Math 16A (LEC 004), Fall 2008.
December 9, 2008.

FINAL EXAM

NAME(print in CAPITAL letters, *first name first*): KEY

NAME(sign): ______________________

ID#: ______________________

Instructions: Each of the first four problems is worth 20 points, while problems 5 to 8 are each worth 30 points. Read each question carefully and answer it in the space provided. YOU MUST SHOW ALL YOUR WORK TO RECEIVE FULL CREDIT. Clarity of your solutions may be a factor in determining credit. Calculators, books or notes are not allowed.

Make sure that you have a total of 10 pages (including this one) with 8 problems. Read through the entire exam before beginning to work.

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8	
TOTAL	

1. Compute derivatives of the following two functions. *Do not simplify!*

(a) $y = \dfrac{\sqrt{x-4}}{3x-1}$

$$y' = \frac{\frac{1}{2}(x-4)^{-1/2}(3x-1) - \sqrt{x-4}\cdot 3}{(3x-1)^2}$$ 10

(b) $y = x^4 \cdot \cos(2x)$

$$y' = 4x^3 \cos(2x) - x^4 \sin(2x)\cdot 2$$ 10

2. Find the equation of the tangent line to the curve $(x+y)^4 + x^2y - y = 0$ at the point $(0, 1)$.

$$4(x+y)^3\left(1+\frac{dy}{dx}\right) + 2xy + x^2\frac{dy}{dx} - \frac{dy}{dx} = 0$$ 12

Plug in $x=0$, $y=1$:

$$4\cdot\left(1+\frac{dy}{dx}\right) - \frac{dy}{dx} = 0$$

$$4 + 3\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{4}{3} \leftarrow \text{slope}$$ 4

Line: $y - 1 = -\frac{4}{3}x$

$$y = -\frac{4}{3}x + 1$$ 4

3. Compute the following limits.

(a) $\lim_{h\to 0} \dfrac{(1+h)^{-4}-1}{h} =$

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = f'(x)$$

Let $f(x) = x^{-4}$, and $x = 1$. Then $f'(x) = -4x^{-5}$ which equals -4 at $x = 1$.

5 5

$= \underline{\underline{-4}}$.

(b) $\lim_{x\to 5} \dfrac{x-5}{x-\sqrt{4x+5}}$ $= \lim_{x\to 5} \dfrac{(x-5)(x+\sqrt{4x+5})}{(x-\sqrt{4x+5})(x+\sqrt{4x+5})}$ 3

$= \lim_{x\to 5} \dfrac{(x-5)(x+\sqrt{4x+5})}{x^2-4x-5}$ 3

$= \lim_{x\to 5} \dfrac{\cancel{(x-5)}(x+\sqrt{4x+5})}{\cancel{(x-5)}(x+1)} = \dfrac{5+\sqrt{25}}{6} = \underline{\underline{\dfrac{5}{3}}}$

3 1

4. Consider the function

$$f(x) = \begin{cases} x^4, & x < 0, \\ ax^2 + b, & 0 \le x \le 1, \\ x^3 + x, & x > 1. \end{cases}$$

(a) Determine the numbers a and b so that $y = f(x)$ is continuous for all x.

Cont. at 0: $\underline{\underline{0 = b}}$

Cont. at 1: $a + b = 2$; $\underline{\underline{a = 2}}$

10

(b) Assuming the values of a and b obtained in (a), determine all the values of x for which $y = f(x)$ is *not* differentiable.

At 0: $\frac{d}{dx}(x^4) = 4x^3$

$\frac{d}{dx}(2x^2) = 4x$

both are 0 at x = 0

5

At 1: $\frac{d}{dx}(2x^2) = 4x$

$\frac{d}{dx}(x^3 + x) = 3x^2 + 1$

both are 4 at x = 1

5

The function is differentiable <u>everywhere</u>.

5. Consider the function $f(x) = \dfrac{8(x-2)}{x^2} = 8x^{-1} - 16x^{-2}$.

(a) Determine the domain of $y = f(x)$, its intercepts, and horizontal and vertical asymptotes. Compute the left and right limits at vertical asymptotes.

Domain: $x \neq 0$,

Intercept: $(2, 0)$

H. a.: $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{8}{x} - \frac{16}{x^2} = 0$,

V. a: $x = 0$; $\lim_{x\to 0+} \frac{8(x-2)}{x^2}$ $\left(\frac{\approx(-16)}{\text{small} > 0}\right) = -\infty$, $\lim_{x\to 0-} \frac{8(x-2)}{x^2} = -\infty$ 5

(b) Determine the intervals on which $y = f(x)$ is increasing and the intervals on which it is decreasing. Identify all local extrema.

$f'(x) = -8x^{-2} + 32x^{-3} = -8x^{-3}(x-4)$ 3

critical nos.: $x = 0$, $x = 4$

	$(-\infty, 0)$	$(0, 4)$	$(4, \infty)$
f'	$-$	$+$	$-$
f	↘	↗	↘

1

$(4, 1)$ local max. 1

(d) Determine the intervals on which $y = f(x)$ is concave up and the intervals on which it is concave down. Identify all inflection points. (*Note*: $3 \cdot 32 = 6 \cdot 16$.)

$f''(x) = 16x^{-3} - 3 \cdot 32\, x^{-4} = 16x^{-4}(x-6)$ 3

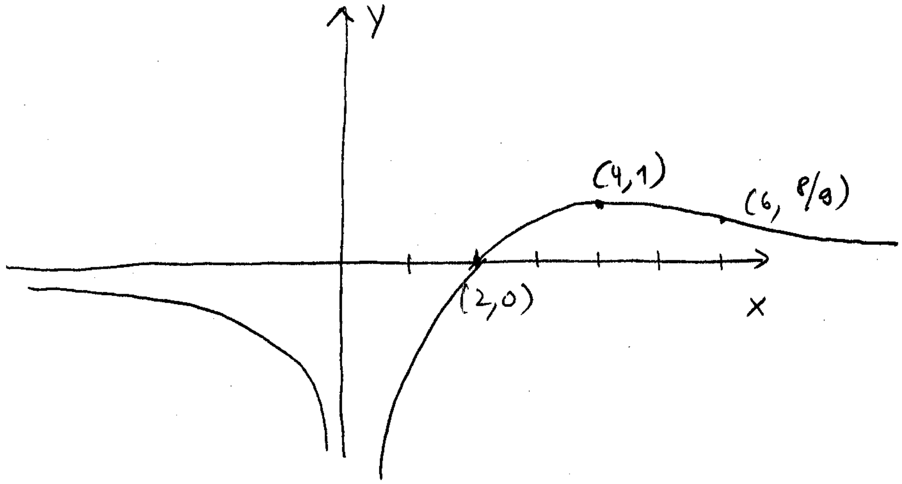
cr. nos.: $x = 0$, $x = 6$

	$(-\infty, 0)$	$(0, 6)$	$(6, \infty)$
f''	$-$	$-$	$+$
f	$\frown$	$\frown$	$\smile$

1

$(6, \frac{8}{9})$ infl. pt. 1

(e) Sketch the graph of $y = f(x)$. Identify all points of importance on the graph.



5

(f) Determine the domain and range of the composite function $y = \sqrt{f(x)}$.

Domain: $x \geq 2$, i.e. $[2, \infty)$,
Range: $[0, 2]$. 5

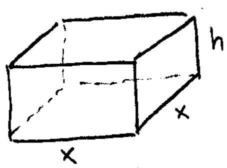
(g) Determine the domain and range of the composite function $y = f(x^2 + 1)$.

Domain: all x

Range: $y = x^2 + 1$ has range $[1, \infty)$. On this interval, f has values in $[-8, 4]$ 5

6. A construction contractor needs to build a small rectangular storage box with a square base, flat roof and volume 1,600 cubic feet. Every square foot of side walls costs \$10 (that is, the cost of side walls is the area multiplied by \$10), and every square foot of the roof costs \$4. Assume that the base does not need to be built, so it costs nothing.

(a) Determine the dimensions of the box that will minimize the building costs. Justify all your conclusions!



$x^2 h = 1{,}600$; $h = \dfrac{1{,}600}{x^2}$) 3

$C = 40\,xh + 4x^2 = 40\,x \cdot \dfrac{1{,}600}{x^2} + 4x^2$

$= \dfrac{64{,}000}{x} + 4x^2$ ← minimize for $x > 0$) 5

$\dfrac{dC}{dx} = -\dfrac{64{,}000}{x^2} + 8x$) 5, $x^3 = 8000$, $x = 20$

$h = \dfrac{1{,}600}{400} = 4$) 2

	(0,20)	(20,∞)
$\frac{dC}{dx}$	−	+

↑ min at 20) 5

(b) Now assume that the height of the box is restricted to be between 5 and 10 feet. With this additional restriction, what dimensions of the box minimize cost?

$5 \le h \le 10$

When $h = 5$, $x^2 = \dfrac{1600}{5} = 320$, $x = \sqrt{320} \approx 17.9$ | 2

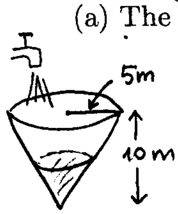
When $h = 10$, $x^2 = \dfrac{1{,}600}{10} = 160$, $x = \sqrt{160} \approx 12.6$

So $\dfrac{dC}{dx}$ is negative between $\sqrt{160}$ and $\sqrt{320}$, 4 10

and cost is minimized at $x = \sqrt{320}$, $h = 5$. 4

7. Water flows into a conical reservoir (with radius of the base 5 m, and height 10 m) as shown in the picture. At one point, the water level in the reservoir is measured to be 4 m, and the rate of inflow of water is measured to be 2 m^3/sec. Find the following two rates. (The volume of a cone is $\frac{1}{3}\pi r^2 h$, where r is the radius of the circular base of the cone and h is its height.)

(a) The rate at which the water level in the reservoir is increasing.

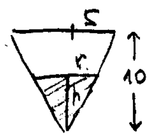


20

$$\frac{r}{h} = \frac{5}{10} = \frac{1}{2}$$

$$r = \frac{1}{2} h$$ 5

Side view:



$$V = \frac{1}{3}\pi r^2 h = \frac{1}{12}\pi h^3$$ 5

$$\frac{dV}{dt} = \frac{1}{12}\pi \cdot 3h^2 \frac{dh}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}$$ 5

Plug in $h = 4$, $\frac{dV}{dt} = 2$, to get

$$\frac{dh}{dt} = \frac{2}{4\pi} = \frac{1}{2\pi} \quad (m/sec)$$ 5

π on top (−5)

$\frac{1}{4\pi}$ −5

(b) The rate at which the area of the water surface in the reservoir is increasing. (*Surface* refers to the top circular surface of the water that is exposed to air.) 10

$$A = \pi r^2 = \frac{1}{4}\pi h^2$$ 4

$$\frac{dA}{dt} = \frac{1}{4}\pi \cdot 2h \cdot \frac{dh}{dt} = \frac{1}{2}\pi h \frac{dh}{dt}$$ 4

Plug in $h = 4$, $\frac{dh}{dt} = \frac{1}{2\pi}$ to get

$$\frac{dA}{dt} = \frac{1}{2}\pi \cdot 4 \cdot \frac{1}{2\pi} = 1 \quad (m^2/sec)$$ 2

8. Your hardware company makes a flash drive called Bolt. The total demand, at zero price, for Bolt is 6000 units per year. At the current price of \$60 per unit, you sell 3000 Bolts. Your fixed operating costs are \$20,000 per year, and each Bolt costs you \$20 to make. As usual, assume that the demand function is linear.

(a) Write down the demand, revenue, cost, and profit functions.

x	p
6000	0
3000	60

$$p - 0 = \frac{60-0}{3000-6000}(x-6000)$$

$$= -\frac{1}{50}(x-6000)$$

12 (5)

10

$$p = -\frac{1}{50}x + 120 \quad (3) \qquad 0 \le x \le 6{,}000 \quad (2)$$

$$P = x\left(-\frac{1}{50}x + 120\right) - 20x - 20{,}000$$

$$= -\frac{1}{50}x^2 + 100x - 20{,}000 \qquad (5)$$

(b) Which price should you charge for Bolt charge to maximize your profit? How many Bolts would you then sell per year?

$$\frac{dP}{dx} = -\frac{1}{25}x + 100 = 0, \quad x = 2500 \ \Big) \ 5 \quad p = 70$$

10

As P is a quadratic function with negative leading coefficient, P has maximum at $x = 2500$) 5

(c) Which production level would maximize the profit per unit sold, that is, $\bar{P} = \dfrac{P}{x}$.

$$\bar{P} = -\frac{1}{50}x + 100 - \frac{20{,}000}{x}$$

10

$$\frac{d\bar{P}}{dx} = -\frac{1}{50} + \frac{20{,}000}{x^2} \ \Big) \ 5 \quad x^2 = 50 \cdot 20{,}000 = 1{,}000{,}000$$

$$x = 1000$$

$$\frac{d^2\bar{P}}{dx^2} = -2 \cdot \frac{20{,}000}{x^3} < 0 \ \Big) \ 5$$

so $\bar{P}$ is concave down, and so $x = 1{,}000$ minimizes $\bar{P}$,