $s_i$ is the true sum of their numbers. For simplicity, you may assume that $a_i \in \mathbb{N}$. Whenever the first disciple knows his number (or, equivalently, the true sum), the exam is over. What will happen? What may happen if $k > n$?

8. Indiana Jones arrives at the ancient temple of the Sphinx. He hears a disembodied voice say: "You see in from of you 4 vases, each of which either contains a key or a deadly poisonous snake. The keys, if any, in the vases are needed for the admission into the temple." "How can I figure out which vases contain the keys," nervously asks Indiana, with bad guys at his heels and his well-known aversion to snakes. "You can ask me any number of yes-no questions, but I am permitted to lie to you at most once," answers the voice. Help Indiana get into the temple with the fewest number of questions. Then generalize to an arbitrary number $n$ of vases.

9. Somebody has picked $n$ integers $0 \leq a_1 < a_2 < \cdots < a_n$. These are unknown to you. The information you get are all $\binom{n}{2}$ sums of these numbers (in, say, increasing order, so you are *not* told which two numbers produce which sum). Can you determine the unknown numbers?

This problem demonstrates the value of selecting the proper tool. It is a very difficult problem to solve by brute force.

10. Another famous problem which has a nice solution when interpreted properly is the following.

You have $n$ lamps and each lamp has a switch. There also are several connections between pairs of lamps. These connections are completely known to you, work the same in both directions, and have the following effect. If you flip a switch on a lamp, you change its state (from on to off or vice versa) and also the state of all lamps connected to it. Initially, all lamps are turned off. Prove that, regardless of the connectivity structure, you can find a sequence of flips that turns all the lights on.

Note that it is important that all the lights are initially off. Several versions of this puzzle exist (and were even sold) as a game, in which you are given a connectivity structure and a configuration of lamp states (some on and some off), and you need to figure out how to turn all of them on.

**Solutions.**

1. Statement #3 *must* be incorrect, since its correctness implies it's incorrect. Now, there could be 0, 1, 2, or 3 incorrect statements, but we know 0 is impossible and so is 3 (which would imply correctness of statement #3). If one statement is incorrect, both #1 and #3 are incorrect, a contradiction again. The only possibility left is that two statements are incorrect, #1 and #3, while #2 is correct.

For the list of 10 statements, you can use the same first statement together these 9 statements:

> #$i$. Exactly $i-1$ statements on this list are false. Here, $i = 2, \ldots, 8$, defining 7 statements.
> #9. Exactly 9 statements on this list are false.
> #10. Exactly 10 statements on this list are false.

The logic then is as follows. As before, statement #10 must be false. Further, there must be at least 8 incorrect statements, as at most one of last 9 is correct. One possibility, 10 incorrect statements, is contradictory as before, since it would imply correctness of the statement #10. If 8 statements are incorrect, all 9 statements #2,..., #10 are incorrect, a contradiction. So, 9 incorrect statements is the only possibility, with statement #9 the only correct one.

Note that the *law of excluded middle*, that is, that every statement can only be either true or false is a very important assumption here. It's of course very easy to make a list of statements for which this law does not hold, such as this single statement

> #1. Statement #1 is false.

This statement is self–contradictory, or pure nonsense in layman's terms.

2. The most commonly given answer to the first question is this: If the member of the tribe you *do not belong to* were asked "Is A the correct road?" what would be the answer? Then the answer a Truthteller gives is no exactly when A *is the correct road*, and the same is true for a Liar.

For a solution that does not need a reference to the other tribe, consider the following two statements:

> A: A is the correct road.
> T: The native is a Truthteller.

The question you ask him is: "Is A the correct road if and only if you are a Truthteller?" (In shorthand: "Is $A \Leftrightarrow T$ true?") If T is correct then he will answer yes if A is true and no otherwise. If T is false, then $A \Leftrightarrow T$ is is false if and only if A is true, in which case the Liar will answer yes. Therefore a yes answer is always given if and only if A is true.

(a) If C is not telling the truth, he is a Normal, and so A is a Liar and B a Truthteller, which is impossible. Therefore, C is telling the truth. Then he can only be a Truthteller. If A is a Normal, then B is a Liar, which means that A's statement is incorrect and so A is not a Normal, a contradiction. Therefore, A is not a Normal, thus he is a Liar, and so B is Normal and his statement is incorrect.

(b) Immediately, A cannot be a Truthteller, because if he is, his statement is incorrect, a contradiction. It follows that C cannot be lying, since otherwise B would have to be a Normal, and A would have to be the only Truthteller. Therefore C is telling the truth, and B could either be a Truthteller or a Liar. If B is a Liar, then A is guilty, so a Truthteller, a contradiction. So, B is a Truthteller and the criminal, while A and C are Normals.

5. Here is my take on the resolution of this paradox; bear in mind that the confusion is based mostly on the meaning of words and is hard to resolve. The judge has issued a statement whose validity leads to a contradiction. *Anything at all can be deduced* assuming its validity, including no hanging. The conclusion is that it is a logically incorrect statement. The extreme version would be:

> #1 You will be hanged tomorrow. #2 You do not have enough information to know whether you will be hanged tomorrow.

Assuming both #1 and #2 leads to an immediate contradiction. So the either #1 or #2 must be false and you have been given no information whatsoever. Imagine that the judge flips a coin and says (without looking at the outcome):

> #1 The outcome is Tails. #2 You do not know whether the outcome is Tails.

Of course you are given no information (and this time you *cannot* be given any), but there is a 50% chance that the judge's word is "validated" after he looks at the outcome.

4. To answer the first question, assume that $i$ foreheads were painted white, $1 \leq i \leq 50$. If $i = 1$, then the one painted white would conclude the color of his forehead during the first day and leave, the second day all others would conclude their colors (since they know that the white would not be able to deduce his color unless they were all black) and then leave. If $i = 2$, then on the second day both whites would conclude their colors and leave, etc. The induction hypothesis then is that on the $i$'th day all whites will deduce their color and leave, and then the same will happen with blacks the next day. We have checked that this is true for $i = 1$. To see the $i \to i + 1$ implication, look at the situation at the dawn of $(i + 1)$'st day. Nobody has left. Therefore, all white disciples deduce that the number of whites is at least $i + 1$, but they see $i$ whites, therefore they deduce their color. Next day, all blacks conclude that the number of whites was at most $i + 1$, hence they deduce their color as well. It follows by induction that all disciples leave on the evening of day 50. The fact that the information is useless for others may be useful to you!

The second question is answered similarly. Assume that the integers are $i, i+1$. If $i = 1$, then the 2–disciple can leave the first night, and the 1–disciple leaves the next night. The hypothesis now is: the $(i + 1)$–disciple leaves after the $i$'th day, and the $i$–disciple the following night. To do the $i \to i + 1$ step, consider the situation on the day $(i + 1)$. Neither of them has left, so the $(i + 1)$–disciple knows that his number is *not* $i - 1$! Therefore he leaves that night, so next morning the $i$–disciple knows that his number has to be $i$ rather than $i + 2$. The 14–disciple leaves after day 13, the following day the 13–disciple leaves.

5. The solution proceeds by adding the highest ranked pirate to the group and inductively using the result for the smaller number of pirates.

After we have figured out the $n = 3$ case, let us assume that $P_4$ enters the scene. He can assure the vote of $P_2$ by giving him one plate (if $P_2$ votes against him, he gets nothing), and that (together with his vote) is enough: the split is $(0, 1, 0, 9)$. Then $P_5$ must offer two gold plates, one each to $P_1$ and $P_3$. And so on up to $P_{20}$ who will give one plate to each of the even numbered pirates $P_2, \ldots, P20$.

What next? $P_{21}$ can still prevent being thrown overboard by giving away all the plates to odd pirates $P1, \ldots, P_{19}$. Then $P_{22}$ must bribe 10 of the 11 even pirates who get nothing under the previous scheme and still saves himself. However, $P_{23}$ is in an impossible position, because he cannot assure the votes of 11 pirates. So he goes overboard. Now $P_{24}$ knows he can be sure of the vote of $P_{23}$, no matter what he proposes, so he can just save himself by assuring the vote of himself, $P_{23}$, and 10 bribed colleagues (among odd pirates $P_1, \ldots, P21$). $P_{25}$ is then lost, as he cannot count on either $P_{24}$ or $P_{23}$ and he can get at most 11 votes. So is $P_{26}$, because the vote of $P_{25}$ plus his plus 10 other votes is still insufficient. Same fate awaits $P_{27}$. However, $P_{28}$ can count on himself, $P_{25}, P_{26}, P_{27}$ and 10 others, just enough.

And so on. The pirates who can save themselves are those numbered $20 + 2^n$, for $n = 0, 1, 2, \ldots,$ while others get thrown overboard. The division of plates alternates between even pirates ($n$ odd) and odd ones ($n$ even).

6. This solution is adapted from the *Car Talk* site. The prisoners all meet, and the leader of the prisoners says, "Okay, guys, here's our strategy. I am the only one of us who can count past two, and I'll be the one responsible for telling the warden we've all been in the switch room when the time comes."

He then proceeds to give instructions to the other inmates. Each of the $n - 1$ prisoners is told, "When you go into the switch room, I want you to turn the light off. If it is already off, then leave it there, and walk out." All the prisoners nod. "Each of you turns the light off *twice, and only twice*. So if you go in there and the light is already off, that doesn't count. I want each of you to actually turn it off two times. You got that?"

One of prisoners asks, "Who's going to be turning the light on?"

"Good question," says the leader. "I am the only one with the authority to turn the light on."

Why does this work?

Each time the leader is taken into the switch room, finds the light off, he knows that at least one prisoner has been in there. After $2n - 3$ dark revisits, he knows that everybody has been there. If one prisoner, say Bob, has not been there, then the light will not have been turned off more than $2(n - 2)$ times. On the other hand, the light will be eventually turned off $2(n - 1)$ times, and only one of these (namely, the first prisoner switching the light off) can fail to cause a dark revisit by the leader.

7. Write down all possible vectors, that is, $n$-tuples of numbers, with the displayed sums. There are finitely many of them. If nobody leaves on the first day, a few will be crossed off (namely, those that have any of their coordinates $\geq s_{k-1}$). In other words, you can cross off any vectors $x$ which have a coordinate $x_i$ such that no other vector $y$ still in play has $y_j = x_j$ for $j \neq i$. At the next step, you repeat this procedure with a smaller set of vectors and the question is whether it is possible to get stuck, that is, to arrive at an *ambiguous* set of vectors $S$, with the property

that every $x \in S$, and every $i$, has a $y \in S$, $y \neq x$, such that $y_j = x_j$ for $j \neq i$. The following lemma excludes this possibility.

**Lemma.** *Take an ambiguous set $S$ of $n$–vectors and the set $A$ of all coordinate sums of vectors in $S$. Then $A$ has at least $n+1$ elements.*

*Proof.* We prove this by induction. The statement is clearly true for $n = 1$, as $S$ must contain at least 2 numbers to be ambiguous. Now take a vector $x$ with the lowest coordinate of all members in $S$. Without loss of generality we can assume this is the first coordinate $x_1$. The set $S'$, consisting of all $(n-1)$–vectors $y'$ with $(x_1, y') \in S$ is ambiguous. Therefore, by the induction hypothesis, the resulting set of sums $A'$ must contain at least $n$ elements. Let $y'_0 \in S'$ have the largest such sum. By ambiguity, there is a $y_1 > x_1$ so that $(y_1, y'_0) \in S$. Now we are done, because the sum of $(y_1, y'_0)$ is clearly strictly larger that the sum of any $(x_0, y')$ for any $y' \in S'$, and so $A$ contains at least $n+1$ numbers. $\square$

If $k > n$, this analysis breaks down. This is easily seen when $n = 1$, but if this is too trivial, consider a $n = 2$ case, and the sums $1, 2, 3$. Then possible ordered pairs are $(0, 3)$, $(0, 2)$, $(0, 1)$, $(1, 2)$, $(1, 1)$. The first step eliminates $(0, 3)$ (there's no other 3 – this means that both $(0, 3)$ and $(3, 0)$ get eliminated). But now no number occurs uniquely and we are stuck. For example, if the numbers painted on students' foreheads are 1 and 2, there's no way to end the exam.

8. With $n$ vases, the number of questions is at least the smallest $k$ for which $(k+1)2^n \leq 2^k$, i.e., $2^{k-n} \geq k+1$. Why? The amount of different states between which one can distinguish by $k$ binary questions is at most $2^k$. But, the number of states with $k$ question is the number of possible contents of vases $(2^n)$ times the number of possible locations of a lie $(k+1)$.

Here is how one devises the questions. Fix an integer $q$, to be determined later. First ask $q$ questions on the contents of the first $q$ vases.

The $(q+1)$–st question is: "Have you lied so far?" If the answer is no, this is a correct answer (otherwise it would be the second lie), and you are faced with the same problem with $n-q$ vases, $k-q-1$ questions. If the answer is yes, it is not necessarily correct, but you know that all the answers from now on are correct. So you have to identify a possible lie among the first $q$ answers and the contents of $n-q$ vases, which means that $2^{k-q-1} \geq (q+1)2^{n-q}$, that is, $2^{k-n-1} \geq (q+1)$.

The rest is proved by induction. We only have to show that there is a choice of $q$ such that $2^{k-n-1} \geq k-q$ and $2^{k-n-1} \geq (q+1)$. These two inequalities can be satisfied if $k - 2^{k-n-1} \leq 2^{k-n-1} - 1$, exactly the inequality we have.

So, seven questions suffice in the case of 4 vases. (Note that $q = 3$ in this case.)

9. Assume that there exist numbers $0 \leq b_1 < \cdots < b_n$ with the same pairwise sums. The trick is to consider $p(z) = z^{a_1} + \cdots + z^{a_n}$ and $q(z) = z^{b_1} + \cdots + z^{b_n}$. Then $p(z)^2 - q(z)^2 = p(z^2) - q(z^2)$, as the cross terms cancel. Moreover, as $p(1) - q(1) = 0$, there exist an integer $k \geq 1$ and a polynomial $r$ with degree between 0 and $n-1$ so that $p(z) - q(z) = (z-1)^k r(z)$ and $r(1) \neq 0$. This way we get

$$(z-1)^k r(z)(p(z) + q(z)) = (z^2 - 1)^k r(z^2)$$

and then

$$r(z)(p(z) + q(z)) = (z+1)^k r(z^2).$$

Plug in $z = 1$ and cancel $r(1)$ to get $2n = 2^k$, $n = 2^{k-1}$ so $n$ is a power of 2! The numbers can be reconstructed whenever $n$ is not a power of 2.

Now for $n = 2$, the numbers obviously cannot be reconstructed, e.g., $1, 2$ and $0, 3$. Neither can they when $n = 4$, e.g., $0, 3, 5, 6$ and $1, 2, 4, 7$. In general, given $a$'s and $b's$ for $n = 2^k$, construct $2^{k+1}$ numbers $a'$ and $b'$ as follows: $a'_i = a_i$ for $i \leq 2^k$ and $a'_i = 2^k + b_{i-2^k}$ for $i > 2^k$, and $b'_i = b_i$ for $i \leq 2^k$ and $b'_i = 2^k + a_{i-2^k}$ for $i > 2^k$.

The claim is that the pairwise sums are the same, so we need to check that every $a'_i + a'_j$ equals to a pairwise sums of $(b')$'s. This is true by induction hypothesis when $i$ and $j$ are either both larger or both smaller than $2^n$. If exactly one of $i$ and $j$ is larger than $2^n$ the claim is trivially true.

The conclusion therefore is that the number can be reconstructed exactly $n$ is not a power of 2.

10. Let $G$ be the connectivity graph, with self-loops at every vertex, reflecting the fact that the switch affects its lamp. Think of the on/off states of the lamps as $n$-dimensional vectors over the field $\mathbb{Z}_2$ with two elements, with the off state represented by 0. For each lamp $r$, write $v_r$ for the vector which has a 1 in position $s$ if the switch of lamp $r$ affects lamp $s$. (In graph theory terminology, $v_r$'s are row or column vectors of the adjacency matrix of $G$. This matrix has 1's on its main diagonal!) The achievable states from all 0's form a vector subspace, specifically the span of the $v_r$. (To see this, note that it does not matter in which order you flip the switches you decide to flip and the resulting state is exactly a linear combination of the corresponding $v_r$.) If the all 1's vector is not in this subspace, then there is some vector $w$ which is orthogonal to all the $v_r$, but not to the all 1's vector. Let $H$ be the induced subgraph of $G$ whose vertices are the rooms which have ones in the corresponding coordinates of $w$ (and edges still given by the connections).

We will now *erase the self-loops* in $H$ and see what can we say about degrees of vertices of $H$. First, for each $r \in H$, $\langle w, v_r \rangle$ equals one plus the degree of $r$ in $H$, but this inner product is 0 by the above construction, so all degrees are odd. Furthermore, the non-zero inner product of $w$ with the all ones vector of course gives the number of vertices in $H$, so this number is also odd. This is impossible, as the sum of degrees in every graph is twice the number of edges, thus even.