Math 108, Fall 2013.

**Writing Tips for Math 108**

Advanced mathematics is primarily based on ideas rather than computations. Communication of ideas needs words and symbols, carefully defined and organized in a coherent narrative. Clarity and precision of written arguments are integral parts of mathematics. Here are a few basic principles:

- A clearly and carefully written argument is much more likely correct than a messy one. Often, a mistake is discovered during writing. "Nothing is proved until written down" is the edict of responsibility for a working mathematician.
- A proof is *not* written for a friendly audience that is willing to give the writer the benefit of the doubt. The reader is an *adversary* that will classify every mistake as fatal and every missing step as evidence that the argument is incomplete.
- While reading the proof, the reader should not be hampered by bad English (e.g., lack of punctuation, incomplete sentences, colloquialisms), stylistic quirks (e.g., page-long sentences), bad handwriting, or poor organization on a page. Faced with such obstacles, the reader is apt to give up and reject the argument as incoherent.
- Everything written down needs to be *correct*, with statements taken *literally* as written. For example, the statement

  Every prime number is odd.

  is false despite being "essentially true," i.e., true for all but one prime. A much worse mistake, likely to be judged very harshly, is this example of carelessness:

  There exists a prime $p$ so that, for every natural number $n$, $p > n$.

  which is a misstatement of:

  For every natural number $n$, there exists a prime $p$ so that $p > n$.

- Every statement needs to be clear and concise. Judiciously chosen and carefully defined notation can be of considerable help. For example,

  Every natural number has a prime between itself and its double.

  is much less clear than

  For every natural number $n$, there is a prime in $(n, 2n]$.

Here is a good example (due to M. Tomforde):

> It is maximum or minimum when it is zero of its derivative.

This sentence should be rewritten as follows:

> If a differentiable function $f$ has a maximum or a minimum at a point $x$, then $f'(x) = 0$.

The commonly used symbol for the end of proof is $\square$, which eliminates the need to say "The proof is now finished." Similarly, one should not begin the proof with

> *Proof*. We now prove the theorem. Let $\epsilon > 0$. Etc.

but rather

> *Proof*. Let $\epsilon > 0$. Etc.

At the end, your argument should *read like* it was easy to write, even if a lot of work is needed to achieve this effect.

- There should be no unnecessary notation and shorthand notation such as $\Rightarrow$ should be avoided. The text should be divided into sentences; *every sentence begins with an English word (that is, not with a symbol) and ends with a period*. For example, this is *not* an improvement upon the formulation in the previous item:

  > $P \subset \mathbb{N}$ is the set of primes. $(\forall n \in \mathbb{N})(\exists p \in P)(p \in (n, 2n])$

  Of course, logical symbols have their place when the subject is mathematical logic.

- A well-written piece of mathematics is almost always the result of many rewrites. A rewrite is an opportunity to check that all notation is defined, that the conclusion is proved, and that the hypotheses are used. It is also a chance to verify that there there are no parts that can be shortened or eliminated, that is, to take the advice of W. Strunk and "make every word tell."

Here are two examples of simple proofs, with some parenthetical comments. These also illustrate two of the most common strategies: *direct proof*, in which the conclusion is established by direct use of hypotheses and possibly earlier theorems; and *proof by exhaustion* (also known as *proof by cases*) in which the proof reduces to analysis of finitely many cases. We will learn additional techniques in this course.

**Statement**. Two natural numbers of different parity sum to an odd number.

*Proof*. Assume that $m$ is an odd number and that $n$ is an even number. We need to show that $m + n$ is odd. [*Use of the first person plural is standard*.] By the hypothesis, there exist an integer $i > 0$ so that $n = 2i$ and another integer $j \geq 0$ so that $m = 2j + 1$. Then

$$m + n = 2i + 2j + 1 = 2(i + j) + 1,$$

which is an odd number. [*Any important, or long, formula should be displayed*.] $\square$

**Statement**. Call a pair $(x, y)$ of natural numbers *good* if the equation $x^2 = y^2+2008$ is satisfied. There exist exactly two good pairs.

*Proof*. We rewrite the equation in the form $x^2 - y^2 = 2008$. Next, we write 2008 as a product of primes, $2008 = 251 \cdot 2^3$, and factor $x^2 - y^2 = (x - y)(x + y)$ to get

$$(x - y)(x + y) = 251 \cdot 2^3.$$

We will now see how the four prime factors of the right-hand side can be distributed to the two factors of the left-hand side. [*This sentence is not part of the formal argument, but helps the reader understand where the proof is going*.] As $x+y > x-y$, and 251 must divide exactly one of the two factors $x-y$ and $x+y$, we conclude that $251|(x + y)$. It remains to consider four possibilities for distributing the 2's.

Case 1: $x - y = 1$, $x + y = 251 \cdot 2^3 = 2008$.
We add the two equations to get $2x = 2009$, so $x$ is not a natural number and there is no solution in this case.

Case 2: $x - y = 2$, $x + y = 251 \cdot 2^2 = 1004$.
Now $2x = 1006$, so $x = 503$ and then $y = 501$.

Case 3: $x - y = 2^2 = 4$, $x + y = 251 \cdot 2 = 502$.
Now $2x = 506$, so $x = 253$ and then $y = 249$.

Case 4: $x - y = 2^3 = 8$, $x + y = 251$.
Now $2x = 259$, and so this is another case with no solution.

[*Format so that the division into cases is clear*.] We conclude that the only good pairs are $(253, 249)$ and $(503, 501)$. $\square$

For your first exercise, write down the proofs of the above two statements in your own words, without looking at (or trying to remember) the proofs written above. Make sure to follow the outlined principles, but expect that your style will differ from that of other people. It is useful to find another person to critique your writing: it is often difficult to spot your own shortcomings. Do not be alarmed if proof-writing seems difficult; this is the reason for this class!