Morita equivalences of torus equivariant spectral triples

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Some of the conditions (not all of them):

- Lipschitz continuity: $[D, a]$ bounded $\forall a \in \mathcal{A}$.
- First order condition: $[[D, a], Jb^* J^{-1}] = 0 \ \forall a, b \in \mathcal{A}$.
- Smoothness: $\mathcal{A}, [D, \mathcal{A}] \subset \bigcap \mathrm{Dom} \delta^k$, $\delta(T) = [|D|, T]$.
- Spectral dimension: k-th eigenvalue of $|D|^{-1}$, ordered from big to small, is of order $\mathcal{O}(k^{-d})$ for an integer d.
- Finiteness: $\mathcal{H}^\infty := \bigcap \mathrm{Dom} D^k$ is f.g. projective over $\mathcal{A}$.

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Canonical example is a spin structure on a manifold M, with spin Dirac operator $\not{D}$.

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Isometric torus action: $\sigma_t(D) = D$.
Compatibility with spin structure: if $\sigma_t(T) = U_t T U_t^{-1}$, $U_t J = J U_{-t}$.

Coactions

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- ρ is injective
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For $\mathbb{T}^n$-action, any smooth action can be translated to continuous coaction:$\sigma_t(a) = e^{2\pi i k \cdot t} a \Rightarrow \rho(a) = a \otimes u^k$.

Strong connections

Because of the results of Dąbrowski, Gosse and Hajac, a right H-comodule algebra A is principal if and only if there exists a strong connection, i.e. there exists a map $\omega : H \to A \otimes H$ such that:

$$\omega(1) = 1 \otimes 1$$
$$\mu \circ \omega = \eta \circ \epsilon$$
$$(\omega \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \rho) \circ \omega$$
$$(S \otimes \omega) \circ \Delta = (\sigma \otimes \mathrm{id}) \circ (\rho \otimes \mathrm{id}) \circ \omega,$$

Flip $\sigma : A \otimes H \to H \otimes A$

Algebra multiplication $\mu : A \otimes A \to A$.

Strong connections II

Lemma

An algebra A, with coaction $\rho : A \to A \otimes U(\mathbb{T}^n)$, has a strong connection if, for each $1 \leq j \leq n$, there exists elements $\sum_i a_i \otimes b_i$, and $\sum_i b'_i \otimes a'_i$ such that $\sum a_i b_i = \sum b'_i a'_i = 1$, $\rho(a_i) = a_i \otimes \mathfrak{t}_j^{-1}$, $\rho(a'_i) = a'_i \otimes \mathfrak{t}_j^{-1}$, $\rho(b_i) = b_i \otimes \mathfrak{t}_j$, and $\rho(b'_i) = b'_i \otimes \mathfrak{t}_j$.

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Proof.

Define strong connection recursively: $\omega(1) = 1 \otimes 1$,

$$\omega(u_j^n) = \sum a_i \omega(u_j^n) b_i.$$

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Example:
S_θ^{2n+1} is a $\mathbb{T}^1$ fibration over S_θ^{2n}:
$\alpha_i \mapsto \alpha_i \otimes u$. $\sum \alpha_i \alpha_i^* = 1$

Spin structures on principal fibrations

Now assume $(\mathcal{A}_0, \mathcal{H}_0, D_h, J_0)$ is a real spectral triple, of spectral dimension d.
Then $(\mathcal{A}, \mathcal{H}, D_h + D_v + Z, J)$ is a real spectral triple, with $[D_v, a_0] = 0$, and $J\mathcal{H}_k = \mathcal{H}_{-k}$, Z commuting with algebra (isometric fibers).

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First order condition $([[D, a], JbJ^{-1}] = 0)$ + strong connection + compact resolvent ensures that $D_v \xi_k = \sum_j (\boldsymbol{\tau}_j \cdot k) A_j \xi_k$, with A_j generator of n-dimensional Clifford algebra, $\boldsymbol{\tau}_j$ basis of $\mathbb{R}^n$. Gives spectral triple of spectral dimension $d + n$.

Morita equivalences

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Two C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ are strongly Morita equivalent if there exists a $(\mathcal{A}, \mathcal{B})$-equivalence bimodule ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$, with $\mathcal{B} = \mathrm{End}_{\mathcal{A}}(\mathcal{E})$ such that:

- $\langle x, y \rangle_{\mathcal{B}} z = x \langle y, z \rangle_{\mathcal{A}}$ for all $x, y, z \in \mathcal{E}$.
- $\langle \mathcal{E}, \mathcal{E} \rangle_{\mathcal{A}}$ spans a dense subset of $\mathcal{A}$, $\langle \mathcal{E}, \mathcal{E} \rangle_{\mathcal{B}}$ of $\mathcal{B}$.

Morita equivalences of spectral triples

Idea: Morita equivalent C^*-algebras contain same topological data (representation theory).
Same geometry? Need Morita equivalence of spectral triples $(\mathcal{A}, \mathcal{H}, D, J)$ and $(\mathcal{A}', \mathcal{H}', D', J')$.

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- Hilbert space: $\mathcal{H}' = \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H} \otimes_{J\mathcal{A}J^{-1}} \bar{\mathcal{E}}$.
- Reality operator: $J'(e \otimes v \otimes \bar{f}) = f \otimes Jv \otimes \bar{e}$.

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Connection is an operator $\nabla_D : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_D$ that satisfies:

- Leibniz rule: $\nabla_D(ea) = \nabla_D(e)a + e \otimes [D, a]$.
- Self-adjointness: $\langle e, \nabla_D f\rangle - \langle \nabla_D e, f\rangle = [D, \langle e, f\rangle_{\mathcal{A}}]$.

The space Ω^1_D is the space of *one forms*: $\mathcal{A}$-bimodule spanned by $\{\sum_i a_i[D, b_i] | a_i, b_i \in \mathcal{A}\}$ where the sum is finite.

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$$D'(e \otimes v \otimes \bar{f}) = \nabla_D(e)v \otimes \bar{f} + e \otimes Dv \otimes \bar{f} + e \otimes v\overline{(\nabla_D f)}.$$

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Examples of Morita equivalences:

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For $\mathcal{A}$ commutative: D also unchanged.

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In fact, Morita equivalence of spectral triples is *not* symmetric.
Well-known counterexample: finite spectral triples. There $(A, \mathcal{H}, D)$ is Morita equivalent to $(A, \mathcal{H}, 0)$.

Morita equivalences of noncommutative tori

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Two of them are isomorphisms, "classical".
Interesting new equivalence σ_2, which is for noncommutative 2-tori:

$$\sigma_2(\begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}) = \begin{pmatrix} 0 & -\frac{1}{\theta} \\ \frac{1}{\theta} & 0 \end{pmatrix}.$$

Trivial deformations

Trivial θ-deformation: strong connection for $U(\mathbb{T}^2)$-coaction generated by 1 generator: $a_k = U_k a_0$, where $U_k^* = U_{-k} = U_k^{-1}$,and A_0 unital commutative.

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Left action of $U'_k a_0$ on $f(t) \otimes b_0$: $e^{-2\pi i k_1 t} f(t + \frac{k_2}{\theta}) \otimes a_0 b_0$.

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Also works (slightly modified) if A_0 is deformed by a cocycle deformation (for example, A_0 is a noncommutative torus itself).

Calculation of Dirac operator

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Morita equivalence of trivial θ-deformations is a symmetric relation.

Proof.

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- Space of connections is free module over $\mathcal{A}$ (Clifford algebra)
- Invariance under torus action: D'_v is uniquely determined.
- D_h unchanged, explicit formula for D'_v.
- Applying σ_2 twice gives back D_v.

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- Only works so far for trivial θ-deformations: no S^n_θ, etc.
- More general fibrations, based on the strong connection?
- Is there a more general principle at work?