Chapter 2

The supremum and infimum

We review the definition of the supremum and and infimum and some of their properties that we use in defining and analyzing the Riemann integral.

2.1. Definition

First, we define upper and lower bounds.

Definition 2.1. A set $A \subset \mathbb{R}$ of real numbers is bounded from above if there exists a real number $M \in \mathbb{R}$, called an upper bound of A, such that $x \leq M$ for every $x \in A$. Similarly, A is bounded from below if there exists $m \in \mathbb{R}$, called a lower bound of A, such that $x \geq m$ for every $x \in A$. A set is bounded if it is bounded both from above and below.

The supremum of a set is its least upper bound and the infimum is its greatest upper bound.

Definition 2.2. Suppose that $A \subset \mathbb{R}$ is a set of real numbers. If $M \in \mathbb{R}$ is an upper bound of A such that $M \leq M'$ for every upper bound M' of A, then M is called the supremum of A, denoted $M = \sup A$. If $m \in \mathbb{R}$ is a lower bound of A such that $m \geq m'$ for every lower bound m' of A, then m is called the or infimum of A, denoted $m = \inf A$.

If A is not bounded from above, then we write $\sup A = \infty$, and if A is not bounded from below, we write $\inf A = -\infty$. If $A = \varnothing$ is the empty set, then every real number is both an upper and a lower bound of A, and we write $\sup \varnothing = -\infty$, $\inf \varnothing = \infty$. We will only say the supremum or infimum of a set exists if it is a finite real number. For an indexed set $A = \{x_k : k \in J\}$, we often write

$$\sup A = \sup_{k \in J} x_k, \qquad \inf A = \inf_{k \in J} x_k.$$

Proposition 2.3. The supremum or infimum of a set A is unique if it exists. Moreover, if both exist, then $\inf A \leq \sup A$.

Proof. Suppose that M, M' are suprema of A. Then $M \leq M'$ since M' is an upper bound of A and M is a least upper bound; similarly, $M' \leq M$, so $M = M'$. If m, m' are infima of A, then $m \geq m'$ since m' is a lower bound of A and m is a greatest lower bound; similarly, $m' \geq m$, so $m = m'$.

If $\inf A$ and $\sup A$ exist, then A is nonempty. Choose $x \in A$, Then

$$\inf A \leq x \leq \sup A$$

since $\inf A$ is a lower bound of A and $\sup A$ is an upper bound. It follows that $\inf A \leq \sup A$. ☐

If $\sup A \in A$, then we also denote it by $\max A$ and call it the maximum of A, and if $\inf A \in A$, then we also denote it by $\min A$ and call it the minimum of A.

Example 2.4. Let $A = \{1/n : n \in \mathbb{N}\}$. Then $\sup A = 1$ belongs to A, so $\max A = 1$. On the other hand, $\inf A = 0$ doesn't belong to A and A has no minimum.

The following alternative characterization of the sup and inf is an immediate consequence of the definition.

Proposition 2.5. If $A \subset \mathbb{R}$, then $M = \sup A$ if and only if: (a) M is an upper bound of A; (b) for every $M' < M$ there exists $x \in A$ such that $x > M'$. Similarly, $m = \inf A$ if and only if: (a) m is a lower bound of A; (b) for every $m' > m$ there exists $x \in A$ such that $x < m'$.

Proof. Suppose M satisfies the conditions in the proposition. Then M is an upper bound and (b) implies that if $M' < M$, then M' is not an an upper bound, so $M = \sup A$. Conversely, if $M = \sup A$, then M is an upper bound, and if $M' < M$ then M' is not an upper bound, so there exists $x \in A$ such that $x > M'$. The proof for the infimum is analogous. ☐

We frequently use one of the following arguments: (a) If M is an upper bound of A, then $M \geq \sup A$; (b) For every $\epsilon > 0$, there exists $x \in A$ such that $x > \sup A - \epsilon$. Similarly: (a) If m is an lower bound of A, then $m \leq \inf A$; (b) For every $\epsilon > 0$, there exists $x \in A$ such that $x < \inf A + \epsilon$.

The completeness of the real numbers ensures the existence of suprema and infima. In fact, the existence of suprema and infima is one way to define the completeness of $\mathbb{R}$.

Theorem 2.6. Every nonempty set of real numbers that is bounded from above has a supremum, and every nonempty set of real numbers that is bounded from below has an infimum.

This theorem is the basis of many existence results in real analysis. For example, once we show that a set is bounded from above, we can assert the existence of a supremum without having to know its actual value.

2.2. Properties

If $A \subset \mathbb{R}$ and $c \in \mathbb{R}$, then we define

$$cA = \{y \in \mathbb{R} : y = cx \text{ for some } x \in A\}.$$

Proposition 2.7. If $c \geq 0$, then

$$\sup cA = c \sup A, \qquad \inf cA = c \inf A.$$

If $c < 0$, then

$$\sup cA = c \inf A, \qquad \inf cA = c \sup A.$$

Proof. The result is obvious if $c = 0$. If $c > 0$, then $cx \leq M$ if and only if $x \leq M/c$, which shows that M is an upper bound of cA if and only if M/c is an upper bound of A, so $\sup cA = c \sup A$. If $c < 0$, then then $cx \leq M$ if and only if $x \geq M/c$, so M is an upper bound of cA if and only if M/c is a lower bound of A, so $\sup cA = c \inf A$. The remaining results follow similarly. □

Making a set smaller decreases its supremum and increases its infimum.

Proposition 2.8. Suppose that A, B are subsets of $\mathbb{R}$ such that $A \subset B$. If $\sup A$ and $\sup B$ exist, then $\sup A \leq \sup B$, and if $\inf A$, $\inf B$ exist, then $\inf A \geq \inf B$.

Proof. Since $\sup B$ is an upper bound of B and $A \subset B$, it follows that $\sup B$ is an upper bound of A, so $\sup A \leq \sup B$. The proof for the infimum is similar, or apply the result for the supremum to $-A \subset -B$. □

Proposition 2.9. Suppose that A, B are nonempty sets of real numbers such that $x \leq y$ for all $x \in A$ and $y \in B$. Then $\sup A \leq \inf B$.

Proof. Fix $y \in B$. Since $x \leq y$ for all $x \in A$, it follows that y is an upper bound of A, so $y \geq \sup A$. Hence, $\sup A$ is a lower bound of B, so $\sup A \leq \inf B$. □

If $A, B \subset \mathbb{R}$ are nonempty, we define

$$\begin{aligned} A + B &= \{z : z = x + y \text{ for some } x \in A,\, y \in B\}, \\ A - B &= \{z : z = x - y \text{ for some } x \in A,\, y \in B\} \end{aligned}$$

Proposition 2.10. If A, B are nonempty sets, then

$$\begin{aligned} \sup(A + B) &= \sup A + \sup B, &\qquad \inf(A + B) &= \inf A + \inf B, \\ \sup(A - B) &= \sup A - \inf B, &\qquad \inf(A - B) &= \inf A - \sup B. \end{aligned}$$

Proof. The set $A + B$ is bounded from above if and only if A and B are bounded from above, so $\sup(A+B)$ exists if and only if both $\sup A$ and $\sup B$ exist. In that case, if $x \in A$ and $y \in B$, then

$$x + y \leq \sup A + \sup B,$$

so $\sup A + \sup B$ is an upper bound of $A + B$ and therefore

$$\sup(A + B) \leq \sup A + \sup B.$$

To get the inequality in the opposite direction, suppose that $\epsilon > 0$. Then there exists $x \in A$ and $y \in B$ such that

$$x > \sup A - \frac{\epsilon}{2}, \qquad y > \sup B - \frac{\epsilon}{2}.$$

It follows that

$$x + y > \sup A + \sup B - \epsilon$$

for every $\epsilon > 0$, which implies that $\sup(A+B) \geq \sup A + \sup B$. Thus, $\sup(A+B) = \sup A + \sup B$.

It follows from this result and Proposition 2.7 that

$$\sup(A - B) = \sup A + \sup(-B) = \sup A - \inf B.$$

The proof of the results for $\inf(A + B)$ and $\inf(A - B)$ are similar, or apply the results for the supremum to $-A$ and $-B$. □

2.3. Functions

The supremum and infimum of a function are the supremum and infimum of its range, and results about sets translate immediately to results about functions.

Definition 2.11. If $f : A \to \mathbb{R}$ is a function, then

$$\sup_A f = \sup \{f(x) : x \in A\}, \qquad \inf_A f = \inf \{f(x) : x \in A\}.$$

A function f is bounded from above on A if $\sup_A f$ is finite, bounded from below on A if $\inf_A f$ is finite, and bounded on A if both are finite.

Inequalities and operations on functions are defined pointwise as usual; for example, if $f, g : A \to \mathbb{R}$, then $f \leq g$ means that $f(x) \leq g(x)$ for every $x \in A$, and $f + g : A \to \mathbb{R}$ is defined by $(f + g)(x) = f(x) + g(x)$.

Proposition 2.12. Suppose that $f, g : A \to \mathbb{R}$ and $f \leq g$. If g is bounded from above then

$$\sup_A f \leq \sup_A g,$$

and if f is bounded from below, then

$$\inf_A f \leq \inf_A g.$$

Proof. If $f \leq g$ and g is bounded from above, then for every $x \in A$

$$f(x) \leq g(x) \leq \sup_A g.$$

Thus, f is bounded from above by $\sup_A g$, so $\sup_A f \leq \sup_A g$. Similarly, g is bounded from below by $\inf_A f$, so $\inf_A g \geq \inf_A f$. □

Note that $f \leq g$ does not imply that $\sup_A f \leq \inf_A g$; to get that conclusion, we need to know that $f(x) \leq g(y)$ for all $x, y \in A$ and use Proposition 2.10.

Example 2.13. Define $f, g : [0, 1] \to \mathbb{R}$ by $f(x) = 2x$, $g(x) = 2x + 1$. Then $f < g$ and

$$\sup_{[0,1]} f = 2, \quad \inf_{[0,1]} f = 0, \qquad \sup_{[0,1]} g = 3, \quad \inf_{[0,1]} g = 1.$$

Thus, $\sup_{[0,1]} f > \inf_{[0,1]} g$.

Like limits, the supremum and infimum do not preserve strict inequalities in general.

Example 2.14. Define $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{if } 0 \leq x < 1, \\ 0 & \text{if } x = 1. \end{cases}$$

Then $f < 1$ on $[0,1]$ but $\sup_{[0,1]} f = 1$.

Next, we consider the supremum and infimum of linear combinations of functions. Scalar multiplication by a positive constant multiplies the inf or sup, while multiplication by a negative constant switches the inf and sup,

Proposition 2.15. Suppose that $f : A \to \mathbb{R}$ is a bounded function and $c \in \mathbb{R}$. If $c \geq 0$, then

$$\sup_A cf = c \sup_A f, \qquad \inf_A cf = c \inf_A f.$$

If $c < 0$, then

$$\sup_A cf = c \inf_A f, \qquad \inf_A cf = c \sup_A f.$$

Proof. Apply Proposition 2.7 to the set $\{cf(x) : x \in A\} = c\{f(x) : x \in A\}$. $\square$

For sums of functions, we get an inequality.

Proposition 2.16. If $f, g : A \to \mathbb{R}$ are bounded functions, then

$$\sup_A (f+g) \leq \sup_A f + \sup_A g, \qquad \inf_A (f+g) \geq \inf_A f + \inf_A g.$$

Proof. Since $f(x) \leq \sup_A f$ and $g(x) \leq \sup_A g$ for evry $x \in [a,b]$, we have

$$f(x) + g(x) \leq \sup_A f + \sup_A g.$$

Thus, $f+g$ is bounded from above by $\sup_A f + \sup_A g$, so $\sup_A(f+g) \leq \sup_A f + \sup_A g$. The proof for the infimum is analogous (or apply the result for the supremum to the functions $-f$, $-g$). $\square$

We may have strict inequality in Proposition 2.16 because f and g may take values close to their suprema (or infima) at different points.

Example 2.17. Define $f, g : [0,1] \to \mathbb{R}$ by $f(x) = x$, $g(x) = 1 - x$. Then

$$\sup_{[0,1]} f = \sup_{[0,1]} g = \sup_{[0,1]} (f+g) = 1,$$

so $\sup_{[0,1]}(f+g) < \sup_{[0,1]} f + \sup_{[0,1]} g$.

Finally, we prove some inequalities that involve the absolute value.

Proposition 2.18. If $f, g : A \to \mathbb{R}$ are bounded functions, then

$$\left|\sup_A f - \sup_A g\right| \leq \sup_A |f - g|, \qquad \left|\inf_A f - \inf_A g\right| \leq \sup_A |f - g|.$$

Proof. Since $f = f - g + g$ and $f - g \leq |f - g|$, we get from Proposition 2.16 and Proposition 2.12 that

$$\sup_A f \leq \sup_A (f - g) + \sup_A g \leq \sup_A |f - g| + \sup_A g,$$

so

$$\sup_A f - \sup_A g \leq \sup_A |f - g|.$$

Exchanging f and g in this inequality, we get

$$\sup_A g - \sup_A f \leq \sup_A |f - g|,$$

which implies that

$$\left| \sup_A f - \sup_A g \right| \leq \sup_A |f - g|.$$

Replacing f by $-f$ and g by $-g$ in this inequality and using the identity $\sup(-f) = -\inf f$, we get

$$\left| \inf_A f - \inf_A g \right| \leq \sup_A |f - g|.$$

□

Proposition 2.19. If $f, g : A \to \mathbb{R}$ are bounded functions such that

$$|f(x) - f(y)| \leq |g(x) - g(y)| \qquad \text{for all } x, y \in A,$$

then

$$\sup_A f - \inf_A f \leq \sup_A g - \inf_A g.$$

Proof. The condition implies that for all $x, y \in A$, we have

$$f(x) - f(y) \leq |g(x) - g(y)| = \max\left[g(x), g(y)\right] - \min\left[g(x), g(y)\right] \leq \sup_A g - \inf_A g,$$

which implies that

$$\sup\{f(x) - f(y) : x, y \in A\} \leq \sup_A g - \inf_A g.$$

From Proposition 2.10,

$$\sup\{f(x) - f(y) : x, y \in A\} = \sup_A f - \inf_A f,$$

so the result follows. □