Real Analysis
Math 125A, Fall 2012
Sample Final Questions

1. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \frac{x^3}{1+x^2}.$$

Show that f is continuous on $\mathbb{R}$. Is f uniformly continuous on $\mathbb{R}$?

Solution.

- To simplify the inequalities a bit, we write

$$\frac{x^3}{1+x^2} = x - \frac{x}{1+x^2}.$$

  For $x, y \in \mathbb{R}$, we have

$$\begin{aligned} |f(x) - f(y)| &= \left| x - y - \frac{x}{1+x^2} + \frac{y}{1+y^2} \right| \\ &\leq |x-y| + \left| \frac{x}{1+x^2} - \frac{y}{1+y^2} \right|. \end{aligned}$$

- Using the inequality $2|xy| \leq x^2 + y^2$, we get

$$\begin{aligned} \left| \frac{x}{1+x^2} - \frac{y}{1+y^2} \right| &= \left| \frac{x - y + xy^2 - x^2 y}{(1+x^2)(1+y^2)} \right| \\ &\leq \left[\frac{1+|xy|}{(1+x^2)(1+y^2)} \right] |x-y| \\ &\leq \frac{1}{2} \left[\frac{1+x^2+1+y^2}{(1+x^2)(1+y^2)} \right] |x-y| \\ &\leq \frac{1}{2} \left(\frac{1}{1+y^2} + \frac{1}{1+x^2} \right) |x-y| \\ &\leq |x-y| \end{aligned}$$

- It follows that

$$|f(x) - f(y)| \leq 2|x-y| \qquad \text{for all } x, y \in \mathbb{R}.$$

  Therefore f is Lipschitz continuous on $\mathbb{R}$, which implies that it is uniformly continuous (take $\delta = \epsilon/2$).

2. Does there exist a differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that $f'(0) = 0$ but $f'(x) \geq 1$ for all $x \neq 0$?

Solution.

- No such function exists.
- We have
$$f'(0) = \lim_{x \to 0} \left[\frac{f(x) - f(0)}{x} \right].$$
The mean value theorem implies that for for every $x \neq 0$, there is some ξ strictly between 0 and x (so $\xi \neq 0$) such that
$$\frac{f(x) - f(0)}{x} = f'(\xi) \geq 1.$$
- Since limits preserve inequalities, it follows that
$$\lim_{x \to 0} \left[\frac{f(x) - f(0)}{x} \right] \geq 1,$$
so we cannot have $f'(0) = 0$.

3. (a) Write out the Taylor polynomial $P_2(x)$ of order two at $x = 0$ for the function $\sqrt{1+x}$. and give an expression for the remainder $R_2(x)$ in Taylor's formula

$$\sqrt{1+x} = P_2(x) + R_2(x) \qquad -1 < x < \infty.$$

(b) Show that the limit

$$\lim_{x \to 0} \left[\frac{1 + x/2 - \sqrt{1+x}}{x^2} \right]$$

exists and find its value.

Solution.

- (a) The function and its derivatives are given by

$$\begin{aligned} f(x) &= \sqrt{1+x}, & f(0) &= 1, \\ f'(x) &= \frac{1}{2}(1+x)^{-1/2}, & f'(0) &= \frac{1}{2}, \\ f''(x) &= -\frac{1}{4}(1+x)^{-3/2}, & f''(0) &= -\frac{1}{4}, \\ f'''(x) &= \frac{3}{8}(1+x)^{-5/2}. \end{aligned}$$

- The Taylor polynomial and remainder are

$$P_2(x) = \sum_{k=0}^{2} \frac{1}{k!} f^{(k)}(0) x^k, \qquad R_2(x) = \frac{1}{3!} f'''(\xi) x^3,$$

  where ξ is between 0 and x, which gives

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}(1+\xi)^{-5/2} x^3$$

  (b) For this part, we only need the Taylor polynomial of order one,

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}(1+\xi)^{-3/2} x^2$$

  where ξ is between 0 and x. Since $\xi \to 0$ as $x \to 0$, it follows that

$$\lim_{x \to 0} \left[\frac{1 + x/2 - \sqrt{1+x}}{x^2} \right] = \frac{1}{8} \lim_{\xi \to 0} (1+\xi)^{-3/2} = \frac{1}{8}.$$

4. (a) Suppose $f_n : A \to \mathbb{R}$ is uniformly continuous on A for every $n \in \mathbb{N}$ and $f_n \to f$ uniformly on A. Prove that f is uniformly continuous on A.

(b) Does the result in (a) remain true if $f_n \to f$ pointwise instead of uniformly?

Solution.

- (a) Let $\epsilon > 0$. Since $f_n \to f$ converges uniformly on A there exists $N \in \mathbb{N}$ such that

  $$|f_n(x) - f(x)| < \frac{\epsilon}{3} \qquad \text{for all } x \in A \text{ and } n > N.$$

  Choose some $n > N$. Since f_n is uniformly continuous, there exists $\delta > 0$ such that

  $$|f_n(x) - f_n(y)| < \frac{\epsilon}{3} \qquad \text{for all } x, y \in A \text{ with } |x - y| < \delta.$$

  Then, for all $x, y \in A$ with $|x - y| < \delta$, we have

  $$|f(x) - f(y)| \le |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(x)| < \epsilon,$$

  which implies that f is uniformly continuous on A.

- (b) The result does not remain true if $f_n \to f$ pointwise. For example, consider $f_n : [0, 1] \to \mathbb{R}$ defined by $f_n(x) = x^n$. Then f_n is uniformly continuous on $[0, 1]$ because it is a continuous function on a compact interval, but $f_n \to f$ pointwise where

  $$f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1, \\ 1 & \text{if } x = 1. \end{cases}$$

  The limit f is not even continuous on $[0, 1]$.

5. Define $f_n : [0, \infty) \to \mathbb{R}$ by

$$f_n(x) = \frac{\sin(nx)}{1 + nx}.$$

(a) Show that f_n converges pointwise on $[0, \infty)$ and find the pointwise limit f.
(b) Show that $f_n \to f$ uniformly on $[a, \infty)$ for every $a > 0$.
(c) Show that f_n does not converge uniformly to f on $[0, \infty)$.

Solution.

- (a) If $x > 0$, then

  $$|f_n(x)| \leq \frac{1}{1 + nx} \to 0 \qquad \text{as } n \to \infty$$

  so $f_n(x) \to 0$. Also, $f_n(0) = 0$ for every n, so $f_n(0) \to 0$. Thus, $f_n \to 0$ pointwise on $[0, \infty)$.

- (b) We have

  $$|f_n(x)| \leq \frac{1}{1 + na} < \frac{1}{na} \qquad \text{for all } a \leq x < \infty,$$

  so given $\epsilon > 0$ take $N = 1/a$ and then $|f_n(x)| < \epsilon$ for all $n > N$, meaning that $f_n \to 0$ uniformly on $[a, \infty)$.

- (c) If (f_n) converges uniformly on $[0, \infty)$, then it must converge to the pointwise-limit 0. Let $x_n = \pi/(2n)$. Then

  $$f_n(x_n) = \frac{1}{1 + \pi/2}.$$

  Therefore, if $0 < \epsilon_0 \leq 1/(1 + \pi/2)$, there exists $x \in [0, \infty)$ such that

  $$f_n(x) \geq \epsilon_0,$$

  which means that f_n does not converge uniformly to 0 on $[0, \infty)$.

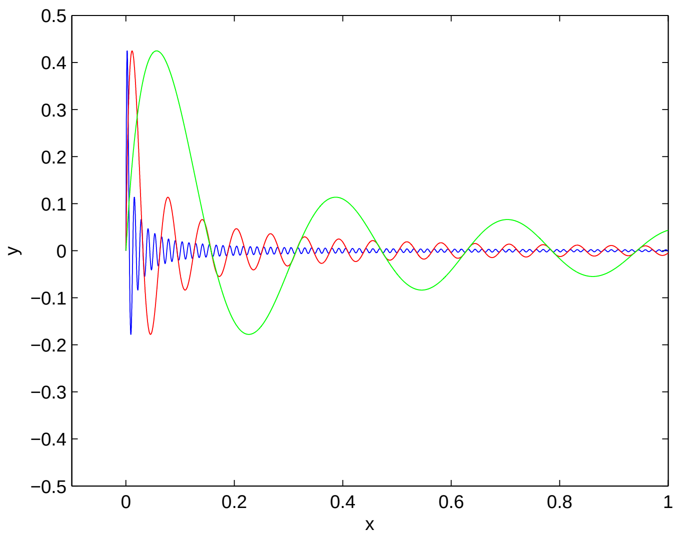

Figure 1: Plot of the function $f_n(x) = \sin(nx)/(1+nx)$ on $[0,1]$ for $n = 20$ (green), $n = 100$ (red), and $n = 500$ (blue).

Remark. The non-uniform convergence of the sequence near $x = 0$ is illustrated in the figure.

We can also write the proof in terms of the sup-norm. Let

$$\|f\|_a = \sup_{x\in[a,\infty)} |f(x)|$$

denote the sup-norm of f on $[a,\infty)$. If $a > 0$, then

$$\|f_n\|_a \le \frac{1}{na} \to 0 \qquad \text{as } n \to \infty,$$

so $f_n \to 0$ uniformly on $[a,\infty)$. If $a = 0$, then

$$\|f_n\|_0 \ge \frac{1}{1+\pi/2} \qquad \text{for every } n \in \mathbb{N},$$

so (f_n) does not converge uniformly to 0 on $[0,\infty)$.

6. Suppose that

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n^3}, \qquad g(x) = \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}.$$

(a) Prove that $f, g : \mathbb{R} \to \mathbb{R}$ are continuous.
(b) Prove that $f : \mathbb{R} \to \mathbb{R}$ is differentiable and $f' = g$.

Solution.

- (a) Since

  $$\left|\frac{\sin nx}{n^3}\right| \le \frac{1}{n^3}, \qquad \sum_{n=1}^{\infty} \frac{1}{n^3} < \infty$$
  $$\left|\frac{\cos nx}{n^2}\right| \le \frac{1}{n^2}, \qquad \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty,$$

  the Weierstrass M-test implies that both series converge uniformly (and absolutely) on $\mathbb{R}$.

- Each term in the series is continuous, and the uniform limit of continuous functions is continuous, so f, g are continuous on $\mathbb{R}$.

- (b) The series for g is the term-by-term derivative of the series for f. Since the series for g converges uniformly, the theorem for the differentiation of sequences implies that f is differentiable and $f' = g$.

7. Let $P = \{2, 3, 5, 7, 11, \ldots\}$ be the set of prime numbers.
(a) Find the radius of convergence R of the power series

$$f(x) = \sum_{p \in P} x^p = x^2 + x^3 + x^5 + x^7 + x^{11} + \ldots$$

(b) Show that

$$0 \le f(x) \le \frac{x^2}{1-x} \qquad \text{for all } 0 \le x < 1.$$

Solution.

- (a) We write the series as

$$f(x) = \sum_{n=2}^{\infty} a_n x^n$$

  where

$$a_n = \begin{cases} 1 & \text{if } n \text{ is prime,} \\ 0 & \text{if } n \text{ isn't prime.} \end{cases}$$

- Then

$$|a_n x^n| \le |x|^n \qquad \text{for every } n = 2, 3, 4, \ldots.$$

  Therefore, if $|x| < 1$ the series converges by comparison with the convergent geometric series $\sum |x|^n$. Furthermore, if $|x| > 1$, the terms in the series do not approach 0. So the radius of convergence of the series is $R = 1$.

- (b) As in (a), and using the sum of the geometric series, we have for $0 \le x < 1$ that

$$0 \le \sum_{p \in P} x^p \le \sum_{n=2}^{\infty} x^n = x^2 \sum_{n=0}^{\infty} x^n = \frac{x^2}{1-x},$$

  which proves the result.

8. Let (X, d) be a metric space.
(a) Define the open ball $B_r(x)$ of radius $r > 0$ and center $x \in X$.
(b) Define an open set $A \subset X$.
(c) Show that the open ball $B_r(x) \subset X$ is an open set.

Solution.

- (a) The open ball is defined by

$$B_r(x) = \{y \in X : d(x, y) < r\}.$$

- (b) A set $A \subset X$ is open if for every $x \in A$ there exists $r > 0$ such that $B_r(x) \subset A$.

- (c) Suppose that $y \in B_r(x)$. We have to show that $B_r(x)$ contains an open ball $B_s(y)$ for some $s > 0$. Choose

$$s = r - d(x, y) > 0.$$

(Draw a picture!) If $z \in B_s(y)$, then by the triangle inequality

$$d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + s = r,$$

meaning that $z \in B_r(x)$. Thus, $B_s(y) \subset B_r(x)$, which proves the result.