Chapter 10

Linear Differential Operators and Green's Functions

We have seen that linear differential operators on normed function spaces are not bounded. Differential operators are important for the study of differential equations and we would like to analyze them in spite of their lack of continuity. There are two main approaches to this problem. One is to use a weak topology, not derived from a norm, with respect to which differential operators are continuous. This is what is done in distribution theory, studied in Chapter 11. The other approach, which we follow in this chapter, is to consider special classes of unbounded operators that are defined on dense linear subspaces of a Hilbert, or Banach, space.

The inverse of a linear differential operator is an integral operator, whose kernel is called the *Green's function* of the differential operator. We may use the bounded inverse to study the properties of the unbounded differential operator. For example, if the inverse is a compact, self-adjoint operator, then the differential operator has a complete orthonormal set of eigenfunctions.

We begin by giving some general definitions for unbounded operators. We will consider unbounded linear operators acting in a Hilbert space, although similar ideas apply to unbounded operators acting in a Banach space.

10.1 Unbounded operators

One of the main new features of unbounded operators, in comparison with bounded operators, is that they are not defined on the whole space. For example, a general continuous function does not have a continuous derivative, so differential operators are defined on a subspace of differentiable functions. The definition of an unbounded linear operator

$$A : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}$$

acting in a Hilbert space $\mathcal{H}$ therefore includes the definition of its domain $\mathcal{D}(A)$. We will assume that the domain of A is a dense linear subspace of $\mathcal{H}$, unless we state explicitly otherwise. If the domain of A is not dense, then we may obtain a densely defined operator by setting A equal to zero on the orthogonal complement of its domain, so this assumption does not lead to any loss of generality.

An operator $\widetilde{A}$ is an *extension* of A, or A is a *restriction* of $\widetilde{A}$, if $\mathcal{D}(\widetilde{A}) \supset \mathcal{D}(A)$ and $\widetilde{A}x = Ax$ for all $x \in \mathcal{D}(A)$. We write this relationship as $\widetilde{A} \supset A$, or $A \subset \widetilde{A}$. From Theorem 5.19, if A is a bounded linear operator on a dense domain $\mathcal{D}(A)$ in $\mathcal{H}$, then A has a unique bounded extension to $\mathcal{H}$. Consequently, it is only useful to consider densely defined operators when the operator is unbounded.

The domain of a differential operator defines the somewhat technical property of the smoothness of the functions on which the operator acts. More importantly, it also encodes any boundary conditions associated with the operator. The following example, which we discuss in greater detail later on, illustrates differential operators and their domains.

Example 10.1 Let $A_k u = u''$ with $k = 1, 2, 3, 4$ be differential operators in $L^2([0,1])$ with domains

$$\begin{aligned}
\mathcal{D}(A_1) &= \left\{u \in C^2([0,1]) \mid u(0) = u(1) = 0\right\}, \\
\mathcal{D}(A_2) &= C^2([0,1]), \\
\mathcal{D}(A_3) &= \left\{u \in H^2\left((0,1)\right) \mid u(0) = u(1) = 0\right\}, \\
\mathcal{D}(A_4) &= H^2\left((0,1)\right).
\end{aligned}$$

Here, $H^2\left((0,1)\right)$ is the Sobolev space of functions whose weak derivatives of order less than or equal to two belong to $L^2([0,1])$. The Sobolev embedding theorem implies that $H^2((0,1)) \subset C^1([0,1])$, so it makes sense to use the pointwise values of u in defining $\mathcal{D}(A_3)$. Then $A_1 \subset A_2 \subset A_4$, and $A_1 \subset A_3 \subset A_4$.

The adjoint of an unbounded operator $A : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}$ is an operator

$$A^* : \mathcal{D}(A^*) \subset \mathcal{H} \to \mathcal{H}.$$

Generalizing the basic property in (8.9) of the adjoint of a bounded linear operator, we want

$$\langle Ax, y\rangle = \langle x, A^*y\rangle \qquad \text{for all } x \in \mathcal{D}(A) \text{ and all } y \in \mathcal{D}(A^*), \tag{10.1}$$

where $\mathcal{D}(A^*)$ is the largest subspace of $\mathcal{H}$ for which (10.1) holds. In more detail, if $y \in \mathcal{H}$, then $\varphi_y(x) = \langle y, Ax\rangle$ defines a linear functional $\varphi_y : \mathcal{D}(A) \to \mathbb{C}$. We say that $y \in \mathcal{D}(A^*)$ if φ_y is bounded on $\mathcal{D}(A)$. In that case, since $\mathcal{D}(A)$ is dense in $\mathcal{H}$, the bounded linear transformation theorem in Theorem 5.19 implies that φ_y has a unique extension to a bounded linear functional on $\mathcal{H}$, and the Riesz representation theorem in Theorem 8.12 implies there is a unique vector $z \in \mathcal{H}$ such that $\varphi_y(x) = \langle z, x\rangle$. Then $\langle y, Ax\rangle = \langle z, x\rangle$ for all $x \in \mathcal{D}(A)$, and we define $A^*y = z$. Summarizing this procedure, we get the following definition.

Definition 10.2 Suppose that $A : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}$ is a densely defined unbounded linear operator on a Hilbert space $\mathcal{H}$. The *adjoint* $A^* : \mathcal{D}(A^*) \subset \mathcal{H} \to \mathcal{H}$ is the operator with domain

$$\mathcal{D}(A^*) = \{y \in \mathcal{H} \mid \text{there is a } z \in \mathcal{H} \text{ with } \langle Ax, y\rangle = \langle x, z\rangle \text{ for all } x \in \mathcal{D}(A)\}.$$

If $y \in \mathcal{D}(A^*)$, then we define $A^*y = z$, where z is the unique element such that $\langle Ax, y\rangle = \langle x, z\rangle$ for all $x \in \mathcal{D}(A)$.

It is possible that $\mathcal{D}(A^*)$ is not dense in $\mathcal{H}$, even if $\mathcal{D}(A)$ is dense, in which case we do not define A^{**} (see Exercise 10.15 for an example).

As we will see below, the adjoint of a differential operator is another differential operator, which we obtain by using integration by parts. The domain $\mathcal{D}(A)$ defines boundary conditions for A, and the domain $\mathcal{D}(A^*)$ defines adjoint boundary conditions for A^*. The boundary conditions ensure that the boundary terms arising in the integration by parts vanish.

A particularly important class of unbounded operators is the class of self-adjoint operators. Self-adjointness includes the equality of the domains of A and A^*. For differential operators, this equality of domains corresponds to the self-adjointness of the boundary conditions.

Definition 10.3 An unbounded operator A is *self-adjoint* if $A^* = A$, meaning that $\mathcal{D}(A^*) = \mathcal{D}(A)$ and $A^*x = Ax$ for all $x \in \mathcal{D}(A)$. An unbounded operator A is *symmetric* if A^* is an extension of A, meaning that $\mathcal{D}(A^*) \supset \mathcal{D}(A)$ and $A^*x = Ax$ for all $x \in \mathcal{D}(A)$.

It is usually straightforward to show that an operator is symmetric, but it may be more difficult to show that a symmetric operator is self-adjoint.

Example 10.4 For the differential operators defined in Example 10.1, we will see that $A_1^* = A_3$, so A_1 is symmetric but not self-adjoint, while $A_3^* = A_3$, so A_3 is self-adjoint. We will also see that $A_2^* = A_4^* = A_5$ where $A_5u = u''$ with domain

$$\mathcal{D}(A_5) = \left\{u \in H^2([0,1]) \mid u(0) = u(1) = u'(0) = u'(1) = 0\right\}.$$

Since A_5 is not an extension of A_2 or A_4, neither A_2 nor A_4 is symmetric. We also have $A_5^* = A_4$, so A_5 is symmetric, but not self-adjoint.

Although differential operators are not continuous, they have a related property called *closedness*.

Definition 10.5 An operator $A : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}$ is *closed* if for every sequence (x_n) in $\mathcal{D}(A)$ such that $x_n \to x$ and $Ax_n \to y$, we have $x \in \mathcal{D}(A)$ and $Ax = y$.

Note carefully the difference between continuous and closed operators. For a continuous operator A, the convergence of the sequence (x_n) implies the convergence of (Ax_n), and

$$\lim_{n\to\infty} Ax_n = A\left(\lim_{n\to\infty} x_n\right). \tag{10.2}$$

For a closed operator A, the convergence of (x_n) does not imply the convergence of (Ax_n); but if both (x_n) and (Ax_n) converge, then (10.2) holds. The *graph* $\Gamma(A)$ of an operator $A : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}$ is the subset of $\mathcal{H} \times \mathcal{H}$ defined by

$$\Gamma(A) = \{(x, y) \mid x \in \mathcal{D}(A) \text{ and } y = Ax\}.$$

An operator is closed if and only if its graph is a closed subspace of $\mathcal{H} \times \mathcal{H}$.

An operator A is *closable* if it has the following property: for every sequence (x_n) of elements in $\mathcal{D}(A)$ such that $x_n \to 0$ and $Ax_n \to y$ for some $y \in \mathcal{H}$, we have $y = 0$. We define the *closure* $\overline{A}$ of a closable operator A to be the operator with domain

$$\begin{aligned}\mathcal{D}(\overline{A}) \quad = \quad &\{x \in \mathcal{H} \mid \text{there is a sequence } (x_n) \text{ in } \mathcal{D}(A) \text{ and a } y \in \mathcal{H} \\ &\text{such that } x_n \to x \text{ and } Ax_n \to y\}.\end{aligned}$$

If $x_n \to x$ and $Ax_n \to y$, then we define $\overline{A}x = y$. Since A is closable, the value y does not depend on the sequence (x_n) in $\mathcal{D}(A)$ that is used to approximate x. The graph of $\overline{A}$ is the closure of the graph of A in $\mathcal{H} \times \mathcal{H}$, and $\overline{A}$ is the smallest closed extension of A. If A is not closable, then the closure of the graph of A is not the graph of an operator, and A has no closed extensions (see Exercise 10.8 for an example). Every symmetric operator is closable (see Exercise 10.2). We say that a symmetric operator A is *essentially self-adjoint* if its closure is self-adjoint.

Example 10.6 The operators A_1 and A_2 in Example 10.1 are not closed because we may choose a sequence of functions $u_n \in C^2([0,1])$ such that $u_n \to u$ and $u_n'' \to v$ in $L^2([0,1])$, where v is not continuous. Hence u is not C^2, and therefore does not belong to the domain of A_1 or A_2. The operators A_3 and A_4 are closed. Both A_1 and A_2 are closable, with $\overline{A_1} = A_3$, and $\overline{A_2} = A_4$. Thus, A_1 is essentially self-adjoint, but A_2 is not.

If $A : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}$ is one-to-one and onto, then we define the inverse operator $A^{-1} : \mathcal{H} \to \mathcal{H}$ by $A^{-1}y = x$ if and only if $Ax = y$. The range of A^{-1} is equal to the domain of A. If A is closed, then the closed graph theorem, which we do not prove here, implies that A^{-1} is bounded.

Proposition 10.7 If $A : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}$ is a densely defined linear operator on a Hilbert space $\mathcal{H}$ with a bounded inverse $A^{-1} : \mathcal{H} \to \mathcal{H}$, then $(A^*)^{-1} = (A^{-1})^*$.

Proof. Since A^{-1} is bounded, it has a bounded adjoint. If $x \in \mathcal{D}(A^*)$ and $y \in \mathcal{H}$, then

$$\langle (A^{-1})^* A^* x, y\rangle = \langle A^* x, A^{-1} y\rangle = \langle x, AA^{-1} y\rangle = \langle x, y\rangle.$$

Therefore $(A^{-1})^* A^* x = x$ for $x \in \mathcal{D}(A^*)$. Moreover, if $x \in \mathcal{H}$ and $y \in \mathcal{D}(A)$, then

$$\langle A^* (A^{-1})^* x, y\rangle = \langle (A^{-1})^* x, Ay\rangle = \langle x, A^{-1} Ay\rangle = \langle x, y\rangle.$$

Since $\mathcal{D}(A)$ is dense in $\mathcal{H}$, it follows that $(A^{-1})^* x \in \mathcal{D}(A^*)$ and $A^*(A^{-1})^* x = x$. □

The definitions of the resolvent set, spectrum, and resolvent operator for an unbounded operator $A : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}$ are analogous to those for a bounded operator. The resolvent set $\rho(A)$ of A consists of the complex numbers λ such that $A - \lambda I$ is a one-to-one, onto map from $\mathcal{D}(A)$ to $\mathcal{H}$, and $(A - \lambda I)^{-1}$ is bounded. The spectrum $\sigma(A)$ is the complement of the resolvent set in $\mathbb{C}$. If $\lambda \in \rho(A)$, then we define the resolvent operator $R_\lambda : \mathcal{H} \to \mathcal{H}$ by

$$R_\lambda = (\lambda I - A)^{-1}.$$

If A is closed, then the closed graph theorem implies that R_λ is bounded whenever $A - \lambda I$ is one-to-one and onto. Unlike bounded operators, unbounded operators may have an empty spectrum (see Exercise 10.13 for an example).

10.2 The adjoint of a differential operator

In this section, we consider differential operators acting on smooth functions, and explain how to determine their adjoints. We discuss the domain of the adjoint in more detail in Section 10.4.

A linear ordinary differential operator of order n is a linear map A that acts on an n-times continuously differentiable function u by

$$Au = \sum_{j=0}^{n} a_j u^{(j)},$$

where $u^{(j)}$ denotes the jth derivative of u, and the coefficients a_j are real or complex-valued functions. Our goal is to study BVPs (boundary value problems) for ODEs of the form

$$Au = f, \qquad Bu = 0, \tag{10.3}$$

where $Bu = 0$ denotes a set of linear boundary conditions.

For concreteness, we assume that all functions are defined on the interval $[0, 1]$, and we consider second-order ordinary differential operators A of the form

$$Au = au'' + bu' + cu, \tag{10.4}$$

where a, b, and c are sufficiently smooth functions on $[0, 1]$. The same ideas apply to linear ordinary differential operators of arbitrary order. We assume, unless stated otherwise, that $a(x) > 0$ for all $0 \leq x \leq 1$, so that A is second-order at every point.

For a second-order differential equation, we expect that we need to impose two boundary conditions to obtain a unique solution, although this is not always sufficient to guarantee uniqueness. Sometimes we may want to consider overdetermined or underdetermined boundary value problems with a larger or smaller number of boundary conditions. We always assume that the boundary condition $Bu = 0$ is a homogeneous system of linear equations that involves the values of u and u' at the endpoints $x = 0, 1$. Higher derivatives of u may be expressed in terms of u and u' by use of the differential equation.

Some common types of boundary conditions are:

$$\begin{aligned} u(0) &= 0, \quad u(1) = 0 && \text{Dirichlet;} \\ u'(0) &= 0, \quad u'(1) = 0 && \text{Neumann;} \\ u(0) &= u(1), \quad u'(0) = u'(1) && \text{periodic;} \\ \alpha_0 u(0) + \beta_0 u'(0) &= 0, \quad \alpha_1 u(1) + \beta_1 u'(1) = 0 && \text{mixed.} \end{aligned}$$

In the mixed boundary condition, α_0, α_1, β_0, and β_1 are complex constants. Instead of imposing conditions that involve the solution at both endpoints, we can impose two conditions at one of the endpoints:

$$\begin{aligned} u(0) &= 0, \quad u'(0) = 0 && \text{initial;} \\ u(1) &= 0, \quad u'(1) = 0 && \text{final.} \end{aligned}$$

For linear problems, nonhomogeneous boundary conditions may be reduced to homogeneous ones by subtraction of any function that satisfies the nonhomogeneous conditions: if $Au = f$, $Bu = b$, and $Bu_p = b$, then $v = u - u_p$ satisfies $Av = g$ and $Bv = 0$, where $g = f - Au_p$. In practice, it may be convenient to retain nonhomogeneous boundary conditions when using Green's formula below, but in developing the general theory it is simplest to assume that all boundary conditions have been reduced to homogeneous ones.

We begin by formulating the adjoint boundary value problem, using the following result.

Proposition 10.8 (Green's) Suppose that A is given by (10.4), where $a \in C^2([0, 1])$, $b \in C^1([0, 1])$, and $c \in C([0, 1])$. Let $\langle \cdot, \cdot \rangle$ denote the usual L^2-inner product,

$$\langle v, u \rangle = \int_0^1 \overline{v(x)} u(x)\, dx,$$

and define A^* by

$$A^* v = (\overline{a} v)'' - \left(\overline{b} v\right)' + \overline{c} v. \tag{10.5}$$

Then, for every $u, v \in C^2([0,1])$, we have

$$\langle v, Au\rangle - \langle A^* v, u\rangle = \left[a\left(\overline{v}u' - \overline{v}'u\right) + \left(b - a'\right)\overline{v}u\right]_0^1. \tag{10.6}$$

Proof. Integration by parts implies that

$$\begin{aligned}
\langle v, Au\rangle &= \int_0^1 \overline{v}\left(au'' + bu' + cu\right)\, dx \\
&= \left[a\overline{v}u' + b\overline{v}u\right]_0^1 + \int_0^1 \left\{-\left(a\overline{v}\right)' u' - \left(b\overline{v}\right)' u + c\overline{v}u\right\}\, dx \\
&= \left[a\overline{v}u' - \left(a\overline{v}\right)' u + b\overline{v}u\right]_0^1 + \int_0^1 \overline{\left(\left(\overline{a}v\right)'' - \left(\overline{b}v\right)' + \overline{c}v\right)} u\, dx,
\end{aligned}$$

which gives (10.6). □

We call A^* in (10.5) the *formal adjoint* of A ("formal" because we have not specified its domain). The adjoint A^* depends on the inner product as well as on A (see Exercise 10.10). We will use the standard L^2-inner product, unless explicitly stated otherwise.

Example 10.9 Let D be the differentiation operator,

$$D = \frac{d}{dx}. \tag{10.7}$$

Then $D^* = -D$, $(iD)^* = iD$, and $\left(D^2\right)^* = D^2$, so D is formally skew-adjoint, while iD and D^2 are formally self-adjoint.

Given boundary conditions B for A, we define adjoint boundary conditions B^* for A^* by the requirement that the boundary terms in (10.6) vanish. Thus, for $v \in C^2([0,1])$, we say that $B^* v = 0$ if and only if

$$\langle v, Au\rangle = \langle A^* v, u\rangle \qquad \text{for all } u \in C^2([0,1]) \text{ such that } Bu = 0.$$

For A given by (10.4), we have $B^* v = 0$ if and only if

$$\left[a\left(\overline{v}u' - \overline{v}'u\right) + \left(b - a'\right)\overline{v}u\right]_0^1 = 0 \qquad \text{for all } u \text{ such that } Bu = 0.$$

We say that the BVP (10.3) is *self-adjoint* if $A = A^*$ and $B = B^*$.

Example 10.10 Suppose that $A = D^2$. Then Green's formula may be written as

$$\langle v, u''\rangle - \langle v'', u\rangle = \left[\overline{v}u' - \overline{v}'u\right]_0^1.$$

If $Bu = 0$ is the Dirichlet conditions $u(0) = u(1) = 0$, then we have

$$\left[\overline{v}u' - \overline{v}'u\right]_0^1 = \overline{v(1)}u'(1) - \overline{v(0)}u'(0).$$

This vanishes for all values of $u'(0)$ and $u'(1)$ if and only if $v(0) = v(1) = 0$. Thus, the Dirichlet boundary value problem for D^2 is self-adjoint. Neumann, mixed, and periodic boundary conditions are also self-adjoint. For initial conditions $u(0) = u'(0) = 0$ the boundary terms reduce to

$$[\overline{v}u' - \overline{v}'u]_0^1 = \overline{v(1)}u'(1) - \overline{v'(1)}u(1).$$

These terms vanish if and only if $v(1) = v'(1) = 0$, so final conditions are the adjoint of initial conditions, and the initial or final value problem for D^2 is not self-adjoint.

If we impose no boundary conditions on u, then we must require that $v(0) = v'(0) = v(1) = v'(1) = 0$. The adjoint of an undetermined boundary value problem is therefore overdetermined, and conversely.

Let us find all the formally self-adjoint, second-order differential operators. Expanding the expression for A^* in (10.5) and equating it with the expression for A in (10.4), we find that

$$au'' + bu' + cu = \overline{a}u'' + (2\overline{a}' - \overline{b})u' + (\overline{a}'' - \overline{b}' + \overline{c})u$$

for every $u \in C^2([0,1])$. We must therefore have

$$a = \overline{a}, \quad b = 2\overline{a}' - \overline{b}, \quad c = \overline{a}'' - \overline{b}' + \overline{c}.$$

These relations are satisfied if and only if a is real, $\operatorname{Re} b = a'$, and $\operatorname{Im} c = -\operatorname{Im} b/2$, where $\operatorname{Re} z$ and $\operatorname{Im} z$ denote the real and imaginary parts of $z \in \mathbb{C}$, respectively. The coefficients of a self-adjoint, second-order ordinary differential operator A are therefore determined by three real functions: a, $\operatorname{Im} b$, and $\operatorname{Re} c$. For operators with real coefficients, there are only two independent real-valued coefficient functions, which we denote by p and q, where $a = -p$, $b = -p'$, and $c = q$. The resulting formally self-adjoint operator, called a *Sturm-Liouville* operator, is given by

$$Au = -(pu')' + qu, \tag{10.8}$$

or $A = -D(pD) + q$. For example, if $p = 1$ and $q = 0$, we get the second-derivative operator $A = -D^2$. By imposing self-adjoint boundary conditions on functions in the domain of a Sturm-Liouville operator, we obtain a self-adjoint operator.

For operators with imaginary coefficients, we find that $a = 0$, $b = 2ir$ and $c = ir'$, which gives

$$Au = i\left(2ru' + r'u\right),$$

or $A = i(rD + Dr)$, since $Dr = rD + r'$. Any real linear combination of these real and imaginary formally self-adjoint operators is formally self-adjoint.

10.3 Green's functions

For concreteness, we consider the Dirichlet boundary value problem for the second-order differential operator A defined in (10.4),

$$Au = f, \qquad u(0) = u(1) = 0, \tag{10.9}$$

where $f : [0,1] \to \mathbb{C}$ is a given continuous function.

We look for a solution of (10.9) in the form

$$u(x) = \int_0^1 g(x,y) f(y)\, dy, \tag{10.10}$$

where $g : [0,1] \times [0,1] \to \mathbb{C}$ is a suitable function, called the *Green's function* of (10.9). If we regard

$$A : \mathcal{D}(A) \subset C([0,1]) \to C([0,1])$$

as an operator in $C([0,1])$ with domain

$$\mathcal{D}(A) = \left\{ u \in C^2([0,1]) \mid u(0) = u(1) = 0 \right\},$$

then the integral operator $G : C([0,1]) \to \mathcal{D}(A)$ given by

$$Gf(x) = \int_0^1 g(x,y) f(y)\, dy \tag{10.11}$$

is the inverse of A.

We can write an equation for the Green's function g in terms of the *Dirac delta function* δ. We give a heuristic discussion here, and use it to motivate the classical definition of the Green's function in Definition 10.11 below. In Chapter 11, we will show that the delta function has a mathematically rigorous interpretation as a distribution.

We regard δ as a "function" on $\mathbb{R}$ that has unit integral concentrated at the origin, meaning that

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1, \qquad \delta(x) = 0 \quad \text{for } x \neq 0.$$

More generally, for any continuous function f, we have

$$\int_{-\infty}^{\infty} \delta(x-y) f(y)\, dy = f(x).$$

Formally, we also have

$$\int_{-\infty}^{x} \delta(y)\, dy = H(x),$$

where H is the *Heaviside step function*, defined by

$$H(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x > 0. \end{cases}$$

The step function is constant on any interval that does not contain the origin and has a jump of one at zero. Conversely, the delta function,

$$\delta = H', \tag{10.12}$$

is the derivative of the step function. We will give a precise meaning to these results when we study distribution theory in Chapter 11.

The Green's function $g(x, y)$ associated with the boundary value problem in (10.9) is the solution of the following problem:

$$Ag(x, y) = \delta(x - y), \qquad g(0, y) = g(1, y) = 0. \tag{10.13}$$

Here, A is a differential operator with respect to x, and y plays the role of a parameter. If u is given by (10.10), then formally differentiating under the integral sign with respect to x, we find that for $0 < x < 1$

$$Au(x) = \int_0^1 Ag(x, y) f(y)\, dy = \int_0^1 \delta(x - y) f(y)\, dy = f(x).$$

Moreover, u satisfies the boundary conditions, since

$$u(0) = \int_0^1 g(0, y) f(y)\, dy = 0, \qquad u(1) = \int_0^1 g(1, y) f(y)\, dy = 0.$$

Thus, (10.10) provides an integral representation of the solution of (10.9).

We may reformulate (10.13) in classical, pointwise terms. From (10.4), (10.12), and (10.13), we want $g(x, y)$ to satisfy the homogeneous ODE (as a function of x) when $x \neq y$, and we want the jump in $a(x)g_x(x, y)$ across $x = y$ to equal one in order to obtain a delta function after taking a second x-derivative. We therefore make the following definition.

Definition 10.11 A function $g : [0, 1] \times [0, 1] \to \mathbb{C}$ is a *Green's function* for (10.9) if it satisfies the following conditions.

(a) The function $g(x, y)$ is continuous on the square $0 \leq x, y \leq 1$, and twice continuously differentiable with respect to x on the triangles $0 \leq x \leq y \leq 1$ and $0 \leq y \leq x \leq 1$, meaning that the partial derivatives exist in the interiors of the triangles and extend to continuous functions on the closures. The left and right limits of the partial derivatives on $x = y$ are not equal, however.

(b) The function $g(x, y)$ satisfies the ODE with respect to x and the boundary conditions:

$$Ag = 0 \qquad \text{in } 0 < x < y < 1 \text{ and } 0 < y < x < 1, \tag{10.14}$$

$$g(0, y) = g(1, y) = 0 \qquad \text{for } 0 \leq y \leq 1. \tag{10.15}$$

(c) The jump in g_x across the line $x = y$ is given by

$$g_x\left(y^+, y\right) - g_x\left(y^-, y\right) = \frac{1}{a(y)}, \tag{10.16}$$

where the subscript x denotes a partial derivative with respect to the first variable in $g(x, y)$, and

$$g_x(y^+, y) = \lim_{x \to y^+} g_x(x, y), \qquad g_x(y^-, y) = \lim_{x \to y^-} g_x(x, y).$$

We will discuss the existence and construction of the Green's function below. First we show that if a function g satisfies the conditions in this definition, then the expression in (10.10) gives a solution of (10.9).

Proposition 10.12 Let A be given by (10.4), where $a, b, c \in C([0, 1])$ and $a(x) > 0$ for all $0 \le x \le 1$. If g satisfies (10.14)–(10.16) and $f \in C([0, 1])$, then Gf given by (10.11) is a solution of (10.9).

Proof. The proof is by direct computation. The only non-trivial part to check is that the function

$$u(x) = \int_0^1 g(x, y) f(y)\, dy$$

satisfies the ODE $Au = f$. We split the integration range into $0 \le y \le x$ and $x \le y \le 1$:

$$Au(x) = \left[a\frac{d^2}{dx^2} + b\frac{d}{dx} + c\right] \left[\int_0^x g(x, y) f(y)\, dy + \int_x^1 g(x, y) f(y)\, dy\right]. \tag{10.17}$$

Leibnitz's formula for the differentiation of an integral with variable limits states that if $\alpha(x)$ and $\beta(x)$ are continuously differentiable functions of x, and $h(x, y)$ is a continuous function of (x, y) on $\alpha(x) \le y \le \beta(x)$ that has a continuous partial derivative $h_x(x, y)$ with respect to x on $\alpha(x) \le y \le \beta(x)$, then

$$\frac{d}{dx}\int_{\alpha(x)}^{\beta(x)} h(x, y)\, dy = \beta'(x) h\left(x, \beta(x)\right) - \alpha'(x) h\left(x, \alpha(x)\right) + \int_{\alpha(x)}^{\beta(x)} h_x(x, y)\, dy. \tag{10.18}$$

Using this formula to compute the derivatives in the expression on the right-hand side of (10.17), we find that

$$\begin{aligned}
\frac{du}{dx} &= \int_0^x g_x(x, y) f(y)\, dy + g(x, x) f(x) + \int_x^1 g_x(x, y) f(y)\, dy - g(x, x) f(x) \\
&= \int_0^1 g_x(x, y) f(y)\, dy, \\
\frac{d^2u}{dx^2} &= \int_0^1 g_{xx}(x, y) f(y)\, dy + \left[g_x(x, x^-) - g_x(x, x^+)\right] f(x).
\end{aligned}$$

Thus,

$$A\left(\int_0^1 g(x,y)f(y)\,dy\right) = \int_0^1 Ag(x,y)f(y)\,dy + a(x)\left[g_x(x,x^-) - g_x(x,x^+)\right]f(x).$$

Since $g(x,y)$ is smooth in $x \le y$ and $x \ge y$, we have $g_x(x,x^-) = g_x(x^+,x)$ and $g_x(x,x^+) = g_x(x^-,x)$. It follows from the properties of g that $Au = f$. □

Thus, we can give an integral representation of the solution of (10.9) if we can construct the associated Green's function. We may write the Green's function in terms of the solutions of the homogeneous equations. When a, b, and c are continuous functions and $a(x) \neq 0$, the homogeneous ODE

$$au'' + bu' + cu = 0 \tag{10.19}$$

has a two-dimensional space of solutions spanned by any linearly independent pair of solutions. For example, we may construct a basis $\{u_1, u_2\}$ of the solution space by solving (10.19) subject to the initial conditions $u(0) = 1$, $u'(0) = 0$ for $u = u_1$ and $u(0) = 0$, $u'(0) = 1$ for $u = u_2$. The solutions exist by the existence theorem for ODEs in Theorem 3.7. The uniqueness of solutions of the initial value problem implies that if u is a solution of (10.19), then $u = u(0)u_1 + u'(0)u_2$, so u is a linear combination of $\{u_1, u_2\}$.

In order to construct a function g satisfying the conditions of Definition 10.11, we choose nonzero solutions v_1 and v_2 of $Av = 0$ such that

$$v_1(0) = 0, \quad v_2(1) = 0. \tag{10.20}$$

The pair $\{v_1, v_2\}$ is linearly independent if and only if the only solution of the homogeneous Dirichlet problem, $Au = 0$ with $u(0) = u(1) = 0$, is $u = 0$. The Green's function g then has the following form:

$$g(x,y) = \begin{cases} C(y)v_1(x)v_2(y) & \text{if } 0 \le x \le y, \\ C(y)v_1(y)v_2(x) & \text{if } y \le x \le 1. \end{cases} \tag{10.21}$$

It is clear that g is continuous, satisfies $Ag = 0$ whenever $x \neq y$, and $g(0,y) = g(1,y) = 0$. The jump condition in (10.16) is satisfied if $C(y)$ is given by

$$C(y) = \frac{1}{a(y)W(y)}, \tag{10.22}$$

where W is the *Wronskian* of v_1 and v_2:

$$W = \begin{vmatrix} v_1 & v_2 \\ v_1' & v_2' \end{vmatrix} = v_1 v_2' - v_2 v_1'. \tag{10.23}$$

If a is nonzero at every point, then the Wronskian of two linearly independent solutions is nonzero at every point, so C in (10.22) is well-defined.

Thus, if the homogeneous Dirichlet problem has only the zero solution, then g defined by (10.21) has all the properties required in Proposition 10.12 and is unique.

Example 10.13 The stationary temperature distribution in a rod of unit length that has both ends kept at a constant zero temperature, with heat loss through its surface proportional to u, and that is subject to a given nonuniform heat source per unit length f, is the solution of

$$-u'' + u = f, \qquad u(0) = u(1) = 0. \tag{10.24}$$

To construct the Green's function we need two linearly independent solutions v_1, v_2 of the homogeneous version of (10.24) that satisfy $v_1(0) = 0$ and $v_2(1) = 0$. The general solution of the homogeneous equation is of the form

$$u(x) = c_1 e^x + c_2 e^{-x}.$$

For v_1 and v_2 we choose the solutions

$$v_1(x) = \sinh x, \qquad v_2(x) = \sinh(1 - x),$$

where

$$\sinh x = \frac{e^x - e^{-x}}{2}.$$

The Wronskian, $W = -\sinh 1$, of these solutions is a nonzero constant, so the solutions are linearly independent combinations of e^x and e^{-x}. The Green's function is given by

$$g(x, y) = \begin{cases} \sinh x \sinh(1 - y)/\sinh 1 & \text{if } 0 \le x \le y \le 1, \\ \sinh y \sinh(1 - x)/\sinh 1 & \text{if } 0 \le y \le x \le 1. \end{cases}$$

We may also write this equation as

$$g(x, y) = \frac{\sinh(x_<) \sinh(1 - x_>)}{\sinh 1},$$

where

$$x_< = \min(x, y), \qquad x_> = \max(x, y).$$

The Green's function is a symmetric function of (x, y), as is always the case for real, self-adjoint boundary value problems.

We can use the Green's function to study the relationship between the solvability of the direct and adjoint BVPs. The following argument shows that the Fredholm alternative in Definition 8.19 applies to linear BVPs for ODEs.

Suppose that the homogeneous BVP,

$$Au = 0, \qquad Bu = 0,$$

has only the zero solution and the coefficient a of the highest derivative never vanishes. Then we can construct its Green's function, and therefore the nonhomogeneous BVP,

$$Au = f, \qquad Bu = 0,$$

has a unique solution $u \in C^2([0,1])$ for every $f \in C([0,1])$. If

$$A^*v = 0, \qquad B^*v = 0,$$

then for every $f \in C([0,1])$ we have

$$\langle f, v\rangle = \langle Au, v\rangle = \langle u, A^*v\rangle = 0.$$

Hence, $v = 0$, and the homogeneous adjoint BVP has only the zero solution. We can then construct the adjoint Green's function $g^*(x,y)$, and the adjoint BVP $A^*v = h$, $B^*v = 0$ has a unique solution $v \in C^2([0,1])$ for every $h \in C([0,1])$.

Since $(A^*)^{-1} = (A^{-1})^*$, the direct and adjoint Green's functions are related by

$$g^*(x,y) = \overline{g(y,x)}. \tag{10.25}$$

If A is self-adjoint, then g is Hermitian symmetric.

If A is singular, then A^* is also singular. In that case, it is possible to define a generalized inverse of A, whose kernel is called the *modified Green's function* of A, and show that the direct BVP is solvable if and only if the right-hand side is orthogonal to the kernel of the adjoint (see Exercise 10.9 for an example, and Stakgold [52] for further discussion).

Finally, we describe the Green's function representation of the solution of a BVP with nonhomogeneous boundary conditions. We begin by giving a formal derivation of the representation. For definiteness, we consider the Dirichlet problem for a real, second-order ODE,

$$Au = f(x), \qquad u(0) = \alpha_0, \quad u(1) = \alpha_1, \tag{10.26}$$

where A is defined in (10.4). A similar derivation applies to other types of boundary conditions. The adjoint Green's function $g^*(x,y)$ satisfies

$$A^*g^* = \delta(x-y), \qquad g^*(0,y) = g^*(1,y) = 0,$$

where A^* is a differential operator in x, and y plays the role of a parameter. Using Green's identity (10.6), we find that

$$\begin{aligned}\int_0^1 &\{g^*(x,y)Au(x) - A^*g^*(x,y)u(x)\}\, dx \\ &= \left[a(x)g^*(x,y)u_x(x) - a(x)g^*_x(x,y)u(x) + \{b(x) - a_x(x)\}\, g^*(x,y)u(x)\right]_{x=0}^1,\end{aligned}$$

where the x-subscript denotes a derivative with respect to x. We have formally that

$$\int_0^1 A^* g^*(x,y) u(x)\, dx = \int_0^1 \delta(x-y) u(x)\, dx = u(y).$$

Hence, using the equations satisfied by g^* and u, and rearranging the result, we get

$$u(y) = \int_0^1 g^*(x,y) f(x)\, dx + \left[a(x) g_x^*(x,y) u(x)\right]_{x=0}^1 .$$

Exchanging x and y in this equation, and using (10.25) to replace g^* by g, we obtain the following Green's function representation of the solution of (10.26):

$$u(x) = \int_0^1 g(x,y) f(y)\, dy + a(1) g_y(x,1)\alpha_1 - a(0) g_y(x,0)\alpha_0.$$

The above derivation of this representation does not constitute a proof. We can, however, verify the correctness of the result directly. From Proposition 10.12, the function

$$u_p(x) = \int_0^1 g(x,y) f(y)\, dy$$

is the solution of the nonhomogeneous equation $Au_p = f$ that satisfies the homogeneous boundary conditions $u_p(0) = u_p(1) = 0$. On the other hand, it follows from (10.20)–(10.23) that

$$u_h(x) = a(1) g_y(x,1)\alpha_1 - a(0) g_y(x,0)\alpha_0$$

is the solution of the homogeneous equation $Au_h = 0$ that satisfies the nonhomogeneous boundary conditions $u_h(0) = \alpha_0$, $u_h(1) = \alpha_1$.

10.4 Weak derivatives

In the previous sections, we considered "classical" differential operators that act on continuously differentiable functions. The resulting differential operators lack a number of desirable properties; for example, they are not closed or self-adjoint. To obtain such operators, we need to extend the domains of the classical differentiation operators to include functions whose weak derivatives belong to L^2. In this section, we define the notion of a weak L^2-derivative in terms of integration against test functions. We show that weakly differentiable functions can be approximated by smooth functions, and we use this fact to study some of their basic properties. We also define the Sobolev spaces H^k of functions with k square-integrable derivatives, and use them to give a precise description of the domains of some simple self-adjoint ordinary differential operators.

We begin by considering functions defined on $\mathbb{R}$. We say that $\varphi : \mathbb{R} \to \mathbb{C}$ is a *test function* if it has compact support and continuous derivatives of all orders. We denote the space of test functions by $C_c^\infty(\mathbb{R})$. The following example shows that there are many test functions.

Example 10.14 The function

$$\varphi(x) = \begin{cases} \exp\left[-1/(1-x^2)\right] & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1, \end{cases}$$

belongs to $C_c^\infty(\mathbb{R})$. All its derivatives exist and are equal to zero at $x = \pm 1$. This function is not analytic at $x = \pm 1$, however, since its Taylor series at these points converge to zero, rather than to the function itself. Rescaling this function,

$$\psi(x) = c\varphi\left(\frac{x - x_0}{\delta}\right),$$

we obtain a test function ψ supported on the interval $|x - x_0| \leq \delta$ whose integral has any desired value.

Before defining weak derivatives, we show that $C_c^\infty(\mathbb{R})$ is dense in $L^2(\mathbb{R})$. To do this, we approximate an L^2-function by its convolution with a smooth approximate identity, a technique called *mollification.*

Let $\varphi \in C_c^\infty(\mathbb{R})$ be a nonnegative test function with support $[-1, 1]$ and

$$\int_{\mathbb{R}} \varphi(x)\, dx = 1.$$

For $\epsilon > 0$, we let

$$\varphi_\epsilon(x) = \frac{1}{\epsilon}\varphi\left(\frac{x}{\epsilon}\right).$$

We call such a function φ_ϵ a *mollifier* or *averaging kernel.* The family $\{\varphi_\epsilon \mid \epsilon > 0\}$ is an approximate identity as $\epsilon \to 0^+$ since the support of φ_ϵ shrinks to the origin and each φ_ϵ has unit integral. If $u \in L^2(\mathbb{R})$, we define the *mollification* $u_\epsilon = \varphi_\epsilon * u$ of u, meaning that

$$u_\epsilon(x) = \int_{\mathbb{R}} \varphi_\epsilon(x - y)u(y)\, dy. \tag{10.27}$$

The function u_ϵ belongs to $C^\infty(\mathbb{R})$ because

$$u_\epsilon^{(k)}(x) = \int_{\mathbb{R}} \varphi_\epsilon^{(k)}(x - y)u(y)\, dy. \tag{10.28}$$

The differentiation under the integral sign is justified by the dominated convergence theorem.

Lemma 10.15 If $u \in L^2(\mathbb{R})$ and u_ϵ is defined by (10.27), where φ_ϵ is a mollifier, then $\|u_\epsilon\| \leq \|u\|$, where $\|\cdot\|$ denotes the L^2-norm.

Proof. Using the fact that φ_ϵ is nonnegative and has unit integral over $\mathbb{R}$, we find from the Cauchy-Schwarz inequality that

$$\begin{aligned} |u_\epsilon(x)| &= \left| \int_{\mathbb{R}} \varphi_\epsilon^{1/2}(x-y)\varphi_\epsilon^{1/2}(x-y)u(y)\,dy \right| \\ &\le \left(\int_{\mathbb{R}} \varphi_\epsilon(x-y)\,dy \right)^{1/2} \left(\int_{\mathbb{R}} \varphi_\epsilon(x-y)|u(y)|^2\,dy \right)^{1/2} \\ &= \left(\int_{\mathbb{R}} \varphi_\epsilon(x-y)|u(y)|^2\,dy \right)^{1/2}. \end{aligned}$$

Squaring this equation and integrating the result with respect to x, we obtain that

$$\int_{\mathbb{R}} |u_\epsilon(x)|^2\,dx \le \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \varphi_\epsilon(x-y)|u(y)|^2\,dy \right) dx.$$

Exchanging the order of integration, which is justified by Fubini's theorem, we find that

$$\|u_\epsilon\|^2 \le \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \varphi_\epsilon(x-y)\,dx \right) |u(y)|^2\,dy = \|u\|^2.$$

□

Using this lemma, we prove that the mollifications u_ϵ converge to u in L^2.

Theorem 10.16 The space $C_c^\infty(\mathbb{R})$ is dense in $L^2(\mathbb{R})$. If $u \in L^2(\mathbb{R})$, then $u_\epsilon \to u$ strongly in $L^2(\mathbb{R})$ as $\epsilon \to 0^+$.

Proof. Suppose that $u \in L^2(\mathbb{R})$. Let $\eta > 0$ be arbitrary. The space $C_c(\mathbb{R})$ of continuous functions with compact support is dense in $L^2(\mathbb{R})$ (see Theorem 12.50), so there is a $v \in C_c(\mathbb{R})$ such that $\|u - v\| < \eta/3$. We define $v_\epsilon = \varphi_\epsilon * v \in C_c^\infty(\mathbb{R})$. Then, from Lemma 10.15, we have

$$\|u_\epsilon - v_\epsilon\| \le \|u - v\| < \eta/3.$$

The supports of v and v_ϵ are contained in a compact set. The argument in the proof of Theorem 7.2 implies that $v_\epsilon \to v$ uniformly as $\epsilon \to 0^+$, and therefore $v_\epsilon \to v$ in $L^2(\mathbb{R})$. There is a $\delta > 0$ such that $\|v - v_\epsilon\| < \eta/3$ for $0 < \epsilon < \delta$, and then

$$\|u - u_\epsilon\| \le \|u - v\| + \|v - v_\epsilon\| + \|v_\epsilon - u_\epsilon\| < \eta.$$

It follows that $u_\epsilon \to u$ in $L^2(\mathbb{R})$. □

To motivate the definition of a weak L^2-derivative, we first consider $u \in C^1(\mathbb{R})$ with a "classical" pointwise, continuous derivative

$$v(x) = u'(x). \tag{10.29}$$

The use of this formula, followed by an integration by parts, implies that

$$\int_{\mathbb{R}} v\varphi\,dx = -\int_{\mathbb{R}} u\varphi'\,dx \qquad \text{for all } \varphi \in C_c^\infty(\mathbb{R}). \tag{10.30}$$

The boundary terms are zero because φ vanishes outside a compact set. Conversely, if $u \in C^1(\mathbb{R})$ satisfies (10.30) for some $v \in L^2(\mathbb{R})$, then another integration by parts implies that

$$\int_{\mathbb{R}} v\varphi \, dx = \int_{\mathbb{R}} u'\varphi \, dx \qquad \text{for all } \varphi \in C_c^\infty(\mathbb{R}).$$

Hence, $v = u'$ because $C_c^\infty(\mathbb{R})$ is dense in $L^2(\mathbb{R})$. Thus, (10.29) and (10.30) are equivalent when u is continuously differentiable. Equation (10.30) makes sense, however, if u and v are only square-integrable, because the derivative acts on the test function. Rewriting the integrals as inner products, we obtain the following definition of a weak derivative.

Definition 10.17 A function $u \in L^2(\mathbb{R})$ has a weak derivative $v = u' \in L^2(\mathbb{R})$ if

$$\langle v, \varphi \rangle = -\langle u, \varphi' \rangle \qquad \text{for all } \varphi \in C_c^\infty(\mathbb{R}).$$

The Sobolev space $H^k(\mathbb{R})$ consists of the functions with k square-integrable weak derivatives,

$$H^k(\mathbb{R}) = \left\{ u \in L^2(\mathbb{R}) \mid u, u', \ldots, u^{(k)} \in L^2(\mathbb{R}) \right\},$$

equipped with the following norm and inner product:

$$\|u\|_{H^k} = \left(\int_{\mathbb{R}} \left\{ |u|^2 + |u'|^2 + \ldots + |u^{(k)}|^2 \right\} dx \right)^{1/2},$$

$$\langle u, v \rangle_{H^k} = \int_{\mathbb{R}} \left\{ \overline{u}v + \overline{u}'v' + \ldots + \overline{u}^{(k)} v^{(k)} \right\} dx.$$

Proposition 10.18 The differentiation operator $D : H^1(\mathbb{R}) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined by $Du = u'$ is closed.

Proof. Suppose that $u_n \to u$ and $Du_n \to v$ in $L^2(\mathbb{R})$. It follows from this convergence and the definition of the weak derivative that for every test function φ,

$$\langle v, \varphi \rangle = \lim_{n\to\infty} \langle u_n', \varphi \rangle = - \lim_{n\to\infty} \langle u_n, \varphi' \rangle = -\langle u, \varphi' \rangle.$$

Hence $u \in H^1(\mathbb{R})$, and $Du = v$, so D is closed. □

The closedness of D implies that $H^k(\mathbb{R})$ is complete and therefore a Hilbert space. If a sequence (u_n) is Cauchy in H^k, then $(u_n^{(j)})$ is Cauchy in L^2 for each $j \le k$. Since L^2 is complete, there are functions $v, v_j \in L^2$ such that $u_n \to v$ and $u_n^{(j)} \to v_j$ as $n \to \infty$. Since D is closed, it follows that $v_j = v^{(j)}$ for each $j \le k$, so $u_n \to v$ in H^k.

An alternative, but equivalent, way to define L^2-derivatives is as the L^2-limit of smooth derivatives. Thus, we say that $u' = v$ if there is a sequence of smooth functions u_n such that $u_n \to u$ and $u_n' \to v$ in L^2. The equivalence of these definitions follows from the following theorem, which shows that any H^k-function can be approximated in the H^k-norm by a test function.

Theorem 10.19 The space $C_c^\infty(\mathbb{R})$ is dense in $H^k(\mathbb{R})$. If $u \in H^k(\mathbb{R})$ and $u_\epsilon = \varphi_\epsilon * u$, where φ_ϵ is a mollifier, then $u_\epsilon \to u$ strongly in $H^k(\mathbb{R})$ as $\epsilon \to 0^+$.

Proof. Suppose that $u \in H^k(\mathbb{R})$, and $u_\epsilon = \varphi_\epsilon * u$, where φ_ϵ is a mollifier. The function $\varphi_{\epsilon,x} : \mathbb{R} \to \mathbb{R}$ defined by

$$\varphi_{\epsilon,x}(y) = \varphi_\epsilon(x-y)$$

is a test function in $C_c^\infty(\mathbb{R})$. It therefore follows from (10.28) and the definition of the weak derivative that

$$\begin{aligned} u_\epsilon^{(j)}(x) &= \int_{\mathbb{R}} \frac{\partial^j}{\partial x^j}\left[\varphi_\epsilon(x-y)\right] u(y)\, dy \\ &= (-1)^j \int_{\mathbb{R}} \frac{\partial^j}{\partial y^j}\left[\varphi_\epsilon(x-y)\right] u(y)\, dy \\ &= \int_{\mathbb{R}} \varphi_\epsilon(x-y) u^{(j)}(y)\, dy \end{aligned}$$

for every $j \le k$ and $x \in \mathbb{R}$. Theorem 10.16 implies that $u_\epsilon^{(j)} \to u^{(j)}$ in $L^2(\mathbb{R})$, so $u_\epsilon \to u$ in $H^k(\mathbb{R})$.

If u does not have compact support, then u_ϵ does not have compact support either. To show that $C_c^\infty(\mathbb{R})$ is dense in $H^k(\mathbb{R})$, we truncate u before mollification. We choose $\psi \in C_c^\infty(\mathbb{R})$ such that

$$\psi(x) = \begin{cases} 1 & \text{if } |x| \le 1, \\ 0 & \text{if } |x| \ge 2, \end{cases}$$

and define $\psi_n(x) = \psi(x/n)$. Then $u_n = \psi_n u$ has compact support, and $u_n \in H^k(\mathbb{R})$ when $u \in H^k(\mathbb{R})$. One can show that $u_n \to u$ in $H^k(\mathbb{R})$ as $n \to \infty$, and we have just proved that $\varphi_\epsilon * u_n \to u_n$ as $\epsilon \to 0^+$. Since $\varphi_\epsilon * u_n \in C_c^\infty(\mathbb{R})$, the density follows. □

As an illustration of the use of mollification, we show that integration by parts holds for H^1-functions.

Proposition 10.20 Suppose that $u, v \in H^1(\mathbb{R})$, then

$$\int_{\mathbb{R}} uv'\, dx = -\int_{\mathbb{R}} u'v\, dx. \tag{10.31}$$

Proof. From Theorem 10.19, there are sequences (u_n) and (v_n) in $C_c^\infty(\mathbb{R})$ such that $u_n \to u$ and $v_n \to v$ in $H^1(\mathbb{R})$. Since u_n and v_n vanish outside a compact set, we have

$$\int_{\mathbb{R}} u_n v_n'\, dx = -\int_{\mathbb{R}} u_n' v_n\, dx.$$

Taking the limit of this equation as $n \to \infty$, and using the continuity of the L^2-inner product with respect to L^2-convergence, we obtain (10.31). $\square$

Example 10.21 Let $A : H^1(\mathbb{R}) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the operator $A = iD$, meaning that $Au = iu'$. We claim that A is self adjoint. For every $u, v \in H^1(\mathbb{R})$, we have

$$\langle Au, v\rangle = -i \int_{\mathbb{R}} \overline{u}' v\, dx = i \int_{\mathbb{R}} \overline{u} v'\, dx = \langle u, Av\rangle.$$

Hence, A is symmetric and $\mathcal{D}(A^*) \supset H^1(\mathbb{R})$. If $v \in \mathcal{D}(A^*)$, then there is a $w \in L^2(\mathbb{R})$ such that

$$\langle iu', v\rangle = \langle u, w\rangle \qquad \text{for all } u \in H^1(\mathbb{R}).$$

Since $H^1(\mathbb{R})$ contains $C_c^\infty(\mathbb{R})$, it follows that

$$\langle \varphi', v\rangle = \langle \varphi, iw\rangle \qquad \text{for all } \varphi \in C_c^\infty(\mathbb{R}),$$

which means that $v \in H^1(\mathbb{R})$ and $w = iv'$. Thus, $\mathcal{D}(A^*) \subset H^1(\mathbb{R})$, so $\mathcal{D}(A^*) = H^1(\mathbb{R})$, and A is self-adjoint.

We now consider functions defined on a bounded open interval $(0,1)$. The space of test functions $C_c^\infty((0,1))$ consists of smooth functions that vanish outside a closed interval contained strictly inside $(0,1)$. A function $v \in L^2([0,1])$ is the weak L^2-derivative of $u \in L^2([0,1])$ if

$$\int_0^1 v\varphi\, dx = -\int_0^1 u\varphi'\, dx \qquad \text{for all } \varphi \in C_c^\infty((0,1)).$$

The Sobolev space $H^k((0,1))$ consists of the functions in $L^2([0,1])$ with k weak derivatives in $L^2([0,1])$.

Theorem 10.22 The space $C^\infty([0,1])$ is dense in $H^k((0,1))$.

Proof. We would like to obtain a smooth approximation of $u \in H^k((0,1))$ by extending u to a function

$$\widetilde{u}(x) = \begin{cases} u(x) & \text{if } x \in (0,1), \\ 0 & \text{if } x \neq (0,1), \end{cases}$$

in $L^2(\mathbb{R})$, and mollifying the extension $\widetilde{u}$. However, $\widetilde{u}$ need not belong to $H^k(\mathbb{R})$ because it may be discontinuous at the endpoints of $(0,1)$, so we cannot conclude immediately that the restriction of $\varphi_\epsilon * \widetilde{u}$ to $(0,1)$ converges to u in $H^k((0,1))$. The proof therefore requires a more complicated argument, which we outline without giving all the details. For $\delta > 0$, we define the stretching map $L_\delta : (0,1) \to (-\delta, 1+\delta)$ by

$$L_\delta(x) = (1+2\delta)\left(x - \frac{1}{2}\right) + \frac{1}{2}.$$

We define $u_\delta \in H^k((-\delta, 1+\delta))$ by $u_\delta = u \circ L_\delta^{-1}$. Then one can show that the restriction of u_δ to $(0,1)$ converges to u in $H^k((0,1))$ as $\delta \to 0^+$. We extend u_δ by zero to obtain $\widetilde{u}_\delta \in L^2(\mathbb{R})$. Let φ_ϵ be a mollifier, and

$$\varphi_{\epsilon,x}(y) = \varphi_\epsilon(x-y).$$

For $x \in (0,1)$ and $\epsilon < \delta$, we have $\varphi_{\epsilon,x} \in C_c^\infty((-\delta, 1+\delta))$. The restriction of $\varphi_\epsilon * \widetilde{u}_\delta$ to $(0,1)$ is therefore a C^∞ function on $[0,1]$ that converges to the restriction of u_δ to $(0,1)$ in $H^k((0,1))$ as $\epsilon \to 0^+$. The result then follows. □

Although $C_c^\infty(\mathbb{R})$ is dense in $H^k(\mathbb{R})$ and $C_c^\infty((0,1))$ is dense in $L^2([0,1])$, it is not true that $C_c^\infty((0,1))$ is dense in $H^k((0,1))$ for $k \geq 1$.

Definition 10.23 The Sobolev space

$$H_0^k((0,1)) = \overline{C_c^\infty((0,1))} \subset H^k((0,1))$$

is the closure of $C_c^\infty((0,1))$ in $H^k((0,1))$.

It follows from the Sobolev embedding theorem below that $H_0^k((0,1))$ consists of the functions in $H^k((0,1))$ whose derivatives of order less than or equal to $k-1$ vanish at the endpoints of $(0,1)$.

In Section 7.2, we proved the Sobolev embedding theorem for periodic functions by using Fourier series. Here we give a different proof, for which we need the following lemma.

Lemma 10.24 Suppose that $h : [0,1] \to \mathbb{R}$ is a continuous function such that

$$\int_0^1 h(x)\, dx = 1.$$

Define $k : [0,1] \times [0,1] \to \mathbb{R}$ by

$$k(x,y) = \begin{cases} \int_0^y h(t)\, dt & \text{if } 0 \leq y \leq x, \\ -\int_y^1 h(t)\, dt & \text{if } x < y \leq 1. \end{cases}$$

If $u \in C^1([0,1])$, then

$$u(x) = \int_0^1 u(y)h(y)\, dy + \int_0^1 k(x,y)u'(y)\, dy \qquad \text{for all } 0 \leq x \leq 1.$$

Proof. If $u \in C^1([0,1])$, the fundamental theorem of calculus implies that, for every $x, y \in [0,1]$,

$$u(x) = u(y) + \int_y^x u'(t)\, dt.$$

Multiplying this equation by $h(y)$ and integrating the result, we obtain that

$$u(x) = \int_0^1 u(y)h(y)\,dy + \int_0^1 \left(\int_y^x u'(t)\,dt\right) h(y)\,dy.$$

Exchanging the order of integration, we find that

$$\begin{aligned}\int_0^1 \left(\int_y^x u'(t)\,dt\right) h(y)\,dy &= \int_0^x \left(\int_a^t h(y)\,dy\right) u'(t)\,dt \\ &\quad - \int_x^1 \left(\int_t^1 h(y)\,dy\right) u'(t)\,dt \\ &= \int_0^1 k(x,t)u'(t)\,dt,\end{aligned}$$

and the result follows. □

Theorem 10.25 (Sobolev embedding) The space $H^1((0,1))$ is a subset of $C([0,1])$. There is a constant $C > 0$ such that

$$\|u\|_\infty \leq C\|u\|_{H^1} \qquad \text{for all } u \in H^1((0,1)). \tag{10.32}$$

Proof. First, suppose that $u \in C^\infty([0,1])$. Then, from Lemma 10.24 and the Cauchy-Schwarz inequality, we find that

$$\begin{aligned}|u(x)| &\leq \left|\int_0^1 u(y)h(y)\,dy\right| + \left|\int_0^1 k(x,y)u'(y)\,dy\right| \\ &\leq \|h\|_{L^2}\|u\|_{L^2} + \|k(x,\cdot)\|_{L^2}\|u'\|_{L^2} \\ &\leq C\|u\|_{H^1},\end{aligned}$$

since $\|k(x,\cdot)\|_{L^2}$ is bounded uniformly in x for a continuous function h. Taking the supremum of this inequality with respect to x, we obtain that $\|u\|_\infty \leq C\|u\|_{H^1}$ for every $u \in C^\infty([0,1])$. Since C^∞ is dense in H^1, it follows that this inequality holds for every $u \in H^1$. Furthermore, every $u \in H^1$ is the uniform limit of a sequence of C^∞-functions, and is therefore continuous. □

Strictly speaking, an element of H^1 is an equivalence class of square-integrable functions that are equal almost everywhere, and the embedding theorem states that each such equivalence class contains a continuous function. An alternative way to state this result is that there is a continuous map, or embedding,

$$J : H^1((0,1)) \to C([0,1])$$

that identifies a function u, regarded as an element of $H^1((0,1))$, with the same function u, regarded as an element of $C([0,1])$. The following theorem shows that this embedding is compact.

Theorem 10.26 (Rellich) A bounded subset of $H^1((0,1))$ is a precompact subset of $C([0,1])$.

Proof. Since $C^\infty([0,1])$ is dense in $H^1((0,1))$, it is sufficient to show that a subset of $C^\infty([0,1])$ that is bounded in $H^1((0,1))$ is precompact in $C([0,1])$. Suppose that $\mathcal{F}$ is a subset of $C^\infty([0,1])$ such that there is a constant M with

$$\|u\|_{H^1} \leq M \qquad \text{for all } u \in \mathcal{F}.$$

From (10.32), the set $\mathcal{F}$ is bounded in $C([0,1])$. Moreover, by the fundamental theorem of calculus and the Cauchy-Schwarz inequality, we have for every $u \in \mathcal{F}$ and $x, y \in [0,1]$ that

$$\begin{aligned} |u(x) - u(y)| &= \left| \int_x^y u'(t)\, dt \right| \\ &= \left| \int_0^1 \chi_{[x,y]}(t) u'(t)\, dt \right| \\ &\leq |x-y|^{1/2} \left(\int_0^1 |u'(t)|^2\, dt \right)^{1/2} \\ &\leq M\, |x-y|^{1/2}. \end{aligned}$$

Here, $\chi_{[x,y]}$ is the characteristic function of the interval $[x,y]$. Thus $\mathcal{F}$ is equicontinuous, and therefore the Arzelà-Ascoli theorem implies that it is precompact in $C([0,1])$. □

A function $u \in C([0,1])$ that satisfies

$$|u(x) - u(y)| \leq M|x-y|^r \qquad \text{for all } x, y \in [0,1]$$

for constants $M > 0$ and $0 < r \leq 1$ is said to be *Hölder continuous* with exponent r. Thus, the proof of Theorem 10.26 shows that every $u \in H^1((0,1))$ is Hölder continuous with exponent $1/2$. For a generalization of this result, see Theorem 12.73.

Proposition 10.27 If A is the second-order ordinary differential operator defined in (10.4), where a, b, c are smooth coefficient functions, then Green's formula (10.6) holds for all $u, v \in H^2((0,1))$.

Proof. If $u, v \in H^2((0,1))$, then there are sequences (u_n), (v_n) in $C^\infty([0,1])$ such that $u_n \to u$ and $v_n \to v$ in $H^2((0,1))$. From Green's formula, we have

$$\langle Au_n, v_n\rangle - \langle u_n, A^* v_n\rangle = \left[a\left(\overline{u_n}' v_n - \overline{u_n} v_n'\right) + (c - a')\overline{u_n} v_n\right]_0^1.$$

Letting $n \to \infty$, we obtain Green's formula for u and v, because $Au_n \to Au$, $Av_n \to Av$ in L^2, and, from the Sobolev embedding theorem, the boundary terms converge pointwise. □

Example 10.28 Let us prove that the second derivative operator $A = -D^2$ with domain

$$\mathcal{D}(A) = \left\{u \in H^2((0,1)) \mid u(0) = u(1) = 0\right\}$$

is self-adjoint. If $v \in \mathcal{D}(A^*)$, then there is a $w \in L^2([0,1])$ such that

$$\langle -u'', v\rangle = \langle u, w\rangle \qquad \text{for all } u \in \mathcal{D}(A).$$

Since $\mathcal{D}(A) \supset C_c^\infty((0,1))$, it follows from the definition of the weak derivative that $v \in H^2((0,1))$ and $w = -v''$. Hence, $\mathcal{D}(A^*) \subset H^2((0,1))$, and $A^* = -D^2$ on its domain. If $u \in \mathcal{D}(A)$ and $v \in H^2((0,1))$, then an integration by parts implies that

$$\langle -u'', v\rangle = \langle u, -v''\rangle + \left[-\overline{u}'v\right]_0^1.$$

Thus, v belongs to $\mathcal{D}(A^*)$ if and only if $v(0) = v(1) = 0$, so $\mathcal{D}(A) = \mathcal{D}(A^*)$ and $A = A^*$.

As the previous example illustrates, the direct verification of self-adjointness may be nontrivial even for the simplest unbounded operators. The following result, which we state without proof, gives a basic criterion for self-adjointness.

Theorem 10.29 Let A be a closed, symmetric operator on a Hilbert space $\mathcal{H}$. Then the following statements are equivalent:

(a) A is self-adjoint;
(b) $\ker(A^* \pm iI) = \{0\}$;
(c) $\operatorname{ran}(A \pm iI) = \mathcal{H}$.

If $m = \dim\ker(A^* - iI)$ and $n = \dim\ker(A^* + iI)$, then the pair (m, n) is called the *deficiency index* of A. Thus, a closed, symmetric operator is self-adjoint if and only if its deficiency index is $(0, 0)$.

10.5 The Sturm-Liouville eigenvalue problem

In this section, we study the *Sturm-Liouville eigenvalue problem*

$$\begin{aligned} -(pu')' + qu &= \lambda u, \\ u(0) = u(1) &= 0, \end{aligned} \tag{10.33}$$

where the coefficients p, q are given real-valued functions, and $\lambda \in \mathbb{R}$. For definiteness, we consider the Dirichlet problem, but other self-adjoint boundary conditions can be analyzed in a similar way. Equation (10.33) is the spectral problem for the self-adjoint *Sturm-Liouville operator* $A : \mathcal{D}(A) \subset L^2([0,1]) \to L^2([0,1])$ defined by

$$Au = -(pu')' + qu, \tag{10.34}$$

$$\mathcal{D}(A) = \left\{u \in H^2((0,1)) \mid u(0) = u(1) = 0\right\}. \tag{10.35}$$

Theorem 10.30 Suppose that $p \in C^1([0,1])$, $q \in C([0,1])$ are real-valued functions and $p(x) > 0$ for all $x \in [0,1]$. There is an orthonormal basis of $L^2([0,1])$ that consists of eigenfunctions of the Sturm-Liouville eigenvalue problem (10.33). The eigenvalues $\lambda_1 < \lambda_2 < \ldots$ are real and simple, and $\lambda_n \to \infty$ as $n \to \infty$.

Proof. We begin by showing that if λ is real and sufficiently negative, then the only solution of (10.33) is $u = 0$, so λ is not an eigenvalue of A. We take the inner product of (10.33) with u, and integrate the result by parts. This gives:

$$\int_0^1 \left\{p\left|u'\right|^2 + q\left|u\right|^2\right\}\, dx = \lambda \int_0^1 |u|^2\, dx. \tag{10.36}$$

We let

$$\alpha = \min_{0\le x\le 1} p(x), \quad \beta = \min_{0\le x\le 1} q(x). \tag{10.37}$$

Since $p > 0$, we have $\alpha > 0$; if $q > 0$, then $\beta > 0$ also, but we may have $\beta \le 0$. Using (10.37) in (10.36), and rearranging the result, we find that

$$\alpha \int_0^1 |u'|^2\, dx + (\beta - \lambda)\int_0^1 |u|^2\, dx \le 0.$$

It follows that if $\lambda < \beta$, then

$$\int_0^1 |u'|^2\, dx = \int_0^1 |u|^2\, dx = 0,$$

so $u = 0$.

This result shows that the kernel of $A - \lambda I$ is zero when $\lambda < \beta$. From what we have shown previously, the Green's function g_λ of $A - \lambda I$ exists. Therefore, λ is in the resolvent set of A, and the self-adjoint resolvent operator R_λ is given by

$$R_\lambda = (\lambda I - A)^{-1} : L^2([0,1]) \to L^2([0,1]),$$
$$R_\lambda f(x) = -\int_0^1 g_\lambda(x,y) f(y)\, dy.$$

Since g_λ is continuous, we certainly have

$$\int_0^1\int_0^1 [g_\lambda(x,y)]^2\, dxdy < \infty,$$

so R_λ is Hilbert-Schmidt and hence compact. The spectral theorem for compact, self-adjoint operators implies that there is an orthonormal basis of $L^2([0,1])$ consisting of eigenvectors $\{u_n \mid n \in \mathbb{N}\}$ of R_λ with eigenvalues $\{\mu_n \mid n \in \mathbb{N}\}$ such that $\mu_n \to 0$ as $n \to \infty$. Since $(\lambda I - A)R_\lambda = I$, we have $u_n \in \mathcal{D}(A)$ and $Au_n = \lambda_n u_n$, where

$$\beta \le \lambda_n = \lambda - \frac{1}{\mu_n},$$

so $\lambda_n \to \infty$ as $n \to \infty$. The Sturm-Liouville operator A therefore has a complete orthonormal set of eigenvectors that forms a basis of $L^2([0,1])$.

If an eigenvalue λ is not simple, then (10.33) has a pair of linearly independent solutions. It follows that every solution of the Sturm-Liouville equation

$$-\left(pu'\right)' + qu = \lambda u$$

vanishes at $x = 0, 1$ since it is a linear combination of eigenvectors. This contradicts the existence of a solution of the initial value problem with nonzero initial data for $u(0)$. □

The compactness of the resolvent may also be obtained as a consequence of Rellich's theorem, in Theorem 10.26. We define a symmetric, sesquilinear form a on $H_0^1\left((0,1)\right)$ by

$$a(u,v) = \int_0^1 \left\{p\bar{u}'v' + q\bar{u}v\right\}\, dx. \tag{10.38}$$

We call a the *Dirichlet form* of A. For $u, v \in \mathcal{D}(A)$, we have

$$a(u,v) = \langle Au, v\rangle = \langle u, Av\rangle.$$

The set $\mathcal{D}(A) \times \mathcal{D}(A)$ is dense in $H_0^1\left((0,1)\right) \times H_0^1\left((0,1)\right)$, and the form extends continuously to the larger space. The associated quadratic form $a(u,u)$ on $H_0^1((0,1))$ is given by

$$a(u,u) = \int_0^1 \left\{p\left|u'\right|^2 + q\left|u\right|^2\right\}\, dx.$$

As we saw above, we have the estimate

$$a(u,u) \geq \alpha \int_0^1 \left|u'\right|^2\, dx + \beta \int_0^1 \left|u\right|^2\, dx.$$

It follows that if $u \in \mathcal{D}(A)$, then

$$\langle (A - \lambda I)u, u\rangle \geq \alpha \int_0^1 \left|u'\right|^2\, dx + (\beta - \lambda) \int_0^1 \left|u\right|^2\, dx.$$

If $\lambda < \beta$, this estimate implies that $(\lambda I - A)^{-1}$ maps bounded sets in L^2 to bounded sets in H_0^1, which are precompact in L^2 by Rellich's theorem. Hence, A has a compact resolvent.

The operator A is diagonal in a basis of eigenvectors. We may therefore solve the Sturm-Liouville BVP

$$\begin{aligned} -\left(pu'\right)' + qu &= \lambda u + f,\\ u(0) = u(1) &= 0, \end{aligned}$$

by expanding u and f with respect to the orthonormal basis of eigenvectors. Assuming that λ is not an eigenvalue of A, the solution is

$$u(x) = \sum_{n=1}^{\infty} \frac{\langle f_n, u \rangle}{\lambda_n - \lambda} u_n(x),$$

where the series converges in $L^2([0,1])$. We may write the operator A as

$$A = \sum_{n=1}^{\infty} \lambda_n u_n \otimes u_n, \quad \mathcal{D}(A) = \left\{ \sum_{n=1}^{\infty} c_n u_n \in \mathcal{H} \;\middle|\; \sum_{n=1}^{\infty} \left(1 + \lambda_n^2\right) |c_n|^2 < \infty \right\},$$

where the sum converges strongly on the domain of A. The resolvent operator of A is

$$R_\lambda = \sum_{n=1}^{\infty} \frac{u_n \otimes u_n}{\lambda - \lambda_n},$$

where the sum converges uniformly for $\lambda \in \rho(A)$, and the Green's function g_λ of $A - \lambda I$ is given by

$$g_\lambda(x, y) = \sum_{n=1}^{\infty} \frac{u_n(x) u_n(y)}{\lambda_n - \lambda},$$

where the sum converges in $L^2([0,1]\times[0,1])$. The resolvent operator and the Green's function, regarded as functions of λ, have poles at the eigenvalues of A.

Example 10.31 The simplest example of a Sturm-Liouville eigenvalue problem is

$$-u'' = \lambda u, \qquad u(0) = u(1) = 0.$$

The eigenfunctions u_n and eigenvalues λ_n, where $n = 1, 2, 3, \ldots$, are given by

$$u_n(x) = \sqrt{2} \sin(n\pi x), \qquad \lambda_n = n^2 \pi^2.$$

The associated eigenfunction expansion is a Fourier sine expansion. Neumann boundary conditions lead to a Fourier cosine expansion. Thus, Theorem 10.30 provides another proof of the completeness of the Fourier basis. In this example, the nth eigenfunction has $n - 1$ zeros inside the interval $(0, 1)$. This property holds for all regular Sturm-Liouville eigenvalue problems (see Coddington and Levinson [6]).

A Sturm-Liouville problem is said to be *regular* if it is posed on a bounded interval $[a, b]$ and $p(x) \neq 0$ for every $a \leq x \leq b$; otherwise, it is said to be *singular*. We have just proved that a regular Sturm-Liouville eigenvalue problem has a complete orthonormal set of eigenvectors. The resolvent operator of a singular Sturm-Liouville operator may or may not be compact and, if it is not compact, then the corresponding Sturm-Liouville eigenvalue problem may have a continuous spectrum as well as, or instead of, a point spectrum.

Example 10.32 The function $p(x) = 1 - x^2$ vanishes at the boundaries of the interval $[-1, 1]$. The corresponding singular Sturm-Liouville eigenvalue problem on $[-1, 1]$, with $q = 0$, is

$$\begin{gathered} -\left[(1-x^2)u'\right]' = \lambda u \qquad \text{for } -1 < x < 1, \\ \int_{-1}^{1} \left\{(1-x^2)\,|u'|^2 + |u|^2\right\}\,dx < \infty. \end{gathered} \tag{10.39}$$

This eigenvalue problem has a complete orthogonal set of eigenvectors, the Legendre polynomials, with eigenvalues $\lambda_n = n(n+1)$ (see Exercise 6.12). Since $\lambda_n = n(n+1) \to \infty$ as $n \to \infty$, the resolvent operator $(-I - A)^{-1}$ is compact. No boundary conditions are required at the singular endpoints. The condition in (10.39) rules out singular solutions which are unbounded at $x = \pm 1$.

More generally, if $m \in \mathbb{N}$ is a positive integer, then the singular Sturm-Liouville problem,

$$\begin{gathered} -\left[(1-x^2)u'\right]' + \frac{m^2}{1-x^2}u = \lambda u \qquad \text{for } -1 < x < 1, \\ \int_{-1}^{1} \left\{(1-x^2)\,|u'|^2 + \frac{|u|^2}{1-x^2}\right\}\,dx < \infty, \end{gathered}$$

has eigenvalues $\lambda_n = n(n+1)$, where $n = m, m+1, \ldots$. The corresponding eigenfunctions are the *Legendre functions* $u = P_n^m$. They may be expressed in terms of the Legendre polynomial $P_n = P_n^0$ as

$$\begin{aligned} P_n^m(x) &= (-1)^m \left(1-x^2\right)^{m/2} \frac{d^m}{dx^m} P_n(x) \\ &= (-1)^m \left(1-x^2\right)^{m/2} \frac{1}{2^n n!} \frac{d^{m+n}}{dx^{m+n}} \left(x^2-1\right)^n. \end{aligned}$$

Example 10.33 An example of a Sturm-Liouville operator on the whole of $\mathbb{R}$ with a compact inverse is the quantum harmonic oscillator,

$$Au = -u'' + x^2 u, \qquad \mathcal{D}(A) = \left\{u \in H^2(\mathbb{R}) \mid x^2 u \in L^2(\mathbb{R})\right\}.$$

Its eigenvectors are the Hermite functions (see Exercise 6.14), which form a complete orthonormal set in $L^2(\mathbb{R})$.

Example 10.34 An example of a Sturm-Liouville operator on $L^2(\mathbb{R})$ with a non-compact inverse is given by

$$Au = -u'' + u, \qquad \mathcal{D}(A) = H^2(\mathbb{R}).$$

The inverse of A is given by

$$A^{-1}u(x) = \frac{1}{2}\int_{\mathbb{R}} e^{-|x-y|}u(y)\,dy.$$

The spectrum of A is $[1, \infty)$ and is continuous. For $\lambda \in \mathbb{C} \setminus [1, \infty)$, the resolvent operator $R_\lambda = (\lambda I - A)^{-1}$ is given by

$$R_\lambda u(x) = -\frac{1}{2\sqrt{1-\lambda}} \int_{\mathbb{R}} e^{-\sqrt{1-\lambda}|x-y|} u(y)\, dy,$$

where we use the branch of the square-root with $\operatorname{Re} \sqrt{z} > 0$ in order to ensure that the kernel decays at infinity. The resolvent operator has a branch cut along the continuous spectrum of A.

10.6 Laplace's equation

Adjoint operators and Green's functions can be defined for partial differential equations as well as ordinary differential equations. If the partial differential operator has a compact, self-adjoint inverse (or resolvent), then it has a complete orthonormal set of eigenvectors. In this section, we discuss Laplace's equation, which is one of the most important linear PDEs. We will consider classical solutions in this section. Weak solutions are discussed further in Section 12.11.

Let Ω be a bounded, open, connected set in $\mathbb{R}^n$, with a sufficiently regular boundary $\partial\Omega$. We will not make the required regularity assumptions precise here (see Gilbarg and Trudinger [15] for a detailed discussion). We denote by $C^k(\Omega)$ the space of functions that are k-times continuously differentiable in Ω, and by $C^k(\overline{\Omega})$ the space of functions whose partial derivatives of order less than or equal to k exist in Ω and extend to a continuous function on the closure $\overline{\Omega}$.

If $\mathbf{F} : \overline{\Omega} \to \mathbb{R}^n$ is a continuously differentiable vector field on $\overline{\Omega}$, then the divergence theorem states that

$$\int_\Omega \nabla \cdot \mathbf{F}\, dx = \int_{\partial\Omega} \mathbf{F} \cdot \mathbf{n}\, dS, \tag{10.40}$$

where $\mathbf{n}$ is the unit outward normal to $\partial\Omega$, and dS is an element of $(n-1)$-dimensional surface area on $\partial\Omega$.

The Laplacian operator $-\Delta$ acting on a function $u(x)$ where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ is given by

$$-\Delta u = -\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}.$$

It is convenient to introduce a minus sign in the definition of the Laplacian operator because Δ is a negative operator. The Dirichlet problem for the Laplacian on Ω is

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega, \\ u &= h && \text{on } \partial\Omega. \end{aligned} \tag{10.41}$$

Other types of boundary conditions, such as Neumann conditions

$$\frac{\partial u}{\partial n} = h \qquad \text{on } \partial\Omega,$$

where $\partial u/\partial n = \nabla u \cdot \mathbf{n}$ is the outward normal derivative of u, can be treated in a similar way. First, we show that the Laplacian is formally self-adjoint.

Theorem 10.35 (Green's) If $u, v \in C^2(\overline{\Omega})$, then

$$\int_\Omega (u\Delta v - v\Delta u)\, dx = \int_{\partial\Omega} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) dS.$$

Proof. The result follows from an integration of the vector identity

$$u\Delta v - v\Delta u = \nabla \cdot (u\nabla v - v\nabla u)$$

over Ω and an application of the divergence theorem. □

If $Bu = 0$ is a boundary condition for the Laplacian, then we define the adjoint boundary condition $B^*v = 0$ by the requirement that the boundary terms in Green's formula vanish. If

$$\langle u, v\rangle = \int_\Omega \overline{u} v\, dx$$

is the L^2-inner product on Ω, then we have that

$$\langle \Delta u, v\rangle = \langle u, \Delta v\rangle \qquad \text{for all } u, v \in C^2(\overline{\Omega}) \text{ such that } Bu = B^*v = 0.$$

For example, the adjoint boundary condition to $u = 0$ is $v = 0$, so the Dirichlet problem for Laplace's equation is self-adjoint.

The n-dimensional δ-function has the formal properties

$$\delta(x) = 0 \quad \text{for } x \neq y, \qquad \int_{\mathbb{R}^n} \delta(x)\, dx = 1, \qquad \int_{\mathbb{R}^n} \delta(x-y) f(y)\, dy = f(y)$$

for any continuous function $f : \mathbb{R}^n \to \mathbb{C}$. The Green's function $g(x, y)$ of the Dirichlet problem for the Laplacian is the distributional solution of

$$\begin{aligned} -\Delta g &= \delta(x-y) \qquad && \text{for } x \in \Omega, \\ g(x,y) &= 0 && \text{for } x \in \partial\Omega. \end{aligned} \tag{10.42}$$

The self-adjointness of the boundary value problem implies that g is symmetric, meaning that $g(x, y) = g(y, x)$.

The Green's function representation of the solution of (10.41) follows formally from Green's formula:

$$\int_\Omega \{g\Delta u - u\Delta g\}\, dx = \int_{\partial\Omega} \left\{ g\frac{\partial u}{\partial n} - u\frac{\partial g}{\partial n} \right\} dS.$$

Using (10.41) and (10.42) in this equation, and indicating the integration variable explicitly, we obtain that

$$\int_\Omega \{-g(x,y)f(x) + u(x)\delta(x-y)\}\, dx = -\int_{\partial\Omega} h(x)\frac{\partial g(x,y)}{\partial n(x)}\, dS(x).$$

Evaluating the integral involving the delta function, exchanging x and y, and using the symmetry of the Green's function, we find that

$$u(x) = \int_\Omega g(x,y)f(y)\, dy - \int_{\partial\Omega} h(y)\frac{\partial g(x,y)}{\partial n(y)}\, dS(y).$$

Thus, we can represent the solution of (10.41) for general data $f : \Omega \to \mathbb{R}$ and $h : \partial\Omega \to \mathbb{R}$ in terms of the Green's function.

To give a nondistributional characterization of the Green's function, we integrate (10.42) over a small ball $B_\epsilon(y)$ of radius ϵ centered at y, use the divergence theorem, and let $\epsilon \to 0^+$. This gives

$$\lim_{\epsilon\to 0^+}\int_{\partial B_\epsilon(y)} \frac{\partial g}{\partial n}(x,y)\, dx = -1 \qquad \text{for every } y \in \Omega, \tag{10.43}$$

where $\partial/\partial n$ is the unit outward normal derivative to the ball. We also have

$$\begin{aligned} -\Delta g &= 0 && \text{for } x, y \in \Omega \text{ and } x \neq y,\\ g(x,y) &= 0 && \text{for } x \in \partial\Omega \text{ and } y \in \Omega. \end{aligned}$$

For most domains Ω, it is not possible to obtain an explicit analytical expression for g. A simple solvable case is that of the free-space Green's function g_f defined on $\mathbb{R}^n$. In view of the rotational invariance of the Laplacian, we look for a solution $g_f = g_f(r)$ that depends only on $r = |x-y|$, where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^n$. The polar form of Laplace's equation implies that

$$\frac{1}{r^{n-1}}\frac{d}{dr}\left(r^{n-1}\frac{dg_f}{dr}\right) = 0 \qquad \text{for } r > 0. \tag{10.44}$$

The solution of (10.44) is

$$\begin{aligned} g_f &= c_2 \log\left(\frac{1}{r}\right) && \text{when } n = 2,\\ g_f &= \frac{c_n}{r^{n-2}} && \text{when } n \geq 3, \end{aligned} \tag{10.45}$$

where c_n is a constant, and we omit an arbitrary additive constant.

The radial derivative of g is constant on any sphere centered at y, so the singularity condition in (10.43) implies that

$$\lim_{r\to 0^+} r^{n-1}\frac{dg_f}{dr} = -\frac{1}{\omega_n}, \tag{10.46}$$

where ω_n is the area of the unit sphere in $\mathbb{R}^n$. This area is given by

$$\omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)},$$

where the *Gamma function* Γ is defined, for $x > 0$, by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt.$$

One can show that

$$\Gamma(x+1) = x\Gamma(x), \qquad \Gamma(1) = 1, \qquad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

Hence, for each $n \in \mathbb{N}$,

$$\Gamma(n) = (n-1)!, \qquad \Gamma\left(n + \frac{1}{2}\right) = \left(k - \frac{1}{2}\right)\left(k - \frac{3}{2}\right)\dots\left(\frac{1}{2}\right)\sqrt{\pi},$$

which gives $\omega_2 = 2\pi$ (the length of the unit circle), and $\omega_3 = 4\pi$ (the area of the unit sphere).

Using (10.45) in (10.46), we find that

$$c_2 = \frac{1}{\omega_2}, \qquad c_n = \frac{1}{(n-2)\omega_n} \quad \text{when } n \geq 3.$$

Thus, the free-space Green's function g_f of Laplace's equation in two and three space dimensions is given by

$$g_f(x,y) = \frac{1}{2\pi} \log\left(\frac{1}{|x-y|}\right) \qquad \text{when } n = 2,$$
$$g_f(x,y) = \frac{1}{4\pi|x-y|} \qquad \text{when } n = 3.$$

In contrast to the one-dimensional case, the Green's function is unbounded at $x = y$.

Returning to the Green's function for Laplace's equation on a bounded domain, we may write the solution of (10.42) in the form

$$g(x,y) = g_f(x-y) + \varphi(x,y),$$

where $\varphi(x,y)$ satisfies

$$-\Delta\varphi = 0 \qquad x \in \Omega,$$
$$\varphi(x,y) = -g_f(x-y) \qquad x \in \partial\Omega.$$

If $y \in \Omega$, then the boundary data $-g_f(x-y)$ is smooth for $x \in \partial\Omega$. The solution of an elliptic PDE, like Laplace's equation, on a smooth domain with smooth boundary data is smooth, and therefore $\varphi(x,y)$ is a C^∞ function on $\Omega \times \Omega$. We have used the free-space Green's function to "subtract off" the singularity in the Green's function on a bounded domain.

The eigenvalue problem for the Laplacian is

$$\begin{aligned} -\Delta u &= \lambda u \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{10.47}$$

We again assume Dirichlet boundary conditions for definiteness, when the eigenvalues are strictly positive. If Ω is a bounded domain with a sufficiently regular boundary, then one can show that the Green's operator is a compact, self-adjoint operator on $L^2(\Omega)$. Consequently, it has a complete orthonormal set of eigenfunctions.

Using the divergence theorem, we find that

$$\begin{aligned} \int_\Omega u\,(-\Delta - \lambda I)\,u\,dx &= \int_\Omega \left\{-\nabla\cdot(u\nabla u) + |\nabla u|^2 - \lambda u^2\right\}\,dx \\ &= \int_\Omega \left\{|\nabla u|^2 - \lambda u^2\right\}\,dx. \end{aligned}$$

The boundary terms vanish because $u = 0$ on $\partial\Omega$. Hence, if λ is an eigenvalue of the Dirichlet problem for $-\Delta$ with eigenfunction u, then

$$\int_\Omega \left\{|\nabla u|^2 - \lambda u^2\right\}\,dx = 0.$$

Since $u \neq 0$, it follows that $\lambda \geq 0$. If $\lambda = 0$, then $\nabla u = 0$ in Ω, so $u = \text{constant}$. The boundary condition implies that $u = 0$, so $\lambda = 0$ is not an eigenvalue of the Dirichlet problem. A similar argument applies to the Neumann problem for the Laplacian, with boundary condition $\partial u/\partial n = 0$ on $\partial\Omega$, except that $\lambda = 0$ is an eigenvalue with constant eigenfunction $u = 1$.

It is not possible to compute the eigenvalues and eigenfunctions of the Laplacian explicitly unless the domain Ω has a special shape. For example, if the boundary of Ω is made up of coordinate surfaces of a coordinate system in which the Laplacian separates, then we may use the method of separation of variables illustrated in the next two examples.

Example 10.36 The eigenvalue problem for the Laplacian with Dirichlet boundary conditions on the rectangle $\Omega = [0,a]\times[0,b] \subset \mathbb{R}^2$ is

$$\begin{aligned} -\left(u_{xx} + u_{yy}\right) &= \lambda u, \qquad 0 < x < a, \quad 0 < y < b, \\ u(0,y) &= u(a,y) = 0, \\ u(x,0) &= u(x,b) = 0. \end{aligned}$$

The eigenfunctions $u = u_{m,n}$ and eigenvalues $\lambda = \lambda_{m,n}$, where $m, n = 1, 2, 3, \ldots$, are given by

$$u_{m,n}(x,y) = \frac{2}{\sqrt{ab}}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right),$$

$$\lambda_{m,n} = \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right).$$

The corresponding eigenfunction expansion is a Fourier sine expansion. The lowest eigenvalue is simple, but higher eigenvalues need not be. For example, in the case of a square, $a = b$, the eigenvalue $\lambda = 50\pi^2/a^2$ has multiplicity 3 corresponding to $(m, n) = (5, 5), (1, 7), (7, 1)$. The Green's function $g(x, y; \xi, \eta)$ satisfies

$$\begin{aligned} &-(g_{xx} + g_{yy}) = \delta(x - \xi)\delta(y - \eta), \\ &g(0, y; \xi, \eta) = g(a, y; \xi, \eta) = 0, \\ &g(x, 0; \xi, \eta) = g(x, b; \xi, \eta) = 0. \end{aligned}$$

The eigenfunction expansion of the Green's function is

$$g(x, y; \xi, \eta) = \frac{4}{ab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin(m\pi x/a) \sin(n\pi y/b) \sin(m\pi\xi/a) \sin(n\pi\eta/b)}{\pi^2 (m^2/a^2 + n^2/b^2)},$$

where the series converges in $L^2(\Omega \times \Omega)$.

Example 10.37 The Dirichlet eigenvalue problem for the Laplacian in the three-dimensional unit ball

$$\Omega = \left\{ x \in \mathbb{R}^3 \mid |x| < 1 \right\}$$

may be solved using spherical polar coordinates (r, θ, φ), where

$$x = r \sin\theta \cos\varphi, \quad y = r \sin\theta \sin\varphi, \quad z = r\cos\theta,$$

and $0 \le r$, $0 \le \theta \le \pi$, $0 \le \varphi \le 2\pi$. The eigenvalue problem (10.47) for Laplace's equation is

$$\begin{aligned} &-\left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial u}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial u}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 u}{\partial \varphi^2} \right] = k^2 u, \quad \text{for } r < 1, \\ &\quad u = 0 \qquad \text{for } r = 1. \end{aligned}$$

Here, we write $\lambda = k^2$, since $\lambda > 0$. First, we separate the radial and angular dependence, and look for solutions of the form $u(r, \theta, \varphi) = R(r)Y(\theta, \varphi)$. This gives

$$\frac{1}{r^2} \left(r^2 R' \right)' + \left(k^2 - \frac{\mu}{r^2} \right) R = 0, \quad r < 1, \qquad R(1) = 0, \tag{10.48}$$

$$-\left[\frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial Y}{\partial \theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2 Y}{\partial \varphi^2} \right] = \mu Y, \tag{10.49}$$

where μ is a constant. Equation (10.49) is the eigenvalue problem for the *Laplace-Beltrami equation* on the unit sphere. The nonzero, square-integrable solutions that are 2π-periodic in φ are parametrized by two integers (l, m), where $l \ge 0$ and

$m = -l, -l+1, \ldots, l-1, l$. The eigenvalues are $\mu = l(l+1)$ and the eigenfunctions are the *spherical harmonics* $Y = Y_l^m$, given by

$$Y_l^m(\theta, \varphi) = (-1)^m \left[\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}\right]^{1/2} P_l^m(\cos\theta)e^{im\varphi}.$$

Here, $P_l^m(x)$ is the Legendre function defined in Example 10.32. The set

$$\{Y_l^m \mid l \geq 0 \text{ and } |m| \leq l\}$$

forms a complete orthonormal basis of $L^2(\partial\Omega)$, where $\partial\Omega = \{x \in \mathbb{R}^3 \mid |x| = 1\}$ is the two-dimensional unit sphere in $\mathbb{R}^3$.

Up to an arbitrary multiplicative constant, the solution of the radial equation (10.48) that is bounded at $r = 0$ is given by

$$R(r) = j_l(kr),$$

where $j_l(x)$ is the lth order *spherical Bessel function* that satisfies

$$u'' + \frac{2}{x}u' + \left[1 - \frac{l(l+1)}{x^2}\right]u = 0.$$

The boundary condition $R(1) = 0$ implies that $j_l(k) = 0$, so that $k = z_{l,n}$ where $x = z_{l,n}$ with $n = 1, 2, \ldots$ is the nth positive zero of $j_l(x)$. The corresponding eigenvalues are therefore $\lambda = \lambda_{l,n}$, where $\lambda_{l,n} = z_{l,n}^2$, and $\lambda_{l,n}$ has a multiplicity of $(2l+1)$ corresponding to the different possible choices of $-l \leq m \leq l$. The eigenfunctions,

$$u_{l,m,n}(r,\theta,\varphi) = j_l\left(\sqrt{\lambda_{l,n}}\,r\right) P_l^m(\cos\theta)e^{im\varphi}, \qquad l \geq 0,\ |m| \leq l,\ n \geq 1,$$

form a complete orthogonal basis of $L^2(\Omega)$.

Finally, we consider two examples of partial differential operators that are not formally self-adjoint.

Example 10.38 The *advection-diffusion operator* is

$$A = \mathbf{a} \cdot \nabla + \Delta,$$

where $\mathbf{a}$ is a smooth vector field. The equation $Au = 0$, or

$$\Delta u + \mathbf{a} \cdot \nabla u = 0$$

describes the steady state of a quantity u, such as temperature or the density of a pollutant, subject to diffusion and advection by a velocity field $\mathbf{a}$. We consider A as acting on $C^2(\overline{\Omega})$, where Ω is a smooth, bounded domain in $\mathbb{R}^n$. For simplicity, we suppose all functions are real-valued. Then, using the vector identity

$$\nabla \cdot (u\mathbf{a}) = \nabla u \cdot \mathbf{a} + u\nabla \cdot \mathbf{a},$$

and the divergence theorem, we find that

$$\begin{aligned}\int_\Omega (\mathbf{a}\cdot\nabla u + \Delta u)\, v\, dx &= \int_\Omega u\left(-\nabla\cdot(\mathbf{a}v) + \Delta v\right)\, dx \\ &\quad + \int_{\partial\Omega}\left(uv\mathbf{a}\cdot\mathbf{n} + v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right)\, dS.\end{aligned}$$

Thus, the formal adjoint of A is

$$A^* = -\nabla\cdot\mathbf{a} + \Delta.$$

Example 10.39 The heat operator is

$$A = -\partial_t + \Delta.$$

We consider A as acting on real-valued functions $u(x,t)$ in $C^2(\overline{\Omega}\times[0,T])$, where Ω is a smooth, bounded domain in $\mathbb{R}^n$, and $T > 0$. Then

$$\begin{aligned}\int_0^T\int_\Omega (-u_t + \Delta u)\, v\, dxdt &= \int_0^T\int_\Omega u\left(v_t + \Delta v\right)\, dxdt \\ &\quad - \int_\Omega [uv]_0^T\, dx + \int_0^T\int_{\partial\Omega}\left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right) dSdt.\end{aligned}$$

Thus, for example, the adjoint problem to the initial value problem for the heat equation,

$$\begin{aligned}&u_t = \Delta u + f \qquad \text{in } \Omega\times[0,T],\\ &u(x,t) = 0 \qquad \text{for } x\in\partial\Omega,\\ &u(x,0) = u_0(x),\end{aligned}$$

is the final value problem for the backward heat equation,

$$\begin{aligned}&-v_t = \Delta v + g \qquad \text{in } \Omega\times[0,T],\\ &v(x,t) = 0 \qquad \text{for } x\in\partial\Omega,\\ &v(x,T) = v_T(x).\end{aligned}$$

10.7 References

For more on unbounded operators and a proof of the closed graph theorem, see Kato [26] or Reed and Simon [45]. For Sturm-Liouville problems, see Coddington and Levinson [6]. For an introduction to Green's functions for PDEs, see Zauderer [57]. An extensive collection of Green's functions for various boundary value problems for PDEs is given in Morse and Feshbach [39]. Mikhlin [38] gives a detailed and careful analytical discussion of Green's functions for Laplace's equation. Further analysis of spectral problems for ODEs and PDEs is given in Vol. 3 of Dautry and Lions [7]. For the definition and properties of special functions, such as the Gamma function, Bessel functions, and spherical harmonics, see Hochstadt [23] or Lebedev [31]. For a summary of formulae and integrals, including ones that involve special functions, see Gradshteyn and Ryzhik [16].

10.8 Exercises

Exercise 10.1 Prove that if A^{**} exists, then it is an extension of A.

Exercise 10.2 Prove that a symmetric operator is closable.

Exercise 10.3 Show that the operator A on $\mathcal{H} = L^2(\mathbb{T})$, with domain

$$\mathcal{D}(A) = \left\{ f(x) = \sum_{n\in\mathbb{Z}} a_n e^{inx} \;\Big|\; \sum_{n\in\mathbb{Z}} n^4 |a_n|^2 < \infty \right\},$$

defined by

$$A\left(\sum_{n\in\mathbb{Z}} a_n e^{inx}\right) = \sum_{n\in\mathbb{Z}} n^2 a_n e^{inx}$$

is a self-adjoint extension of the classical differentiation operator $-d^2/dx^2$ with domain $C^2(\mathbb{T})$.

Exercise 10.4 Let $M : \mathcal{D}(M) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the multiplication operator $Mf = xf$ with

$$\mathcal{D}(M) = \left\{ f \in L^2(\mathbb{R}) \mid xf \in L^2(\mathbb{R}) \right\}.$$

Show that M is self-adjoint.

Exercise 10.5 Suppose that $\{e_n \mid n \in \mathbb{N}\}$ is an orthonormal basis of a separable Hilbert space $\mathcal{H}$, and $\lambda_n \in \mathbb{R}$. For $x \in \mathcal{H}$, let $x_n = \langle e_n, x\rangle \in \mathbb{C}$, so

$$x = \sum_{n=1}^{\infty} x_n e_n.$$

Define an operator $A : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}$ by

$$\mathcal{D}(A) = \left\{ x \in H \;\Big|\; \sum_{n=1}^{\infty} (1 + \lambda_n^2)|x_n|^2 < \infty \right\},$$

$$A\left(\sum_{n=1}^{\infty} x_n e_n\right) = \sum_{n=1}^{\infty} \lambda_n x_n e_n.$$

Prove that A is self-adjoint.

Exercise 10.6 Let A and B be two linear operators on a Hilbert space $\mathcal{H}$ with domains $\mathcal{D}(A)$ and $\mathcal{D}(B)$, respectively, and assume $\mathcal{D}(A) \cap \mathcal{D}(B)$ is dense. Define an operator C by $\mathcal{D}(C) = \mathcal{D}(A) \cap \mathcal{D}(B)$ and $Cx = Ax + Bx$ for all $x \in \mathcal{D}(C)$. Prove that C^* is an extension of $A^* + B^*$. Define $\mathcal{D}(AB)$ and $\mathcal{D}(B^*A^*)$ by

$$\begin{aligned} \mathcal{D}(AB) &= \{x \in \mathcal{D}(B) \mid Bx \in \mathcal{D}(A)\} \\ \mathcal{D}(B^*A^*) &= \{x \in \mathcal{D}(A^*) \mid A^*x \in \mathcal{D}(B^*)\} \end{aligned}$$

and assume that $\mathcal{D}(AB)$ and $\mathcal{D}(B^*A^*)$ are dense. Define operators AB and B^*A^* on their respective domains in the obvious way. Prove that $(AB)^*$ is an extension of B^*A^*.

Exercise 10.7 Prove that the adjoint of a densely defined, unbounded operator in a Hilbert space is closed.

Exercise 10.8 Let $\{x_n \mid n \in \mathbb{N}\}$ be an orthonormal basis of a separable Hilbert space $\mathcal{H}$, and let y an element of $\mathcal{H}$ that is not a linear combination of a finite number of basis elements x_n. Define a linear operator A in $\mathcal{H}$, whose domain $\mathcal{D}(A)$ consists of finite linear combinations of the x_n and y, as follows:

$$A\left(\sum_{n=1}^{N} a_n x_n + by\right) = by, \quad \mathcal{D}(A) = \left\{\sum_{n=1}^{N} a_n x_n + by \;\middle|\; a_n, b \in \mathbb{C}\right\}.$$

Show that A is not closable.

Exercise 10.9 Consider a singular self-adjoint BVP,

$$\begin{aligned} &-(pu')' + qu = f, \\ &u(0) = u(1) = 0. \end{aligned}$$

Suppose that the null space of the homogeneous problem is one-dimensional with orthonormal basis $\{\varphi\}$. Define the modified Green's operator $G : L^2([0,1]) \to L^2([0,1])$ where $u = Gf$ if and only if u satisfies the problem

$$\begin{aligned} &-(pu')' + qu = f - \langle \varphi, f \rangle \varphi, \\ &u(0) = u(1) = 0, \qquad \langle \varphi, u \rangle = 0. \end{aligned}$$

Prove that G is well defined, and show that G is an integral operator of the form

$$Gf(x) = \int_0^1 g(x,y) f(y)\, dy.$$

Compute the modified Green's function g in terms of φ.

Exercise 10.10 Let $r : [0,1] \to \mathbb{R}$ be a smooth, nonnegative function. Let $\mathcal{H}$ be the Hilbert space of (equivalence classes) of Lebesgue measurable functions $u : [0,1] \to \mathbb{C}$ such that

$$\int_0^1 r(x)|u(x)|^2\, dx < \infty,$$

with the inner product

$$\langle u, v\rangle = \int_0^1 r(x)\overline{u(x)}v(x)\, dx.$$

Determine the formally self-adjoint, second-order differential operators on $\mathcal{H}$.

Exercise 10.11 Prove that the Wronskian $W(x)$ of the Sturm-Liouville operator (10.8) satisfies $p(x)W(x) = \text{constant}$. Verify directly that the Green's function is symmetric.

Exercise 10.12 The following linearized BBM (Benjamin-Bona-Mahoney) equation for $u(x,t)$, where $x, t \in \mathbb{R}$, arises in the analysis of water waves:

$$-u_{xxt} + u_t = u_x,$$
$$u(x,0) = u_0(x).$$

Use a Green's function to reformulate this equation as an evolution equation

$$u_t = Ku,$$

for a suitable integral operator $K : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, and deduce that there is a global in time solution with $u(\cdot,t) \in L^2(\mathbb{R})$ for any initial data $u_0 \in L^2(\mathbb{R})$. Show that the L^2-norm of u is conserved.

Exercise 10.13 For $k = 1,2,3$, let $A_k : \mathcal{D}(A_k) \subset L^2([0,1]) \to L^2([0,1])$ be the first-order differential operators $A_k u = iu'$ with domains

$$\begin{aligned}
\mathcal{D}(A_1) &= H^1((0,1)),\\
\mathcal{D}(A_2) &= \left\{u \in H^1((0,1)) \mid u(0) = u(1)\right\},\\
\mathcal{D}(A_3) &= \left\{u \in H^1((0,1)) \mid u(0) = 0\right\}.
\end{aligned}$$

Show that the spectrum of A_1 is $\mathbb{C}$, the spectrum of A_2 is the set $\{2n\pi \mid n \in \mathbb{Z}\}$, and the spectrum of A_3 is empty.

Exercise 10.14 Consider the operators A_1, A_2 defined by $A_k u = iu'$, with

$$\mathcal{D}(A_1) = \left\{u \in H^1((0,1)) \mid u(0) = u(1)\right\},$$
$$\mathcal{D}(A_2) = \left\{u \in H^1((0,1)) \mid u(0) = u(1) = 0\right\}.$$

Show that both operators are closed and symmetric. Compute $\operatorname{ran}(A_k \pm i)$ and $\ker(A_k^* \pm i)$. Use Theorem 10.29 to determine whether or not these operators are self-adjoint.

Exercise 10.15 Let φ be a nonzero function in $L^2(\mathbb{R})$ and define an operator $A : \mathcal{D}(A) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by

$$Au = \left(\int_{\mathbb{R}} u(x)\, dx \right) \varphi, \qquad \mathcal{D}(A) = L^1(\mathbb{R}) \cap L^2(\mathbb{R}).$$

Show that A is a closed, unbounded operator that is densely defined in $L^2(\mathbb{R})$. Show that

$$\mathcal{D}(A^*) = \{\varphi\}^\perp$$

and $A^* = 0$ on $\mathcal{D}(A^*)$, so the domain of A^* is not dense in $L^2(\mathbb{R})$.

Exercise 10.16 If $u \in H^1((0,\infty))$ and $u(0) = 0$, prove *Hardy's inequality*:

$$\int_0^\infty \frac{|u|^2}{x^2}\, dx \le 4 \int_0^\infty |u'|^2\, dx.$$

Exercise 10.17 Suppose that u_1 and u_2 are two solutions of the Dirichlet problem for Laplace's equation

$$\begin{aligned} -\Delta u &= f \qquad x \in \Omega, \\ u &= h \qquad x \in \partial\Omega, \end{aligned}$$

where Ω is a smooth, bounded domain in $\mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$ and $h : \partial\Omega \to \mathbb{R}$ are given functions. Show that if $v = u_1 - u_2$ then

$$\int_\Omega |\nabla v|^2\, dx = 0,$$

and deduce that the solution is unique. What can you say about solutions of the Neumann problem, with boundary condition

$$\frac{\partial u}{\partial n} = h \qquad x \in \partial\Omega?$$

Exercise 10.18 According to Maxwell's equations, the magnetic field $\mathbf{B}$ generated in three-dimensional space by a steady current distribution $\mathbf{J}$ satisfies

$$\operatorname{curl} \mathbf{B} = \mathbf{J}, \qquad \operatorname{div} \mathbf{B} = 0.$$

A mathematically identical problem arises in fluid mechanics in reconstructing an incompressible velocity field $\mathbf{u}$, with $\operatorname{div} \mathbf{u} = 0$, from the vorticity $\omega = \operatorname{curl} \mathbf{u}$. Derive the *Biot-Savart law*,

$$\mathbf{B}(\mathbf{x}) = \int \frac{\mathbf{J}(\mathbf{y}) \times (\mathbf{x} - \mathbf{y})}{4\pi\, |\mathbf{x} - \mathbf{y}|}\, d\mathbf{y}.$$

HINT. Write $\mathbf{B} = \operatorname{curl} \mathbf{A}$ and derive a Laplace equation for $\mathbf{A}$.

Exercise 10.19 Let N_n be the $n \times n$ Jordan block

$$N_n = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix},$$

and let $c_n = n^{-1/2}(1, 1, \dots, 1)^T \in \mathbb{C}^n$. Show that for each $n \in \mathbb{N}$ and $t \geq 0$:

$$\left\| e^{tN_n} \right\| \leq e^t; \quad \|N_n c_n - c_n\| \leq n^{-1/2}; \quad \left\| e^{tN_n} c_n - e^t c_n \right\| \leq n^{-1/2} t e^t.$$

Let $\mathcal{H} = \bigoplus_{n=1}^{\infty} \mathbb{C}^n$, meaning that $x \in \mathcal{H}$ is of the form

$$x = (x_1, x_2, \dots, x_n, \dots), \qquad x_n \in \mathbb{C}^n, \qquad \sum_{n=1}^{\infty} |x_n|^2 < \infty.$$

Here, $|\cdot|$ denotes the Euclidean norm on $\mathbb{C}^n$. Let $A_n = N_n + inI_n$, where I_n is the $n \times n$ identity, and define $A : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}$ by

$$A = \bigoplus_{n=1}^{\infty} A_n, \quad A(x_1, x_2, \dots, x_n, \dots) = (A_1 x_1, A_2 x_2, \dots, A_n x_n, \dots),$$

where $\mathcal{D}(A) = \{x \in \mathcal{H} \mid Ax \in \mathcal{H}\}$. We define the associated C_0-semigroup $T(t) = e^{tA}$ for $t \geq 0$, where $T(t) : \mathcal{H} \to \mathcal{H}$, by

$$T(t)(x_1, x_2, \dots, x_n, \dots) = \left(e^{tA_1} x_1, e^{tA_2} x_2, \dots, e^{tA_n} x_n, \dots\right).$$

Show that the spectrum of A is $\{in \in \mathbb{C} \mid n \in \mathbb{N}\}$, and consists entirely of eigenvalues, so it is contained in the left-half plane $\{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda \leq 0\}$. Show, however, that the spectral radius of $T(t)$ is greater than or equal to e^t, so the spectral mapping theorem does not hold for A.
HINT. Consider the action of $T(t)$ on the vectors $(0, 0, \dots, 0, c_n, 0, \dots) \in X$. This example of an operator with arbitrarily large Jordan blocks illustrates some of the pathologies that can arise for unbounded, nonnormal operators on a Hilbert space.

Exercise 10.20 Consider heat flow in a rod with rapidly varying thermal conductivity $a_n(x) = a(nx)$, where $n \in \mathbb{N}$ and $a(y)$ is a strictly positive periodic function with period one, assumed continuous for simplicity. If the ends of the rod are held at an equal fixed temperature, and there is a given heat source $f(x)$ per unit length, the temperature $u_n(x)$ satisfies the boundary value problem

$$-\frac{d}{dx}\left(a_n(x)\frac{d}{dx}u_n\right) = f(x), \quad 0 < x < 1, \qquad u_n(0) = u_n(1) = 0.$$

Integrate this ODE and solve for $u_n(x)$. Let $H_0^1([0,1])$ be the Sobolev space

$$H_0^1([0,1]) = \{u : [0,1] \to \mathbb{R} \mid u, u' \in L^2([0,1]),\, u(0) = u(1) = 0\},$$

with the inner product

$$\langle u, v \rangle = \int_0^1 u'v'\, dx.$$

Show that $u_n \rightharpoonup u$ weakly in $H_0^1([0,1])$, where u is the solution of the *homogenized equation*

$$-\frac{d}{dx}\left(a^h \frac{d}{dx} u\right) = f(x), \quad 0 < x < 1, \qquad u(0) = u(1) = 0,$$

and the effective conductivity a^h is the harmonic mean of the original conductivity,

$$\frac{1}{a^h} = \int_0^1 \frac{1}{a(y)}\, dy.$$