Math 16A — 002, Fall 2009.
Dec. 10, 2009.

FINAL EXAM

NAME(print in CAPITAL letters, *first name first*): KEY

NAME(sign): ______________________

ID#: ______________________

Instructions: Each of the first four problems is worth 20 points, while problems 5 to 8 are each worth 30 points. Read each question carefully and answer it in the space provided. YOU MUST SHOW ALL YOUR WORK TO RECEIVE FULL CREDIT. Clarity of your solutions may be a factor in determining credit. Calculators, books or notes are not allowed.

Make sure that you have a total of 10 pages (including this one) with 8 problems. Read through the entire exam before beginning to work.

1	
2	
3	
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5	
6	
7	
8	
TOTAL	

1. Compute derivatives of the following two functions. *Do not simplify!*
(a) $y = x^4 \cdot \sqrt{1 - x^3}$

$$y' = 4x^3 \sqrt{1-x^3} + x^4 \cdot \frac{1}{2}(1-x^3)^{-1/2} \cdot (-3x^2)$$ 10

(b) $y = \dfrac{\tan(4x)}{x^5 + 1}$

$$y' = \frac{\sec^2(4x) \cdot 4 \cdot (x^5+1) - 5x^4 \cdot \tan(4x)}{(x^5+1)^2}$$ 10

2. Find the equation of the tangent line to the curve $(x^2+y)^3+xy+y^4=2$ at the point $(0,1)$.

$$3(x^2+y)^2 \cdot \left(2x+\frac{dy}{dx}\right) + x\cdot\frac{dy}{dx} + y + 4y^3\frac{dy}{dx} = 0$$ 10

Plug in $x=0,\ y=1$:

$$3\cdot\frac{dy}{dx} + 1 + 4\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{1}{7}$$ ✓

Line : $y-1 = -\frac{1}{7}x$ ✓

$$\underline{y = -\frac{1}{7}x+1}$$

3. Compute the following limits.

(a) $\lim_{h\to 0} \frac{(16+h)^{1/4}-2}{h}$ $= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = f'(x)$ 10

for $f(x) = x^{1/4}$ 5

at $x = 16$. so $f'(x) = \frac{1}{4}x^{-3/4}$ 3

Answer : $\frac{1}{4}(16)^{-3/4} = \underline{\underline{\frac{1}{32}}}$. 2

(b) $\lim_{x\to 3} \frac{x+2-\sqrt{8x+1}}{x-3}$ $\cdot \frac{(x+2+\sqrt{8x+1})}{(x+2+\sqrt{8x+1})}$ 3

$= \lim_{x\to 3} \frac{(x+2)^2-(8x+1)}{(x-3)(x+2+\sqrt{8x+1})}$ 2

$= \lim_{x\to 3} \frac{x^2+4x+4-8x-1}{(x-3)(x+2+\sqrt{8x+1})}$ $\leftarrow = x^2-4x+3 = (x-3)(x-1)$ 2

$= \lim_{x\to 3} \frac{x-1}{x+2+\sqrt{8x+1}} = \frac{2}{5+5} = \underline{\underline{\frac{1}{5}}}$.

3

4. Consider the function

$$f(x) = \begin{cases} x^2, & x < -1, \\ ax + b, & -1 \le x \le 1, \\ -3x^2, & x > 1. \end{cases}$$

(a) Determine the numbers a and b so that $y = f(x)$ is continuous for all x.

Cont. at -1: $1 = -a + b$

Cont. at 1: $-3 = a + b$

10

$2b = -2$, $\underline{\underline{b = -1}}$, $\underline{\underline{a = -2}}$.

$$f(x) = \begin{cases} x^2, & x < -1 \\ -2x - 1, & -1 \le x \le 1 \\ -3x^2, & x > 1 \end{cases}$$

(b) Assuming the values a and b obtained in (a), determine all the values of x for which $y = f(x)$ is *not* differentiable.

Diff at -1: is $2x = -2$ at $x = -1$? Yes. 5

Diff. at 1: is $-2 = -6x$ at $x = 1$? No. 5

10

Answer. Differentiable everywhere except at $x = 1$. (-2)

5. Consider the function $f(x) = \dfrac{36(2-x)}{x^3} = 36(2x^{-3} - x^{-2})$.

(a) Determine the domain of $y = f(x)$, its intercepts, and horizontal and vertical asymptote. Compute the left and right limits at the vertical asymptote.

Domain: $x \neq 0$. (1)

x-intercept: $(2, 0)$. No y-intercept. (1)

h.a. $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{-36x}{x^3} = 0$ (1), $\lim\limits_{x\to-\infty} f(x) = 0$, $\boxed{\text{h.a. } y = 0}$

v.a. $x = 0$, $\lim\limits_{x\to 0+} f(x) = \infty$ $\left(\dfrac{(\approx 72)}{(\text{small} > 0)}\right)$, $\lim\limits_{x\to 0-} f(x) = -\infty$ $\left(\dfrac{(\approx 72)}{\text{small} < 0}\right)$ 5

1 1

(b) Determine the intervals on which $y = f(x)$ is increasing and the intervals on which it is decreasing. Identify all local extrema. (Note: $\frac{36}{3^3} = \frac{4}{3}$.)

$$f'(x) = 36(-6x^{-4} + 2x^{-3}) = 72x^{-4}(-3 + x)$$ 3

$f'(x) = 0$ at $x = 3$, undef. at $x = 0$

	$(-\infty, 0)$	$(0, 3)$	$(3, \infty)$
sign of f'	+	−	+
f	↗	↘	↗

(2) 5

$(3, -\frac{4}{3})$ local min

(c) Determine the intervals on which $y = f(x)$ is concave up and the intervals on which it is concave down. Identify all inflection points. (Note: $\frac{72}{4^3} = \frac{9}{8}$.)

$$f''(x) = 36(+24x^{-5} - 6x^{-4})$$ (3)

$$= 6 \cdot 36 \cdot x^{-5}(4 - x)$$ $= 0$ at $x = 4$

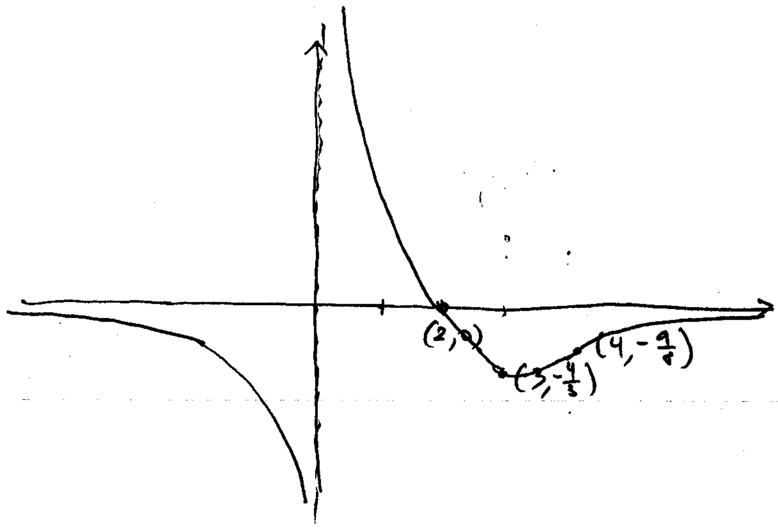
undef. at $x = 0$ 5

	$(-\infty, 0)$	$(0, 4)$	$(4, \infty)$
sign of f''	−	+	−
f	⌒	⌣	⌒

(2)

$(4, -\frac{9}{8})$ pt. of inflection

(d) Sketch the graph of $y = f(x)$. Identify all points of importance on the graph.



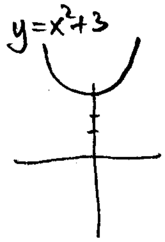
(e) Determine the domain and range of the composite function $y = \sqrt{f(x)}$.

Domain : $(0, 2]$ 2

Range : $[0, \infty)$ 3

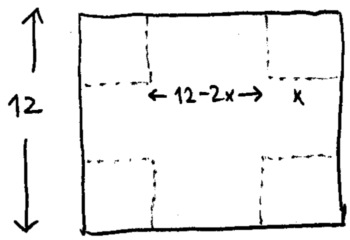
(f) Determine the domain and range of the composite function $y = f(x^2 + 3)$.

Domain : all x 2



Range : x^2+3 has values in $[3, \infty)$; on $[3, \infty)$ f has values in $[-\frac{4}{3}, 0)$ 3

6. You want to build an open box from a square piece of carton, 12 inches by 12 inches, by cutting out equal squares from each corner and turning up the sides.
(a) Find the largest possible volume that such a box can have.



$V = (12-2x)^2 \cdot x$, 8

maximize on $0 \le x \le 6$

$$\frac{dV}{dx} = 2(12-2x)\cdot(-2)\cdot x + (12-2x)^2$$

$= (12-2x)(-4x+12-2x)$ (6)

$= (12-2x)(12-6x)$

$\frac{dV}{dx} = 0$ when $x = 2, 6$. (20)

(6)

x	V
0	0
2	128
6	0

← the largest volume is $\underline{\underline{128}}$ (in^3)

(b) Now assume that the height of the box is restricted to be at least 3 inches. What is the largest volume under this additional restriction?

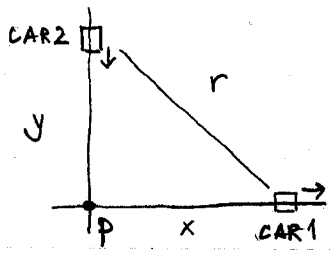
$\frac{dV}{dx} < 0$ when x is in $(3, 6)$ (5) (10)

so $V \searrow$ and the largest volume is when $x = 3$, that is, $V = 36 \cdot 3 = \underline{\underline{108}}$ (in^3).

(5)

7. Car 1 is headed East (away from point P) and Car 2 is headed South (towards point P). At one moment, Car 1 is 4 miles from point P traveling at 60 mph, while Car 2 is 3 miles from point P traveling at 70 mph.

(a) At what rate is the distance between the two cars changing at this moment? (*Note*: be careful about signs.)



$r^2 = x^2 + y^2$ 5

When $x = 4$, $y = 3$, $\frac{dx}{dt} = 60$, $\frac{dy}{dt} = -70$ 3

and $r = \sqrt{16 + 9} = 5$.

$$2r \frac{dr}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}$$ (20)

10

$$\frac{dr}{dt} = \frac{x}{r} \frac{dx}{dt} + \frac{y}{r} \frac{dy}{dt}$$

Plug in: $\frac{dr}{dt} = \frac{4}{5} \cdot 60 + \frac{3}{5}(-70)$

2

$$= \frac{240 - 210}{5} = \underline{\underline{6}} \quad (m/h)$$

(b) At what rate is the area of the triangle determined by P and the two cars changing at the same moment?

$A = \frac{1}{2} x y$ (3)

$$\frac{dA}{dt} = \frac{1}{2}\left(\frac{dx}{dt} \cdot y + x \cdot \frac{dy}{dt}\right)$$ (5) (10)

Plug in:

$$\frac{dA}{dt} = \frac{1}{2}\left(60 \cdot 3 + 4(-70)\right)$$ (2)

$$= \frac{1}{2}(180 - 280) = -50 \quad (m^2/h)$$

8. A maker of sunglasses estimates the total market demand (at zero price) for its new Zeroglare sunglasses to be 1,000. The demand drops to zero at the price of $200. Assume that the demand function is linear between prices of 0 and $200. The costs of *starting* the production of these sunglasses are $2,000; after that, each Zeroglare costs $100 to make.
(a) Write down the demand, revenue, cost, and profit as functions of x, the number of units produced.

x	p
1000	0
0	200

$$p - 200 = \frac{200-0}{0-1000}\,x = -\frac{1}{5}x$$

$$p = -\frac{1}{5}x + 200 \qquad (0 \le x \le 1000) \;(5) \qquad (10)$$

$$C = 100x + 2000 \qquad (1)$$

$$R = x\left(-\frac{1}{5}x + 200\right) = -\frac{1}{5}x^2 + 200x \qquad (2)$$

$$P = -\frac{1}{5}x^2 + 100x - 2000 \qquad (2)$$

(b) Which production level, and what price, of Zeroglare maximizes the manufacturer's profit?

$$\frac{dP}{dx} = -\frac{2}{5}x + 100 \;(5) = 0 \quad \text{at} \quad \frac{2}{5}x = 100, \; x = \frac{500}{2} = 250$$

	(0, 250)	(250, 1000)
$\frac{dP}{dx}$	+	−
P	↗	↘

The profit is maximized at $\underline{x = 250}$, $\underline{p} = -\frac{1}{5}\cdot 250 + 200 = \underline{150}$ (5) (10)

(c) Determine the production level that will maximize the profit per unit produced, that is, $\bar{P} = \dfrac{P}{x}$.

$$\bar{P} = -\frac{1}{5}x + 100 - \frac{2000}{x} \qquad (5) \qquad (10)$$

$$\frac{d\bar{P}}{dx} = -\frac{1}{5} + \frac{2000}{x^2} = 0 \quad \text{when} \quad x^2 = 10{,}000$$

$$\underline{x = 100}$$

$$\frac{d^2\bar{P}}{dx^2} = -\frac{4000}{x^3} < 0 \quad \text{for } x > 0, \; \bar{P} \text{ is } \underline{\text{concave down}} \qquad (5)$$

so $\underline{x = 100 \text{ is the max}}$.