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ON THE DEFORMATION THEORY OF REPRESENTATIONS OF FUNDAMENTAL GROUPS OF COMPACT HYPERBOLIC 3-MANIFOLDS

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WE CONSTRUCT compact hyperbolic 3-manifolds M_1, M_2 and an irreducible representation $\rho_1 : \pi_1(M_1) \to SO(3)$ so that singularity of the representation variety of $\pi_1(M_1)$ into $SO(3)$ at ρ_1 is not quadratic. We prove that for any semi-simple Lie group **G** the singularity of the representation variety of $\pi_1(M_2)$ into **G** at the trivial representation is not quadratic. Copyright © 1996 Elsevier Science Ltd

1. INTRODUCTION

In this paper we continue the discussion of the deformation theory of representations in relation to the deformation theory of mechanical linkages that we began in [9]. Goldman and Millson in [7] prove that for the fundamental group of any compact Kähler manifold M and a compact Lie group **G** the only singularities of the representation variety $Hom(\pi_1(M), \mathbf{G})$ are quadratic. In this paper we study the possible singularities of representation varieties of uniform lattices in the group $SO(3, 1)$. Note that according to Carlson and Toledo [3] lattices in $SO(n, 1)$ $(n > 2)$ cannot be isomorphic to fundamental groups of compact Kähler manifolds. Thus the results of [7] are not applicable in our case. We construct cocompact reflection groups $\Gamma_j \subset SO(3, 1)$ and irreducible representations $\rho_j : \Gamma_j \to SO(3)$ so that $\rho_1(\Gamma_1)$ is Zariski dense and $\rho_2(\Gamma_2)$ is finite, such that the singularities of the varieties $Hom(\Gamma_j, SO(3))$ at ρ_j and $V(\Gamma_j, SO(3)) = Hom(\Gamma_j, SO(3))/SO(3)$ at $[\rho_j]$ are *strongly nonquadratic* (see Section 2 for definitions).

We prove this by finding nonzero classes $\zeta \in H^1(\Gamma_j, so(3))$ such that the first obstructions $[\zeta, \zeta] \in H^2(\Gamma_j, so(3))$ to the "integrability" of ζ are trivial, but the vectors ζ are not tangent to any curve in $V(\Gamma_j, SO(3))$ since the second obstructions to the integrability of ζ are nonzero.

In Section 5 we prove that *strongly nonquadratic* singularities are inherited by normal subgroups of finite index. Thus by taking finite-index subgroups we prove the following

THEOREM 10.7. *There exists a compact hyperbolic* 3*-manifold* M_1 *and an irreducible infinite representation* $\rho : \pi_1(M_1) \to SO(3)$ *such that the singularities of the varieties* $Hom(\pi_1(M_1), SO(3))$ *at* ρ_1 *and* $V(\pi_1(M_1), SO(3))$ *at* $[\rho_1]$ *are not quadratic.*

In Section 6 we prove that for a group Γ strongly nonquadratic singularity at the trivial representation into $SO(3)$ implies that for any semi-simple Lie group **G** the variety $Hom(\Gamma, \mathbf{G})$ again has a nonquadratic singularity at the trivial representation. Thus,

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THEOREM 10.8. *There exists a compact hyperbolic 3-manifold* M_2 *such that for any semi-simple Lie group* $\mathbf{G}$ *the varieties* $Hom(\pi_1(M_2), G)$ *and* $V(\pi_1(M_2), G)$ *have nonquadratic singularities at the trivial representation.*

To the best of our knowledge these are the first examples of this sort.

Our examples are based on constructions of mechanical linkages in $\mathbb{S}^2$, which are not rigid at the first and second order, but some of the first order deformations cannot be extended to deformations of order 3.

These examples contrast sharply with the result of [8]: for any cocompact reflection group $\Gamma \subset SO(3,1)$ the variety $Hom(\Gamma, SO(4,1))$ is *smooth* at the point $id: \Gamma \hookrightarrow SO(4,1)$. We discuss vanishing of the cup-product $H^1(\Gamma, so(4,1)_{Ad}) \times H^1(\Gamma, so(4,1)_{Ad}) \to H^2(\Gamma, so(4,1)_{Ad})$ in Section 11.

2. VARIETIES WITH NONQUADRATIC SINGULARITIES

In this section we prove a simple but useful criterion for detection of higher order singularities.

Suppose that $x = (x_1, \dots, x_n)$ and $f_1(x), \dots, f_m(x)$ are homogeneous quadratic polynomials with coefficients in a field $\mathbf{k}$ and

$$R = \frac{\mathbf{k}[x_1, \dots, x_n]}{(f_1, \dots, f_m)}. \tag{1}$$

Let ψ_0: $R \to \mathbf{k}$ be the evaluation at zero.

LEMMA 2.1. *Suppose that* $\psi_1 : R \to \mathbf{k}[t]/(t^2)$ *is a homomorphism lifting* ψ_0 *and* $\psi_2 : R \to \mathbf{k}[t]/(t^3)$ *is a homomorphism lifting* ψ_1. *Then there exists*

$$\psi_\infty : R \to \mathbf{k}[t] \tag{2}$$

lifting ψ_1.

Proof. Let $\psi_1(x_i) = a_i t$, $\psi_2(x_i) = a_i t + b_i t^2$. Then

$$f_j(a_1 t + b_1 t^2, \dots, a_n t + b_n t^2) \equiv 0 (\mathrm{mod}\, t^3), \quad j = 1, \dots, m, \tag{3}$$

Since all polynomials f_j are quadratic we conclude:

$$f_j(a_1 t, \dots, a_n t) = 0. \tag{4}$$

Therefore we take $\psi_\infty = \psi_1$. □

Let V be a variety defined over $\mathbf{k}$, $o \in V$ be a point and $\hat{\mathcal{O}}_{V,o}$ the complete local ring. We denote by

$$J_o^m(V) = Hom_{\mathbf{k}\text{-}alg}(\hat{\mathcal{O}}_{V,o}, \mathbf{k}[t]/t^{m+1}) \tag{5}$$

the mth order jet space at $o \in V$ and by $\pi: J_o^m \to T_o(V)$ the natural projection to the Zariski tangent space.

We say that V has a *nonquadratic singularity* at o if the complete local ring of V at o is nor formally isomorphic to the complete local ring of zero in an affine variety W given by homogeneous quadratic equations.

LEMMA 2.2. *Suppose that $\xi \in J_o^2(V)$ has the property that $\pi(\xi)$ is not tangent to any formal curve in V. Then V has a nonquadratic singularity at o.*

Proof. Suppose to the contrary that V has a quadratic singularity and W is the corresponding variety given by the quadratic equations

$$f_1 = 0, \dots, f_m = 0 \tag{6}$$

Let $\zeta \in J_0^2(W)$ be the image of ξ under this isomorphism. Then ζ corresponds to a pair (ψ_1, ψ_2) as in Lemma 2.1. It follows that the homomorphism ψ_∞ given by Lemma 2.1 will define a curve tangent to $\pi(\zeta)$. This contradiction proves that V has a nonquadratic singularity. $\square$

Suppose V is a variety such that there exists $\xi \in J_o^2(V)$ with the property that $\pi(\xi) \notin \pi(J_o^3(V))$. In this case we say that the variety V has a *strongly nonquadratic singularity* at the point o. The tangent vector $\pi(\xi)$ is said to be obstructed at the third order but not at the second order.

It follows from Lemma 2.2 that the existence of a *strongly nonquadratic singularity* of V at a point o implies the nonquadratic singularity of V at o.

3. COMPUTATION OF $H^2(\Gamma, \mathfrak{g})$

Let Γ be a finitely-presented group and ρ: $\Gamma \to \mathbf{G}$ be a representation into the group $\mathbf{G}$ of real points of a linear algebraic group defined over $\mathbb{R}$. Denote by $\mathfrak{g}$ the Lie algebra of $\mathbf{G}$. There exists a smooth compact 4-manifold M such that $\Gamma = \pi_1(M)$. We let $p{:}X \to M$ be the universal cover (see [16, p. 180]), hence Γ acts freely and properly on X. We consider $\mathfrak{g}$ as Γ-modulus via the adjoint representation $Ad \circ \rho$. In this section we show how to compute $H^i(\Gamma, \mathfrak{g})$, $i = 0$, 1, 2 in terms of differential forms on M. Proposition 3.2 may be of independent interest.

Let P be the principal $\mathbf{G}$-bundle with flat connection ω_0 associated to ρ and adP the associated flat bundle of Lie algebras (with the fiber isomorphic to $\mathfrak{g}$). Let $\mathscr{A}^\cdot(M, adP)$ (resp. $\mathscr{A}^\cdot(X, p^*adP)$) be the differential graded Lie algebra of smooth adP-valued forms on M (resp. smooth p^*adP-valued forms on X). Let $U_1, \dots, U_N$ be a cover of M by contractible open sets such that all the intersections of the U_i's are contractible. We let $\mathscr{U} = \{U_1, \dots, U_N\}$. The inverse image $p^{-1}(U_i)$ is a countable disjoint union of contractible sets permuted simply-transitively by Γ. We choose an indexing of the components by Γ such that $U_{i,\mu\gamma} = \mu U_{i,\gamma}$. Thus

$$p^{-1}(U_i) = \bigcup\nolimits_{\gamma \in \Gamma} U_{i,\gamma}. \tag{7}$$

We let $\tilde{\mathscr{U}} = \{U_{i,\gamma},\ i = 1, \dots, N, \gamma \in \Gamma\}$ be the resulting cover of X. Then all the intersections of the $U_{i,\gamma}$'s are also contractible.

Let S_q (resp. $\tilde{S}_q$) denote the q simplices in $Nerve(\mathscr{U})$ (resp. $Nerve(\tilde{\mathscr{U}})$). If $\sigma \in S_q$ (resp. $\sigma \in \tilde{S}_q$) we let U_σ (resp. $U_{\tilde{\sigma}}$) denote the corresponding q-fold intersection. Now

$$U_{i_0,\gamma_0} \cap \dots \cap U_{i_q,\gamma_q} \tag{8}$$

Hence each q-simplex of $Nerve(\tilde{\mathscr{U}})$ corresponds to a unique q-simplex $\pi(\sigma)$ on $Nerve(\mathscr{U})$ and we obtain a simplicial map

$$\pi: Nerve(\hat{\mathscr{U}}) \to Nerve(\mathscr{U}). \tag{9}$$

Now let $\sigma = (i_0, i_1, \ldots, i_q)$ be a q-simplex of $Nerve(\mathscr{U})$. The inverse image $p^{-1}(U_\sigma)$ is a countable union of components permuted simply-transitively by Γ. Each of these components corresponds to a unique simplex in $\pi^{-1}(\sigma)$. Thus Γ acts simply-transitively on $\pi^{-1}(\sigma)$. Therefore, we may choose a Γ-equivalent bijection $F\colon \tilde{S}_q \to S_q \times \Gamma$. We write $U_{\tilde{\sigma}} = U_{\sigma,\gamma}$ with $F(\tilde{\sigma}) = (\sigma, \gamma)$. Hence $\mu U_{\sigma,\gamma} = U_{\sigma,\mu\gamma}$, $\mu \in \Gamma$.

Let $\mathscr{A}^q_M$ (resp. $\mathscr{A}^q_X$) denote the sheaves associated to the q-forms on M with values in adP (resp. q-forms on X with values in p^*adP). We let $p_{\sigma,\gamma}$ denote the restriction $p \mid U_{\sigma,\gamma}$. We have an induced isomorphism of sections

$$p^*_{\sigma,\gamma}\colon \Gamma(U_\sigma, \mathscr{A}^q_M) \to \Gamma(U_{\sigma,\gamma}, \mathscr{A}^q_X). \tag{10}$$

Let $C^p(U, \mathscr{A}^q_M)$ and $C^p(U, \mathscr{A}^q_X)$ be the corresponding Čech cochain groups. Hence

$$C^p(\mathscr{U}, \mathscr{A}^q_M) = \prod_{\sigma \in S_p} \Gamma(U_\sigma, \mathscr{A}^q_M) \tag{11}$$

$$C^p(\tilde{\mathscr{U}}, \mathscr{A}^q_X) = \prod_{\tilde{\sigma} \in \tilde{S}_p} \Gamma(U_{\tilde{\sigma}}, \mathscr{A}^q_X). \tag{12}$$

The group Γ acts on $C^p(\tilde{\mathscr{U}}, \mathscr{A}^q_X)$ by

$$(\mu \cdot \omega)_{\sigma,\gamma} = (\mu^{-1})^* \omega_{\sigma,\mu^{-1}\gamma}. \tag{13}$$

Let G, H be groups and V be an H-module. Then we will define the induced G-module $Ind^G_H V$ with the underlying vector space

$$Hom_{\mathbb{R}(H)}(\mathbb{R}(G), V) = \{T : G \to V : T(gh) = hT(q)\} \tag{14}$$

equipped with the G-action

$$g_0 T(g) = T(g_0^{-1} g). \tag{15}$$

We recall Shapiro's Lemma [2, Ch. 3, Proposition 6.2]:

$$H^p(G, Ind^G_H V) = H^p(H, V) \tag{16}$$

LEMMA 3.1. *The Γ-modules $C^p(\tilde{\mathscr{U}}, \mathscr{A}^q_X)$ satisfy*

$$H^i(\Gamma, C^p(\tilde{\mathscr{U}}, \mathscr{A}^q_X)) = 0, \quad \text{for all } p, q \text{ and } i > 0.$$

Proof. Denote by e the trivial subgroup of Γ. We claim that there is an isomorphism of Γ-modules

$$\varphi\colon Ind^\Gamma_e C^p(\mathscr{U}, \mathscr{A}^q_M) \to C^p(\tilde{\mathscr{U}}, \mathscr{A}^q_X). \tag{17}$$

Indeed, we define $\varphi(T)$ for $T \in Hom(\mathbb{R}(\Gamma), C^p(\mathscr{U}, \mathscr{A}^q_M))$ by

$$\varphi(T)_{\sigma,\gamma} = p^*_{\sigma,\gamma} T(\gamma)_\sigma. \tag{18}$$

We claim that φ is a Γ-module isomorphism. Indeed we have

$$\varphi(\mu T)_{\sigma,\gamma} = p^*_{\sigma,\gamma}(\mu T)(\gamma)_\sigma = p^*_{\sigma,\gamma} T(\mu^{-1}\gamma)_\sigma \tag{19}$$

and

$$(\mu\varphi(T))_{\sigma,\gamma} = (\mu^{-1})^* \varphi(T)_{\sigma,\mu^{-1}\gamma} = (\mu^{-1})^* p^*_{\sigma,\mu^{-1}\gamma} T({}^{\mu^{-1}}\gamma)_\sigma. \tag{20}$$

But $p_{\sigma,\mu-1\gamma} \circ \mu^{-1} = p_{\sigma,\gamma}$ and the claim follows.

The lemma follows from Shapiro's Lemma (taking $H = e$). □

PROPOSITION 3.2.

$$H^p(\Gamma, \mathscr{A}^j(X, p^*adP)) = 0, \quad p > 0$$

Proof. We consider the Eilenberg–MacLane, Čech double complex $C^{\cdot\cdot}$ with $C^{p,q} = C^p(\Gamma, C^q(\tilde{\mathscr{U}}, \mathscr{A}^j_X))$. Let $T\cdot C^{\cdot\cdot}$ be the total complex. We claim $H^i(TC^{\cdot\cdot}) = 0, i > 0$. To see this we filter $T\cdot C^{\cdot\cdot}$ by q. Then $F^iT\cdot C^{\cdot\cdot}$ is the subcomplex such that

$$(F^iT\cdot C^{\cdot\cdot})^n = \bigoplus_{q \geq i} C^{n-q,q}. \tag{21}$$

The E_1-term of the resulting spectral sequence is given by

$$E_1^{p,q} = H^p(\Gamma, C^q(\tilde{\mathscr{U}}, \mathscr{A}^j_X)), \tag{22}$$

By Lemma 3.1 $E_1^{p,q} = 0$, $p > 0$. Also

$$E_1^{0,q} = H^0(\Gamma, C^q(\tilde{\mathscr{U}}, \mathscr{A}^j_X)) = C^q(\mathscr{U}, \mathscr{A}^j_X). \tag{23}$$

Since $\mathscr{A}^j_M$ is a fine sheaf the E_2-term of the spectral sequence has only one nonzero term and

$$E_2^{0,0} = \mathscr{A}^j(M, adP)$$

The claim follows by the basic theorem on the spectral sequences associated to a double complex, [15, Theorem 3.10].

We now filter $C^{\cdot\cdot}$ by p and find that the E_1-term is given by

$$E_1^{p,q} = C^p(\Gamma, H^p(\tilde{\mathscr{U}}, \mathscr{A}^j_X)). \tag{24}$$

Hence the E_1-term is concentrated on the p-axis with

$$E_1^{p,0} = C^p(\Gamma, \mathscr{A}^p(X, p^*adP)). \tag{25}$$

Hence the E_2-term is concentrated on the p-axis with

$$E_2^{p,0} = H^p(\Gamma, \mathscr{A}^j(X, p^*adP)). \tag{26}$$

But by the fundamental theorem on the spectral sequences associated to a double complex we have

$$H^p(\Gamma, \mathscr{A}^j(X, p^*adP)) = H^p(TC^{\cdot\cdot}) = 0, \quad p > 0. \tag{27}$$

□

Remark 3.3. In the case $p = 1$ and $dim(M) = 2$ Proposition 3.2 was proven by Kra in [12].

We can now prove the result we need.

PROPOSITION 3.4. *There are canonical morphisms* $\varphi^p : H^p(\Gamma, \mathfrak{g}) \to H^p(M, adP)$ *such that*

(i) φ^1 *is an isomorphism*;

(ii) φ^2 *is a monomorphism onto the kernel of*

$$p^* : H^2(M, adP) \to H^2(X, p^*adP).$$

Proof. We consider the Eilenberg–MacLane de Rham double complex $C^{\cdot\cdot}$ with $C^{p,q} = C^p(\Gamma, \mathscr{A}^q(X, p^*adP))$. Here $C^{p,q}$ is the group of *inhomogeneous* cochains on Γ with values in the Γ-module $\mathscr{A}^q(X, p^*adP)$—see [14, Ch. 4, §5]. We let $(T^{\cdot}C^{\cdot\cdot}, D)$ be the associated total complex. We first claim that

$$H^{\cdot}(T^{\cdot}C^{\cdot\cdot}) = H^{\cdot}(M, adP).$$

To see this filter $T^{\cdot}C^{\cdot\cdot}$ by q. Then the resulting spectral sequence has $E_1^{p,q} = \mathscr{A}^q(X, p^*adP))$. Hence by Proposition 3.2 we have $E_1^{p,q} = 0$, $p > 0$. Since we have $E_1^{0,q} = \mathscr{A}^q(X, p^*adP)$ the claim follows.

We will need to make explicit how the isomorphism

$$\psi\colon H^2(M, adP) \to H^2(T^{\cdot}C^{\cdot\cdot})$$

is obtained. A class in $H^2(T^{\cdot}C^{\cdot\cdot})$ is represented by a cocycle (for the double complex)

$$\begin{aligned}(a,b,c) \in C^0(\Gamma, \mathscr{A}^2(X, p^*adP)) &\oplus C^1(\mathit{\Gamma}, \mathscr{A}^1(X, p^*adP))\\ &\oplus C(\Gamma, \mathscr{A}^0(X, p^*adP)) = T^2C^{\cdot\cdot}.\end{aligned} \tag{28}$$

The cocycle condition is equivalent to

$$da = 0, \quad \delta a = db, \quad \delta b = dc, \quad \delta c = 0 \tag{29}$$

The isomorphism ψ is induced by the map of cochains $\psi : \mathscr{A}^2(X, p^*adP) \to T^2C^{\cdot\cdot}$ given by $\psi(\omega) = (p^*\omega, 0, 0)$. The content of the previous argument is that given a cocycle $(a,b,c) \in T^2C^{\cdot\cdot}$ we can find a cochain $(e,f) \in C^0(\Gamma, \mathscr{A}^1(X, p^*adP)) \oplus C^1(\Gamma, \mathscr{A}^0(X, p^*adP)) = T^1C^{\cdot\cdot}$ such that

$$(a, b, c) - D(e, f) = (a', 0, 0)$$

for some $a' \in C^0(\Gamma, \mathscr{A}^1(X, p^*adP))$. Since $(a', 0, 0)$ is a cocycle in the total complex $da' = 0$ and $\delta a' = 0$ whence $a' = p^*\omega$ with ω a closed adP-valued 1-form on M.

We now filter $T^{\cdot}C^{\cdot\cdot}$ by p. We find that $E_1^{p,0} = C^p(\Gamma, g)$ and consequently $E_2^{p,0} = H^p(\Gamma, \mathfrak{g})$. We define φ^p to be the composition

$$E_2^{p,0} \to E_\infty^{p,0} = F^pH^p(T^{\cdot}C^{\cdot\cdot}) \subset H^p(M, adP). \tag{30}$$

Since $H^1(X) \otimes \mathfrak{g} = 0$, it is immediate that φ^1 is an isomorphism and φ^2 is a monomorphism. It remains to identify the image of φ^2.

By general results on the spectral sequences associated to a double complex [15], Theorem 2.1, the image of φ^2 is the subspace of $H^2(M, adP)$ consisting of classes of filtration level 2 for the filtration induced via the isomorphism ψ from the filtration $T^2C^{\cdot\cdot}$ by p. Hence $\omega \in Im(\varphi^2)$ if and only if $\psi(\omega)$ is cohomologous to a cocycle in $T^2C^{\cdot\cdot}$ of the form $(0, 0, c)$. We now prove that $\psi(\omega)$ is cohomologous to a cocycle of the form $(0, 0, \mathrm{c})$ if and only if $p^*\omega$ is exact in $\mathscr{A}^2(X, p^*adP)$. Suppose first that $\psi(\omega)$ is cohomologous to such a cocycle. Then there exists $(e,f) \in T^1C^{\cdot\cdot}$ such that $de = p^*\omega$, $\delta e = df$, $\delta f = 0$. Now $e \in \mathscr{A}^2(X, p^*adP)$ so $p^*\omega$ is exact.

Now suppose that $p^*\omega$ is exact. Then there exists $e \in \mathscr{A}^1(X, p^*adP)$ such that $de = p^*\omega$. For each $\gamma \in \Gamma$, $\delta e(\gamma)$ is a closed p^*adP-valued 1-form on X. Since X is simply-connected there exists $f(\gamma)$, a smooth section of p^*adP, such that $df(\gamma) = \delta e(\gamma)$. Put $c = \delta f$. Then $c(\mu, \gamma)$ is a parallel section of p^*adP for $\mu, \gamma \in \mathit{\Gamma}$ and defines an element of $Z^2(\Gamma, \mathfrak{g})$. The cochain (e, f) gives a cohomology form $\psi(\omega) = (p^*\omega, 0, 0)$ to $(0, 0, c)$ and the proposition follows. □

4. THE MASSEY TRIPLE PRODUCT

Let Γ be a finitely-presented group. We assume that $\Gamma = \pi_1(M)$ where M is a smooth compact 4-manifold as in the previous section so that $M = X/\Gamma$.

We recall that the first cohomology group $H^1(\Gamma, \mathfrak{g})$ is isomorphic to the Zariski tangent space of the representation variety $V(\Gamma, \mathbf{G})$ at the point $[\rho]$.

In this section we show that the nonzero tangent vector $\zeta \in H^1(\Gamma, \mathfrak{g})$ is obstructed at third order, but not at second order if and only if the cup-product

$$[\zeta, \zeta] \in H^2(\Gamma, \mathfrak{g}) \tag{31}$$

vanishes but the Massey triple product

$$\langle \zeta | \zeta | \zeta \rangle \in H^2(\Gamma, \mathfrak{g})/I \tag{32}$$

is nonzero. Here $I \subset H^2(\Gamma, so(3))$ is the subspace

$$I = \{[\eta, \zeta] : \eta \in H^2(\Gamma, \mathfrak{g})\}. \tag{33}$$

It will be crucial for us that we can compute $H^i(\Gamma, \mathfrak{g})$, $i = 1, 2$ and the deformation space of ρ in terms of differential forms.

Let P be the principal $\mathbf{G}$-bundle with that connection ω_0 associated to ρ and *adP* the associated flat bundle of Lie algebras (with the fiber isomorphic to $\mathfrak{g}$). We then consider the differential graded Lie algebra of forms

$$\mathscr{A}^{\cdot}(M, adP). \tag{34}$$

Choose a point $x_0 \in M$ and define an augmentation

$$\varepsilon : \mathscr{A}^{\cdot}(M, adP) \to \mathfrak{g} \tag{35}$$

as follows. For $\xi \in \mathscr{A}^0(M, adP)$ define $\varepsilon(\xi) = \xi(x_0)$. For $\eta \in \mathscr{A}^0(M, adP)$, $i > 0$ we define $\varepsilon(\eta) = 0$. We let $\mathscr{A}^{\cdot}(M, adP)_0$ denote the kernel of ε. We abbreviate $\mathscr{A}^{\cdot}(M, adP)_0$ to $L^{\cdot}$. It follows that we have a map

$$H^1(L^{\cdot}) \xrightarrow{\tau} Z^1(\Gamma, \mathfrak{g}). \tag{36}$$

Here $Z^1(\Gamma, \mathfrak{g})$ is the space of Eilenberg–MacLane 1-cocycles. The map τ is the isomorphism induced by the *period* map

$$\tau : \mathscr{A}^1(M, adP) \to C^1(\Gamma, \mathfrak{g}) \tag{37}$$

(here $C^1(\Gamma, \mathfrak{g})$ is the space of Eilenberg–MacLane 1-cochains). The map τ is defined as follows. Let $p : X \to M$ be the universal cover. Choose a base-point $\tilde{x}_0 \in X$ over x_0. Let $\eta \in \mathscr{A}^1(M, adP)$. Define $\tau(\eta) \in C^1(\Gamma, \mathfrak{g})$ by

$$\tau(\eta)(\gamma) = \int_{\tilde{x}_0}^{\gamma\tilde{x}_0} p^*\eta. \tag{38}$$

Here we identify $p^*\eta$ with a $\mathfrak{g}$-valued 1-form on X by parallel translation from $\tilde{x}_0$.

We will need another description of the period map τ. We define

$$w : H^1(L^{\cdot}) \to Z^1(\Gamma, \mathfrak{g}) \tag{39}$$

as follows. Given $[\eta] \in H^1(L^\cdot)$ choose a representing closed 1-form $\eta \in L^\cdot$. Let $\tilde{\eta} = p^*\eta$. Let $f\colon X \to \mathfrak{g}$ be the unique function satisfying

(i) $f(\tilde{x}_0) = 0$;
(ii) $df = \tilde{\eta}$.

Define $w([\eta]) \in Z^1(\Gamma, \mathfrak{g})$ by

$$w([\eta])(\gamma) = f(x) - Ad\rho(\gamma) f(\gamma^{-1}x).$$

We observe that w is well-defined. Indeed, if $[\eta]$ is exact in $L^\cdot$ then $w([\eta]) = 0$,

Lemma 4.1. $w = \tau$,

Proof. We have $f(x) = \int_{\tilde{x}_0}^{x} \tilde{\eta}$ whence

$$w([\eta])(\gamma) = \int_{\tilde{x}_0}^{x} \tilde{\eta} - \rho(\gamma) \int_{\tilde{x}_0}^{\gamma^{-1}x} \tilde{\eta} = \int_{\tilde{x}_0}^{x} \tilde{\eta} - \int_{\tilde{x}_0}^{\gamma^{-1}x} \rho(\gamma)\tilde{\eta}$$

$$= \int_{\tilde{x}_0}^{x} \tilde{\eta} - \int_{\tilde{x}_0}^{\gamma^{-1}x} \gamma^*\tilde{\eta} = \int_{\tilde{x}_0}^{x} \tilde{\eta} - \int_{x}^{\gamma\tilde{x}_0} \tilde{\eta} = \int_{\tilde{x}_0}^{\gamma\tilde{x}_0} \tilde{\eta} \qquad \square$$

Let $[\zeta]$, $[\eta] \in H^1(\Gamma, \mathfrak{g})$, choose differential forms $\zeta, \eta \in L^1$ representing these classes. Define $[\zeta, \eta]$ to be the wedge product of these forms where we use the Lie bracket in $\mathfrak{g}$ to multiply the coefficients of these forms. The corresponding class $M_2([\zeta]) := [\zeta, \zeta] \in \mathbb{H}^2(\Gamma, \mathfrak{g})$ is called the cup-product of ζ with itself.

Now let $\mathcal{Q} \subset Z^1(\Gamma, \mathfrak{g})$ be the quadratic cone consisting of those classes ζ such that $[\zeta, \zeta] = 0$ in $H^2(\Gamma, \mathfrak{g})$. Let $\tilde{\mathcal{Q}} \subset H^1(L^\cdot)$ be the quadratic cone consisting of those classes η such that $[\eta, \eta] = 0$ in $H^2(L^\cdot)$. The next lemma is a consequence of [5], Lemma 4.1.

Lemma 4.2. *The period map τ carries the cone $\tilde{\mathcal{Q}}$ onto the cone $\mathcal{Q}$.*

We now define the Massey triple product $\langle \zeta | \zeta | \zeta \rangle$ as follows. Choose a closed form $\eta_1 \in L^1$ representing ζ. Since $\zeta \in \mathcal{Q}$ there exists $\eta_2 \in L^1$ such that $d\eta_2 = [\eta_1, \eta_1]$.

Lemma 4.3. $[\eta_1, \eta_2] \in H^2(\Gamma, \mathfrak{g})$.

Proof. By Proposition 3.4 it suffices to prove that $p^*[\eta_1, \eta_2]$ is exact. Since X is simply-connected there exists $v_1 \in \mathcal{A}^1(X, p^*adP)$ such that $dv_1 = p^*\eta_1$. We will abbreviate $p^*\eta_j$ to $\tilde{\eta}_j$ henceforth. The graded Jacobi identity [7, §1.1], implies

$$[v_1, [\tilde{\eta}_1, \tilde{\eta}_1]] = 2[\tilde{\eta}_1, [v_1, \tilde{\eta}_1]] \tag{40}$$

and hence

$$d[v_1, [v_1, \tilde{\eta}_1]] = \frac{3}{2}[v_1, [\tilde{\eta}_1, \tilde{\eta}_1]]. \tag{41}$$

To conclude we have only to observe that

$$d[v_1, \tilde{\eta}_2] = [\tilde{\eta}_1, \tilde{\eta}_2] + [v_1, [\tilde{\eta}_1, \tilde{\eta}_1]]. \tag{42}$$

$\square$

Define $M_3(\zeta) = \langle\zeta|\zeta|\zeta\rangle$ to be the class of $[\eta_1, \eta_2]$ in $H^2(\Gamma, \mathfrak{g})/I$. We recall that

$$I = \{[\eta, \zeta]: \eta \in H^1(\Gamma, \mathfrak{g})\} \tag{43}$$

is the ideal generated by $[\zeta]$.

LEMMA 4.4. *$\langle\zeta|\zeta|\zeta\rangle$ is well-defined.*

Proof. We check that $[\eta_1, \eta_2]$ is closed. Indeed

$$d[\eta_1, \eta_2] = [\eta_1, d\eta_2] = [\eta_1, [\eta_1, \eta_1]] = 0. \tag{44}$$

The last equality follows from the graded Jacobi identity in $L^{\cdot}$. The reader will check that $\langle\zeta|\zeta|\zeta\rangle$ is independent of choice of the forms η, η_2. □

One defines the higher Massey n-fold product operations M_n similarly (see [6]), we will need them only for $n = 2, 3$.

We now relate the operations M_n to infinitesimal deformations of representations. Let A_n denote the truncated polynomial ring $\mathbb{R}[t]/(t^{n+1})$. If $m < n$ we have a surjection

$$\Pi_{m,n}: A_n \to A_m.$$

We abbreviate $\Pi_{m-1,m}$ to Π_m. Observe that the set $Hom(\Gamma, \mathbf{G})(A_n)$ of A_n-points of the affine variety $Hom(\Gamma, \mathfrak{g})$ is the set of "curves"

$$\rho_t = \rho_0 + \rho_1 t + \cdots + \rho_n t^n \tag{45}$$

such that

$$\rho_t(xy) \equiv \rho_t(x)\rho_t(y)(\operatorname{mod} t^{n+1}). \tag{46}$$

We let $Hom(\Gamma, \mathbf{G})_\rho(A_n)$ denote the subset of the above set such that $\rho_0 = \rho$ where $\rho: \Gamma \to \mathbf{G}$ is a fixed representation. We have the induced maps

$$\Pi_n: Hom(\Gamma\ \mathbf{G})_\rho(A_n) \to Hom(\Gamma, \mathbf{G})_\rho(A_{n-1}) \tag{47}$$

obtained by dropping the last term. We use $\Pi_{1,n}$ to project $Hom(\Gamma, \mathbf{G})_\rho(A_n)$ into $Hom(\Gamma, \mathbf{G})_\rho(A_1)$. We will denote the image of $Hom(\Gamma, \mathbf{G})_\rho(A_n)$ by $Hom^1(\Gamma, \mathbf{G})_\rho(A_n)$, it consists of infinitesimal deformations of the representation ρ which are "integrable up to order n". By [5, §4.4] we have natural bijections of sets:

$$Hom(\Gamma, \mathbf{G})_\rho(A_1) \cong Z^1(\Gamma, \mathfrak{g}) \tag{48}$$

$$Hom^1(\Gamma, \mathbf{G})_\rho(A_2) \cong \mathcal{Q}. \tag{49}$$

The bijections in (48) and (49) are obtained as follows. Let $\rho_t = \rho + \rho_1 t \in Hom(\Gamma, \mathbf{G})_\rho(A_1)$. Define $c \in Z^1(\Gamma, \mathfrak{g})$ by

$$c(\gamma) = \rho_1(\gamma)\rho(\gamma)^{-1}.$$

The reader will verify that c satisfies the cocycle identity

$$c(\gamma_1\gamma_2) = c(\gamma_1) + Ad\rho(\gamma_1)c(\gamma_2).$$

If there exists $\rho_2: \Gamma \to \mathbf{G}$ such that $\rho + \rho_1 t + \rho_2 t^2 \in Hom(\Gamma, \mathbf{G})(A_2)$ then it is easily checked [7, §4.4], that $[c, c]$ is exact, whence $c \in \mathcal{Q}$. The map Π_2 is just the correspondence $\rho_t \mapsto c$.

We now wish to identify $Hom^1(\Gamma, \mathbf{G})(A_3)$ as well as the map Π_3. Define $\mathscr{C} \subset \mathscr{Q}$ by

$$\mathscr{C} = \{\zeta \in \mathscr{Q}: \langle \zeta | \zeta | \zeta \rangle = 0\} \tag{50}$$

LEMMA 4.5. *There is a canonical bijection $Hom^1(\Gamma, \mathbf{G})(A_3) \cong \mathscr{C}$ corresponding to the map Π_3.*

Proof. We replace the infinitesimal deformation theory of ρ with the equivariant deformation theory of the flat connection ω_0. Precisely, we replace the groupoid $\mathscr{R}'_{A_n}(\rho)$ of [7, §6.4], with the equivalent groupoid $\mathscr{F}'_{A_n}(\omega_0)$ of [7, §6.4]. The objects of $\mathscr{F}'_{A_n}(\omega_0)$ are infinitesimal deformations (parameterized by $Spec(A_n)$) of ω_0 and the morphisms are infinitesimal deformations of the identity in the group of gauge transformations. By Corollary 6.4 of [7] the holonomy map induces a canonical bijection *hol* from the set of isomorphism classes $Iso\,\mathscr{F}'_{A_n}(\omega_0)$ to the set $Hom(\Gamma, \mathbf{G})(A_n)$. For $n = 1$ we obtain the bijection w. For $n = 2$ we obtain the bijection τ between $\tilde{\mathscr{Q}}$ and $\mathscr{Q}$ of Lemma 4.2. Thus to prove the lemma we have to solve the following problem.

Let $\eta_1 \in \tilde{\mathscr{Q}}$ with $hol(\eta_1) = \tau(\eta_1) = \zeta$. Hence there exists $\eta_2 \in L^1$ such that $\omega_2 = \eta_1 t + \eta_2 t^2$ satisfies

$$d\omega_2 + \frac{1}{2}[\omega_2, \omega_2] \equiv 0 \pmod{t^3} \tag{51}$$

Find necessary and sufficient conditions that there exist v_2, $v_3 \in L^1$ such that $\omega_3 = \eta_1 t + v_2 t^2 + v_3 t^3$ satisfies

$$d\omega_3 + \frac{1}{2}[\omega_3, \omega_3] \equiv 0 \pmod{t^4} \tag{52}$$

For any choice of v_2, v_3 we have:

$$dv_2 = [\eta_1, \eta_1], \qquad dv_3 = [\eta_1, v_2]. \tag{53}$$

Hence there exists a closed form α such that $v_2 = \eta_2 + \alpha$. We find that v_2, v_3 exist as above if and only if there is a closed 1-form $\alpha \in L^1$ and a 1-form $\eta_3 \in L^1$ such that

$$d\eta_3 = [\eta_1, \eta_2] + [\eta_1, \alpha]. \tag{54}$$

The latter equation holds if and only if the cohomology class of $[\eta_1, \eta_2]$ belongs to I. □

Thus we obtain the main result of this section.

THEOREM 4.6. *Let Γ be a finitely-presented group as above and $\rho: \Gamma \to \mathbf{G}$ be a representation. Then the varieties $Hom(\Gamma, \mathbf{G})$ and $V(\Gamma, \mathbf{G})$ have strongly nonquadratic singularities at the points ρ and $[\rho]$ if and only if there exists $\zeta \in H^1(\Gamma, so(3))$ such that:*

$$[\zeta, \zeta] = 0 \text{ in } H^2(\Gamma, \mathfrak{g})$$

$$\langle \zeta | \zeta | \zeta \rangle \neq 0 \text{ in } H^2(\Gamma, \mathfrak{g})/I.$$

5. NONQUADRATIC SINGULARITIES FOR REPRESENTATIONS OF SUBGROUPS OF FINITE INDEX

In this section we will prove that strongly nonquadratic singularities of representation varieties are inherited by normal subgroups of finite index.

THEOREM 5.1. *Suppose that Γ is a finitely presented group as in Section* 4, Γ' *is a torsion-free normal subgroup in* Γ *of finite index. Let* $\mathbf{G}$ *be a semisimple Lie group such that the representation variety* $Hom(\Gamma, \mathbf{G})$ *has strongly nonquadratic singularity at a point* ρ. *Then the varieties* $Hom(\Gamma', \mathbf{G})$ *and* $V(\Gamma', \mathbf{G})$ *also have strongly nonquadratic singularities at the points* $\rho' = \rho|_{\Gamma'}$ *and* $[\rho']$, *respectively.*

Proof. According to Theorem 4.6 there exists a class $\zeta \in H^1(\Gamma, \mathfrak{g})$ such that $M_2(\zeta) = 0$ but $M_3(\zeta) \neq 0$ in $H^2(\Gamma, \mathfrak{g})/I$. Let ζ' be the image of ζ under the inclusion

$$H^1(\Gamma, \mathfrak{g}) \to H^1(\Gamma', \mathfrak{g}).$$

Let $I' \subset H^2(\Gamma', \mathfrak{g})$ be defined by

$$I' = \{[\eta', \zeta']: \eta' \in H'(\Gamma', \mathfrak{g})\}. \tag{55}$$

Our goal is to prove that $\langle \zeta' | \zeta' | \zeta' \rangle \neq 0$ in $H^2(\Gamma', \mathfrak{g})/I'$ (in this case Theorem 4.6 would imply that we have a strongly nonquadratic singularity). Let $\Delta = \Gamma/\Gamma'$ and $M = X/\Gamma'$. Denote by

$$\mathscr{A}^{\cdot}(M, adP)^{\Delta}$$

the subalgebra of invariants in $\mathscr{A}^{\cdot}(M, adP)$.

We let η_1, η_2 be Δ-invariant 1-forms on M so that $[\eta_1, \eta_1] = d\eta_2$ and $[\eta_1] = \zeta$. Since $M_3(\zeta) \neq 0$ for each η_3, $\xi \in \mathscr{A}^1(M, adP)^{\Delta}$ with $d\xi = 0$ we have the property:

$$[\eta_2, \eta_1] \neq [\xi, \eta_1] + d\eta_3. \tag{56}$$

We now define the Reynolds operator

$$R: \mathscr{A}^1(M, adP) \to \mathscr{A}^1(M, adP)^{\Delta} \tag{57}$$

by the formula:

$$R(\eta) = \frac{1}{|\Delta|} \sum_{\gamma \in \Delta} \gamma^* \eta. \tag{58}$$

Then R is a morphism of complexes, $R(\mu) = \mu$ for $\mu \in \mathscr{A}^1(M, adP)^{\Delta}$ and R satisfies the Reynolds identity

$$R([\mu, \nu]) = [\mu, R(\nu)], \qquad \mu \in \mathscr{A}^1(M, adP)^{\Delta}, \qquad \nu \in \mathscr{A}^1(M, adP). \tag{59}$$

Suppose now that $M_3(\zeta') = 0$ in $H^2(\Gamma', \mathfrak{g})/I'$. Then there exists $\xi' \in \mathscr{A}^1(M, adP)$ with $d\xi' = 0$ and $\eta'_3 \in \mathscr{A}'(M, adP)$ such that

$$[\eta_2, \eta_1] = [\xi', \eta_1] + \mathrm{d}\eta'_3. \tag{60}$$

We apply the operator R to this formula and use the fact that η_2, η_1 are Δ-invariants to obtain

$$[\eta_2, \eta_1] = [R(\xi'), \eta_1] + dR(\eta'_3). \tag{61}$$

Since $R(\xi')$ and $R(\eta'_3)$ are Δ-invariants this contradicts the property (56). □

Remark 5.2. In the proof we used heavily the fact that the singularity is strongly nonquadratic which means that it suffices, to consider only 2- and 3-fold Massey products. In this case of higher-order singularities one may need more complicated calculations.

6. SINGULARITIES NEAR THE TRIVIAL REPRESENTATION

In this section we will prove that if $Hom(\Gamma, SO(3))$ has a strongly non-quadratic singularity at the trivial representation $\mathbf{1}$ then $(Hom(\Gamma, \mathbf{G}), \mathbf{1})$ also has a strongly nonquadratic singularity at $\mathbf{1}$ for all semi-simple groups $\mathbf{G}$.

To begin with we may replace $SO(3)$ by $SU(2)$ since $\mathbf{1}$ has the same (local) deformation theory in the two groups (both deformation problems are controlled by $\mathscr{A}^{\cdot}(M) \otimes so_3$ where $\pi_1(M) = \Gamma$).

Let $\mathfrak{g}$ be the Lie algebra of a semi-simple group $\mathbf{G}$. It must contain $so(3)$ and this inclusion induces a monomorphism of the controlling differential graded Lie algebras for the trivial representation $\mathscr{A}^{\cdot}(M) \otimes so(3) \hookrightarrow \mathscr{A}^{\cdot}(M) \otimes \mathfrak{g}$. We will identify $\mathscr{A}^{\cdot}(M) \otimes so(3)$ with the image of this embedding. Since $so(3)$ is semi-simple we may find an $so(3)$-invariant complement $\mathfrak{m}$ to $so(3)$ in $\mathfrak{g}$. We have $[so(3), \mathfrak{m}] \subset \mathfrak{m}$.

Now we can prove

THEOREM 6.1. *Suppose that the* $(Hom(\Gamma, SO(3)), \mathbf{1})$ *has a strongly nonequadratic singularity. Then for any semi-simple Lie group* $\mathbf{G}$ *the germ* $(Hom(\Gamma, \mathbf{G}), \mathbf{1})$ *also has a strongly nonquadratic singularity.*

Proof. According to Theorem 4.6 there exists a class $[\zeta] \in H^1(\Gamma, so(3))$ such that $[\zeta, \zeta] = d\eta_1$ in $H^2(\Gamma, so(3))$ and

$$\langle \zeta | \zeta | \zeta \rangle \neq 0 \text{ in } H^2(\Gamma, so(3))/I_{so(3)}. \tag{62}$$

Here $I_{so(3)}$ denotes the ideal $\{[\zeta, \omega] : \omega \in H^1(\Gamma, so(3))\}$. Assume that $\langle \zeta | \zeta | \zeta \rangle = 0$ in $H^2(\Gamma, \mathfrak{g})/I_{\mathfrak{g}}$. Then there exists a closed form $\beta \in \mathscr{A}^1(M) \otimes \mathfrak{g}$ such that $[\zeta, \eta_1] + [\zeta, \beta]$ is trivial in $H^2(\Gamma, \mathfrak{g})$. We write $\beta = \beta' + \beta''$ with $\beta' \in \mathscr{A}^1(M) \otimes so(3)$ and $\beta'' \in \mathscr{A}^1(M) \otimes \mathfrak{m}$. We obtain

$$[\zeta, \eta_1 + \beta'] + [\zeta, \beta'']$$

is trivial in $H^2(\Gamma, \mathfrak{g})$. However, the first summand belongs to $\mathscr{A}^2(M) \otimes so(3)$ and the second one to $\mathscr{A}^2(M) \otimes \mathfrak{m}$ since $[so(3), \mathfrak{m}] \subset \mathfrak{m}$. We also have the splitting of complexes

$$\mathscr{A}^{\cdot}(M) \otimes \mathfrak{g} = \mathscr{A}^{\cdot}(M) \otimes so(3) \oplus \mathscr{A}^{\cdot}(M) \otimes \mathfrak{m}.$$

Thus the sum of the closed forms $[\zeta, \eta_1 + \beta']$, $[\zeta, \beta'']$ is exact if and only if both forms are exact. This means $[\zeta, \eta_1 + \beta'] = 0$ in $H^2(\Gamma, so(3))$. This contradicts our assumption that $M_3(\zeta) \neq 0$ in $H^2(\Gamma, so(3))$. □

7. CONSTRUCTION OF LATTICES

In this and following two sections we will construct lattices in $SO(3, 1)$ and their representations which give representation varieties with strongly nonquadratic singularities.

Start with a graph Λ in $\mathbb{S}^2$ which is drawn in Fig. 1. We assign numbers $n_j \in \mathbb{Z}$ to edges of Λ as in Fig. 1; we shall omit the number 2 using the standard convention for Dynkin diagrams. If the label $m = 4$ then we denote the labelled graphs by Λ_2, if $m = 7$ then we denote the labelled graph by Λ_1. (Instead of the number 7 here one can choose any prime number $m \geqslant 7$.)

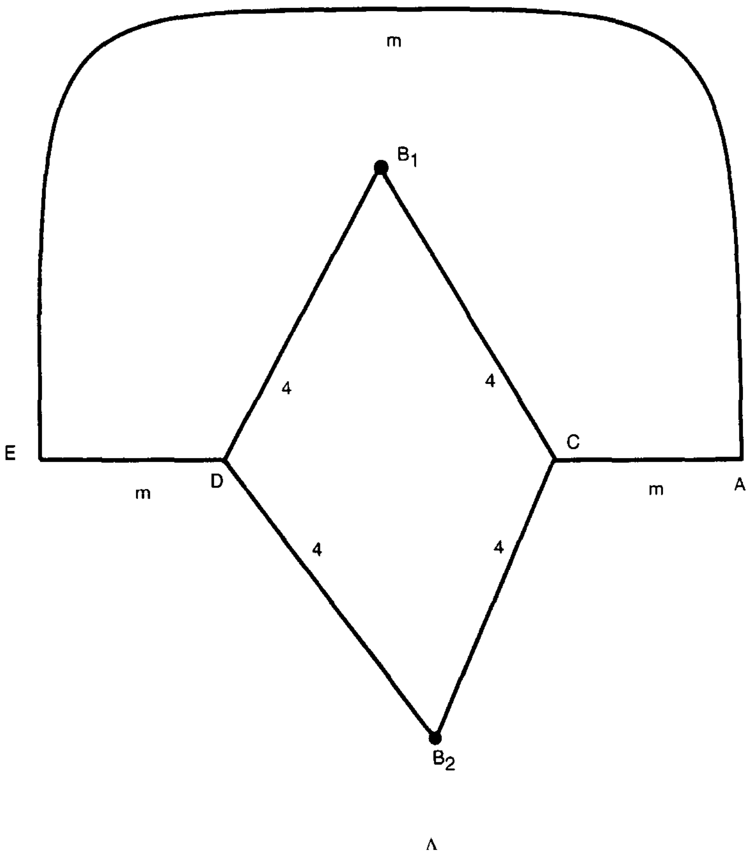

Fig. 1.

Then we add extra edges and vertices Q, F_1, F_2 to Λ to triangulate the complementary regions of $\mathbb{S}^2 - \Lambda$. Denote the result by $\Lambda^{\#}$ (Fig. 2). Finally, we add 14 extra vertices $Z_1, \ldots, Z_{14}$ to the graph $\Lambda^{\#}$ as on Fig. 3 (we omit the labels Z_j). All the edges added to the graph $\Lambda^{\#}$ have the label 2. The result is a labelled planar graph $\Pi = \Pi_j$, $j = 1, 2$.

Consider the graph Π^* dual to Π. We assign integers to the edges of Π^* as follows. If the edge e^* of Π^* intersects an edge e of Π then we assign to e^* the same number which is *assigned to e.*

Lemma 7.1. *There exists a compact finitely-sided convex polyhedron* $\Phi = \Phi_j$ *in* $\mathbb{H}^3$ ($j = 1$, 2) *whose faces correspond to complementary regions of the graph* Π_j^* *and the dihedral angle at each edge e of* Π^* *is equal to* π/n *if the number n is assigned to e.*

Proof. All vertices of Π^* have valency 3 since Π was the 1-skeleton of a triangulation. Then, by examining the graphs Λ, $\Lambda^{\#}$ and Π, we conclude that for each simple closed loop $\ell \subset \Pi$:

(a) either the number of edges in ℓ is 3 and ℓ bounds a triangle in $\mathbb{S}^2 - \Pi$ or the edges of ℓ are labelled as $(4, 4, k)$ with $k = 4, 7$;
(b) or the number of edges in ℓ is 4 and a label on at least one edge of ℓ is > 3;
(c) or the number of edges in ℓ is 4 and one of components of $\mathbb{S}^2 - \ell$ contains exactly one edge;
(d) or the number of edges in ℓ is at least 5.

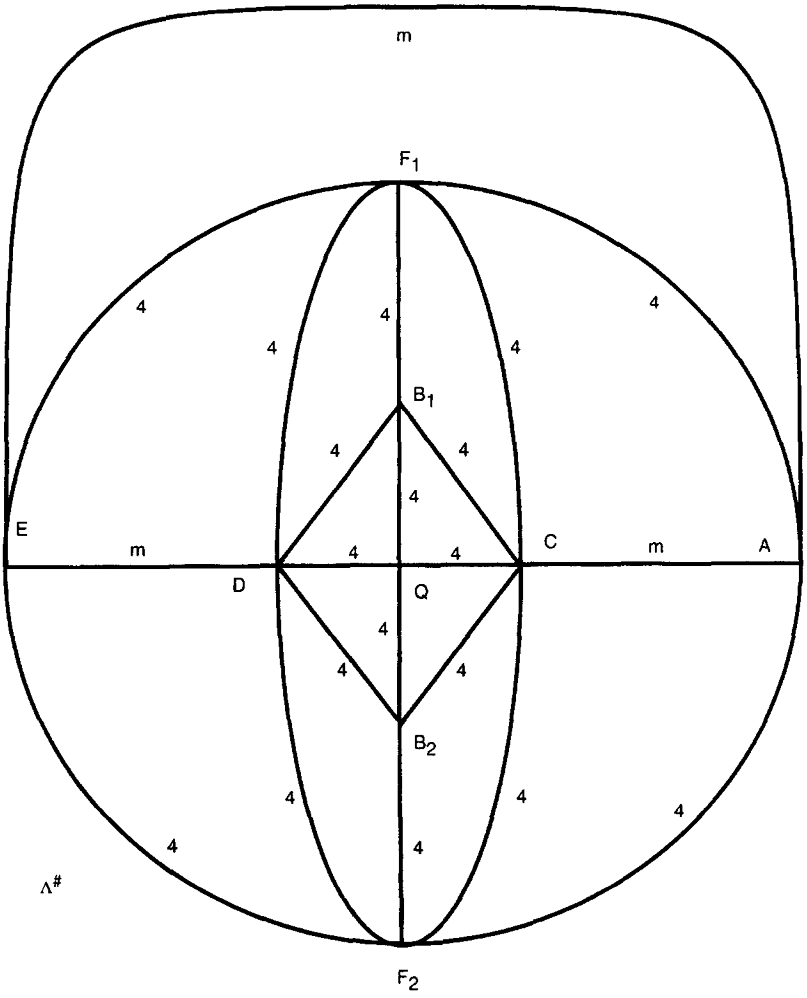

Fig. 2.

Then the existence of Φ follows from Andreev's theorem (see [18, Theorem 13.6.1]). □

We label faces of Φ by the letters $A, B_j, C, \ldots$ which denote corresponding vertices of the dual graph Π. According to Poincarés theorem on fundamental polyhedra [13], the group $\Gamma = \Gamma_j$ generated by reflections τ_S in faces of the polyhedron Φ_j is discrete and Φ_j is the fundamental polyhedron of Γ. Hence Γ is a uniform lattice. The system of relations in Γ can be described as follows. Suppose that S, Q are two faces of $\mathbb{S}^2 - \Pi^*$ which have a common edge e with the label q. Then the product of reflections $\tau_S \cdot \tau_Q$ in the faces S, Q has order q.

8. CONSTRUCTION OF LINKAGES

We construct geodesic maps $\phi_i\colon \Lambda \to \Sigma^2$, $i = 1, 2$, as follows. Consider the unit sphere Σ^2 in $\mathbb{R}^3$ with center at zero. Choose the following points on Σ^2:

$$D = C = (0, 1, 0), \qquad B_j = (0, 1/\sqrt{2}, (-1)^j/\sqrt{2}), \quad j = 1, 2,$$

$$A = (\sin t, \cos t, 0), \qquad E = (\sin 2t, \cos 2t, 0),$$

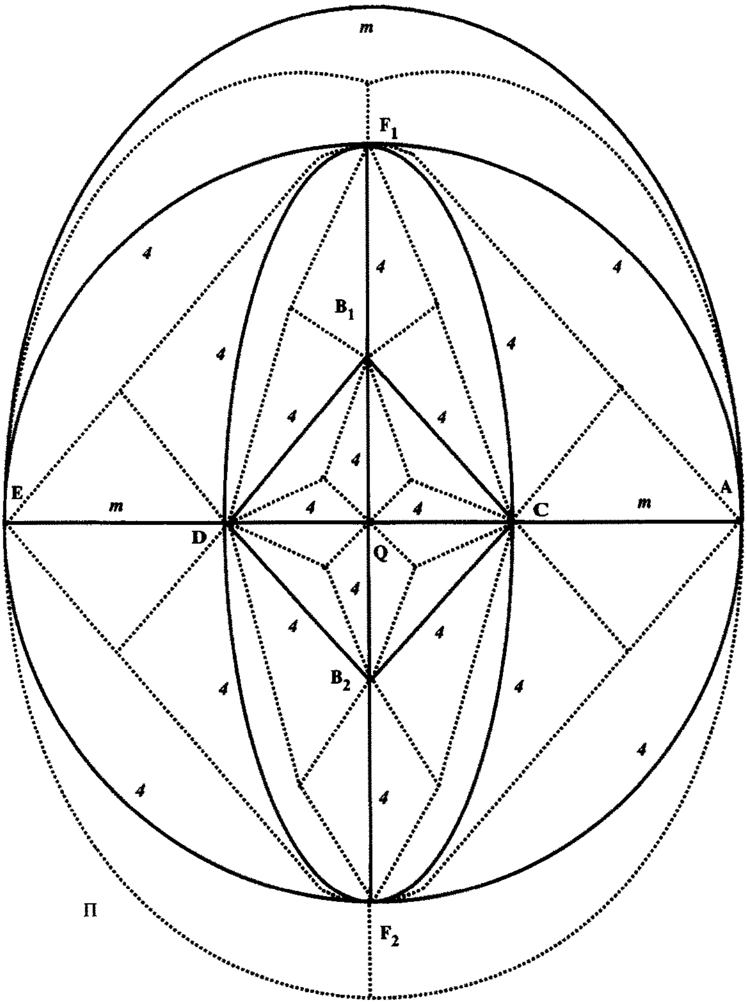

Fig. 3.

where $t = \pi/m$, $t \notin \mathbb{Z}\pi/2$, $m = 4, 7$. Hence the vectors D, E are linearly independent. Note that the number m here is the same as the label of the segments $[A, C]$, $[A, E]$, $[E, D]$ in the graphs Λ_i, $i = 1, 2$. (See Fig. 4.)

If two vertices of Λ are connected by an edge then connect the corresponding points of $\phi_i(\Lambda)$ by the shortest geodesic segment on Σ^2. We introduce a path metric on Λ by pull-back of the spherical metric via ϕ_i. $\Lambda^{(0)}$ shall denote the set of vertices of Λ. The graph Λ is an *abstract mechanical linkage* and $\phi_i(\Lambda) \subset \Sigma^2$ is its realization in 2-sphere.

9. DEFORMATIONS OF MECHANICAL LINKAGES

In this section we drop the index i for the linkage Λ_i and the map ϕ_i since the arguments will be independent of the choice of $m = 4, 7$. Consider the deformation variety $Def(\Lambda)$ of the linkage Λ in the sphere Σ^2 which is the space of all geodesic maps $h : \Lambda \to \Sigma^2$ which are isometries on all edges (we do not divide out by the group $SO(3)$).

The space $Def(\Lambda)$ has a natural structure of an algebraic variety which can be described as follows. We shall regard points of the unit sphere Σ^2 as unit vectors in $\mathbb{R}^3$. Denote by v the number of vertices in Λ and by ε the number of edges.

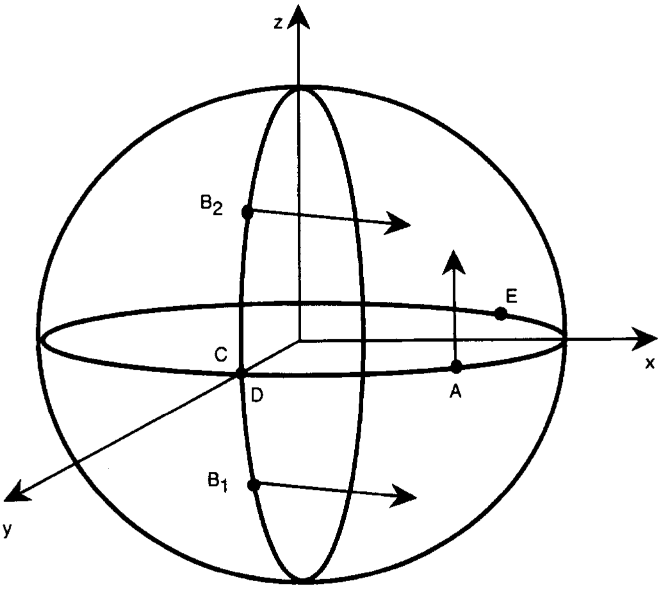

Fig. 4.

Define the polynomials map

$$R: \mathbb{R}^{3v} \times \mathbb{R}^{3v} \to \mathbb{R}^{v+\varepsilon} \tag{63}$$

by the formula

$$R(p, q) = (\dots, p_i \cdot q_j, \dots) \tag{64}$$

where $q = (q_1, \dots, q_v)$, $p = (p_1, \dots, p_v) \in \mathbb{R}^{3v}$ and the dot product appears if either the vertices v_i, v_j of Λ are connected by an edge or $i = j$.

Now let $p = (p_1^0, \dots, p_v^0)$ be the collection of unit vectors corresponding to the configuration $\phi(\Lambda^{(0)})$ and $r^0 = R_{(p,p)} \in R^{\varepsilon+v}$. Then the identification $Def(\Lambda) = R^{-1}(r^0)$ gives the deformation space a structure of the algebraic variety.

Thus the Zariski tangent space $T_p(Def(\Lambda))$ to $Def(\Lambda)$ is given by the kernel of the map $R(p, \bullet): \mathbb{R}^{3v} \to \mathbb{R}^{v+\varepsilon}$. Elements of $T_p(Def(\Lambda))$ are called *infinitesimal deformations of* $\phi(\Lambda)$. An infinitesimal deformation q' is called trivial if it belongs to the kernel of the projection $T(Def(\Lambda)) \to T(Def(\Lambda)/SO(3))$. This means that there exists an element of the Lie algebra $\zeta \in so(3) = \mathbb{R}^3$ such that $q'_j = \zeta \times p_j$, where $\bullet \times \bullet$ is the vector product in $\mathbb{R}^3$.

The second order jet space $J_p^2(Def(\Lambda))$ of the variety $Def(\Lambda)$ is described as follows. Let $q' \in T_p(Def(\Lambda))$, $q'' \in \mathbb{R}^v$. Then $(q', q'') \in J_p^2(Def(\Lambda))$ iff

$$R(q', q') + R(p, q'') = 0. \tag{65}$$

The elements (q', q'') are called *second order deformations* of the configuration p; in such case q'' is called the *acceleration* of the deformation (q', q''). An element $q' \in T_p(Def(\Lambda))$ is called *second order integrable* if there exists q'' such that $(q', q'') \in J_p^2(Def(\Lambda))$. Similarly we can define higher order deformations. Suppose that $p: [0, 1] \to \Sigma^2$ is a smooth curve such that $p(0) = p$ so that

$$\frac{d^m}{dt^m} p(t)|_{t=0} = q^{(m)}$$

and the identities $p_j(t) \cdot p_i(t) = p_j(0) \cdot p_i(0)$ are satisfied up to the order m of $t \to 0$ for each v_i, $v_j \in \Lambda$ connected by an edge. Then the vector $\mathbf{q} = (q' = q^{(1)}, q'' = q^{(2)}, q''' = q^{(3)}, \dots, q^{(m)})$ is an infinitesimal deformation of order m. These deformations belong to the mth order jet space $J_p^m(Def(\Lambda))$ of the variety $Def(\Lambda)$ at p. An infinitesimal deformation $q' \in T_p(Def(\Lambda))$ is called mth order integrable if there exists $\mathbf{q} \in J_p^m(Def(\Lambda))$ such that $q^{(1)} = q'$.

THEOREM 9.1. *There exists a nontrivial infinitesimal deformation*

$$q' \in T_\phi(Def(\Lambda))$$

which is 2nd order integrable but is not 3rd order integrable.

Proof. The infinitesimal deformation q' is given by the following velocities:

$$D' = C' = E' = 0, \qquad B'_j = (1. 0, 0) \quad (j = 1, 2), \qquad A' = (0, 0, 1).$$

We choose the acceleration vectors in p'' as follows:

$$D'' = E'' = 0, \qquad B''_j = (0, 0, (-1)^{j+1}/\sqrt{2}), \quad (j = 1, 2)$$

$$C'' = (1 - 1/\sin^2 t, 0, 0), \qquad A'' = (1/\sin t - 2 \sin t, \ -2\cos t, 0).$$

Then direct calculations show that:

$$B_j \cdot B''_j = -1, \qquad A \cdot A'' = -1, \qquad A'' \cdot E = 0, \qquad A'' \cdot C + C'' \cdot A = 0$$

$$B''_j \cdot C = B''_j \cdot D = B_j \cdot C'' = 0$$

Thus $(q', p'') \in J_\phi^2(Def(\Lambda))$.

Now we will prove that there is no 3-jet (q', q'', q''') in $J_\phi^3(Def(\Lambda))$. Suppose that such a jet exists. We will retain notations A'', B'', etc. for its components..

PROPOSITION 9.2. *The deformation (q'', q''') can be chosen so that $E'' = 0$.*

Proof. Recall that $E \cdot E = 1$, $E \cdot E'' = 0$ since $(q', q'') \in J_\phi^2(Def(\Lambda))$. Hence a direct calculation shows that there exists a skew-symmetric matrix S with the property: $E'' = -SE$. We define a 1-parameter family of orthogonal transformation by

$$Q_t = \exp(t^2 S).$$

The curve $p(t) = p(0) + q't + q''t^2/2 + q'''t^3/6$ is order 3 tangent to the variety $Def(\Lambda)$ at the point $p(0) = \phi$. The same is true for the curve $q(t) = Q_t(p(t))$. The curve $q(t)$ has the same first derivative as $p(t)$ but the restriction of the deformation $q(t)$ to the vertex E has zero second derivative $SE + E''$. Thus instead of (p, q'', q''') we can take the 3-jet (p', p'', p''') of the curve $p(t)$. ☐

In what follows we shall assume that $E'' = 0$ which will simplify our calculations. Let $\Delta'' = C'' - D''$, $\Delta''' = C''' - D'''$. The scalar products $B_j \cdot C$ and $B_j \cdot D$ must be preserved up to the third order, thus

$$C''' \cdot B_j + 3C''' \cdot B'_j + C \cdot B'''_j = 0 \tag{66}$$

$$D''' \cdot B_j + 3D'' \cdot B'_j + D \cdot B'''_j = 0 \tag{67}$$

This implies that $\Delta''' \cdot B_j = \Delta'' \cdot B'_j$. The vectors B_j, $j = 1, 2$ are linearly independent, $B'_1 = B'_2$ and $\Delta''' \cdot C = 0$, thus we conclude $\Delta''' \cdot B_j = 0$, $j = 1, 2$. It follows that $\Delta'' \cdot B'_j = 0$. However, $\Delta'' \cdot C = 0$ and the vectors B'_j, C are linearly independent. Hence $\Delta'' = (0, 0, \lambda)$. The scalar product $D \cdot E$ must be preserved up to the second order, therefore $D'' \cdot E = 0$, $E \cdot (0, 0, \lambda) = 0$ which implies $C'' \cdot E = 0$. The vectors C, E are linearly independent, thus the equality $C'' \cdot C = 0$ implies that $C'' = (0, 0, \gamma)$. We have $0 = A \cdot C'' + A'' \cdot C$, thus $A'' \cdot C = 0$ and on the other hand $A'' \cdot E = 0$. It follows that $A'' \cdot A = 0$ since the vector A is a linear combination of C, E. Recall, however, that the scalar product $A \cdot A$ must be preserved up to the second order, hence $A \cdot A'' + A' \cdot A' = 0$. We conclude that $A' \cdot A' = 0$ which contradicts the assumption that A' is the unit vector $(0, 0, 1)$. □

Note that the infinitesimal deformation q' is nontrivial since it is not extendable to a third order deformation of the linkage.

10. REPRESENTATION VARIETIES WITH NONQUADRATIC SINGULARITIES

Let $\Gamma = \Gamma_i$ be one of two reflection groups constructed in Section 7. We define a representation $\rho: \Gamma \to SO(3)$ as follows. Suppose that a face of the fundamental polyhedron Φ is labelled by a letter S which is the label of a vertex S in the graph Λ. Then we let $\rho(\tau_S)$ be the rotation of order two around the vector $\phi(S) \subset \Sigma^2$. Otherwise (if $S \in \Pi$ is not a vertex of Λ), we let $\phi(\tau_S) = 1$. It follows from the list of relations of the group Γ and the geometry of Λ that ρ is a homomorphism (see the proof of Lemma 10.4). (For instance, the rotations $\rho(\tau_A)\rho(\tau_E)$, $\rho(\tau_D)\rho(\tau_E)$, $\rho(\tau_A)\rho(\tau_D)$ have order m.)

LEMMA 10.1. *If $t = \pi/m = \pi/4$ then the group $\rho_i(\Gamma_i)$ is finite. If $m = 7$ then the group $\rho_i(\Gamma_i)$ is infinite. Moreover, in the latter case, the group $\rho_i(\Gamma_i)$ is Zariski dense in $SO(3)$,*

Proof. We first assume that $t = \pi/m = \pi/4$. It is easy to see that the finite collection of vectors $\{(\varepsilon_1, \varepsilon_2, \varepsilon_3), \varepsilon_j \in \{0, 1, -1\}\}$ is invariant under the generators of the group $\rho_2(\Gamma_2)$. These vectors span $\mathbb{R}^3$, thus the group $\rho_2(\Gamma_2)$ is finite. Suppose now that $m \geqslant 7$ is a prime number. Then the group $\rho_1(\Gamma_1)$ contains the rotation $\rho_1(Q) = \rho_1(\tau_C \circ \tau_A)$ of order m around the axis z. The rotation $\rho_1(\tau_B Q \tau_B)$ has axis different from z and the same order m. On the other hand, if $K \subset SO(3)$ is a finite subgroup which contains an element of prime order $m \geqslant 7$ then this is a dihedral group and axes of all such elements in K must coincide. We conclude that $\rho_1(\Gamma_1)$ is infinite. The representation ρ is irreducible, thus $\rho_1(\Gamma_1)$ contains two elements of infinite order with different axes, thus $\rho_1(\Gamma_1)$ is Zariski dense in $SO(3)$. □

REMARK 10.2. It follows that for $m \geqslant 7$ the group $\rho_1(\Gamma_1)$ is dense in $SO(3)$ in the classical topology.

THEOREM 10.3. *For each $i = 1, 2$ the representation variety $Hom(\Gamma_i, SO(3))$ has a strongly nonquadratic singularity at the point ρ_i and the quotient variety $V(\Gamma_i, SO(3))$ has a strongly nonquadratic singularity at the point $[\rho_i]$.*

Proof. We again drop the index i for the groups Γ_i and representations ρ_i. Denote by $\Delta \subset \Gamma$ the reflection group generated by τ_S, $S \in \Lambda$. Theorem 3.2 of [9] implies that there exists an isomorphism Ψ between the germ of the variety $Hom(\Delta, SO(3))$ near ρ and the germ of the variety $Def(\Lambda)$ near $\phi(\Lambda)$. The last variety has nontrivial elements of $T_{\phi}(Def(\Lambda))$ which can be extended to second order jets, but are not extendable to third order jets. Hence, the same holds for the variety $Hom(\Delta, SO(3))$ at ρ. Since Ψ is $SO(3)$-invariant it induces an isomorphism of quotient germs. Consequently, the germ of $V(\Delta, SO(3))$ at $[\rho]$ also has an infinitesimal deformation with the above properties.

LEMMA 10.4. *The restriction map*

$$Res: Hom(\Gamma, SO(3)) \to Hom(\Delta, SO(3)) \tag{68}$$

is an isomorphism of germs of these varieties near the representation ρ.

Proof. Let N be the normal subgroup of Γ generated by the set Ω of reflections in the faces of the polyhedron Φ corresponding to the vertices Q, F_i, Z_j that were erased in passing from Π to Λ. The composition $\varphi: \Delta \to \Gamma \to \Gamma/N$ is clearly a surjection. We claim that it is also an injection. Let Ξ be the set of reflections in the faces of Φ not included in the set Ω above (the "rest of generators of Γ"). Then Γ has a presentation of the form:

$$\Gamma = \langle \Xi, \Omega : \xi^2, \omega^2, (\xi_i \xi_j)^{2m_{ij}}, (\omega_k \omega_l)^{2n_{kl}}, (\xi_s \omega_r)^{2p_{sr}} \rangle. \tag{69}$$

Here ξ runs through Ξ, ω through Ω and the numbers n_{kl}, m_{ij}, p_{sr} are determined by the labels of edges of the graph Γ.

The above presentation for Γ induces a presentation for Γ/N by adding the extra relation $\omega = 1$ for all $\omega \in \Omega$. We can then eliminate the relations $(\omega_k \omega_l)^{2n_{kl}}$, $(\xi_s \omega_r)^{2p_{sr}}$ since generators ξ have order 2. We obtain the following presentation for Γ/N:

$$\Gamma/N = \langle \Xi : \xi^2, (\xi_i \xi_j)^{2m_{ij}} \rangle. \tag{70}$$

Now it is clear that φ is an isomorphism since Δ has the same representation $\varphi(\xi_j) = \xi_j$. The isomorphism $\varphi: \Delta \to \Gamma/N$ induces an isomorphism of varieties

$$Hom(\Gamma/N, SO(3)) \to Hom(\Delta, SO(3))$$

(see Remark below).

We now prove that the quotient map $\Gamma \to \Gamma/N$ induces an isomorphism of germs $(Hom(\Gamma/N, SO(3)), \rho) \to Hom(\Gamma, SO(3)), \rho)$. Indeed, $(Hom(\Gamma/N, SO(3))$ is the inverse image of the trivial representation under the restriction map $Hom(\Gamma, SO(3)) \to Hom(\Omega, SO(3))$, where $\langle \Omega \rangle$ is the subgroup generated by elements in Ω. Since $\rho|_{\langle \Omega \rangle}$ is the trivial representation 1, we obtain an induced fiber square of germs

$$\begin{tikzcd}
(Hom(\Gamma/N, SO(3)), \rho) \arrow[r] \arrow[d] & (Hom(\Gamma, SO(3)), \rho) \arrow[d] \\
\{1\} \arrow[r] & (Hom(\langle \Omega \rangle, SO(3)), 1)
\end{tikzcd}$$

We claim that the trivial representation is an isolated point of $Hom(\langle \Omega \rangle, SO(3))$. Indeed, $T_1(Hom(\langle \Omega \rangle, SO(3)))$ is the space of 1-cocycles $Z^1(\langle \Omega \rangle, so(3))$. But since $\langle \Omega \rangle$ is generated by elements of order 2 and acts trivially on $so(3)$ we have $Z^1(\langle \Omega \rangle, so(3)) = 0$. Hence $T_1(Hom(\langle \Omega \rangle, SO(3))) = \{0\}$ and the claim follows. Hence the bottom arrow of the above square is an isomorphism and consequently the top one is also. ☐

COROLLARY 10.5. *The map Res induces a map*

$$\overline{Res}: V(\Gamma, SO(3)) \to V(\Delta, SO(3)) \tag{71}$$

which is an isomorphism of germs near $[\rho]$.

Proof. Follows from $SO(3)$-invariance of the map *Res*. ☐

Remark 10.6. In the above proof we have used the fact that the isomorphism of groups $\varphi: \Delta \to \Gamma/N$ induces an isomorphism of representation varieties. Since the description of a representation variety $\mathrm{Hom}(\Gamma, H)$ depends on a presentation of the abstract group Γ this is not obvious. We prove that now. The coordinate ring R of a representation variety represents the functor of points $A \to Hom(\Gamma, H)(A)$ where A is an affine k-algebra. But since $Hom(\Gamma, H)(A) = Hom(\Gamma, H(A))$, a homomorphism of abstract groups induces a natural transformation of the above functors. Hence an isomorphism of abstract groups induces a natural isomorphism of functors and so the representing objects (the two coordinate rings) are isomorphic.

This discussion concludes the proof of Theorem 10.3. ☐

Now we can prove the two main theorems of the paper.

THEOREM 10.7. *There exists a co-compact torsion-free lattice* Γ'_1 *in* $SO(3, 1)$ *and an irreducible representation* $\rho_1 : \Gamma'_1 \to SO(3)$ *such that the varieties* $Hom(\Gamma'_1, SO(3))$ *and* $V(\Gamma'_1, SO(3))$ *have nonquadratic singularities at* ρ_1 *and* $[\rho_1]$, *respectively.*

Proof. Take any torsion-free normal subgroup of finite index $\Gamma'_1 \subset \Gamma_1$ where Γ_1 is as in Theorem 10.3. Then the assertion follows from Theorems 10.3, 4.6, 5.1. ☐

THEOREM 10.8. *There exists a co-compact torsion-free lattice* Γ'_2 *in* $SO(3, 1)$ *such that for any semi-simple Lie group* **G** *the varieties* $Hom(\Gamma'_2, \mathbf{G})$ *and* $V(\Gamma'_2, \mathbf{G})$ *have nonquadratic singularities at the trivial representation* **1** *and its conjugacy class* **[1]**, *respectively.*

Proof. We have constructed a lattice $\Gamma_2 \subset SO(3, 1)$ and a *finite* representation $\rho_2: \Gamma_2 \to SO(3)$ with a strongly nonquadratic singularity of the germ $(Hom(\Gamma_2, SO(3)), \rho_2)$. Take any torsion-free normal subgroup of finite index $\Gamma'_2 \subset \Gamma_2$ such that $\rho_2(\Gamma'_2) = 1$. Then the assertion follows from Theorems 4.6, 5.1, 6.1. ☐

11. DEFORMATION THEORY NEAR THE IDENTITY REPRESENTATION

Suppose that $\Gamma \subset SO(3, 1)$ is a co-compact lattice, ρ is the identity representation $\Gamma \hookrightarrow SO(3, 1)$. We are interested in the germ $(V(\Gamma, SO(4, 1)), [\rho])$. Recall that the embedding $SO(3, 1) \hookrightarrow SO(4, 1)$ corresponds to the totally geodesic embedding $\mathbb{H}^3 \hookrightarrow \mathbb{H}^4$. Denote by τ the reflection in $\mathbb{H}^4$ which fixes $\mathbb{H}^3$ pointwise. Then the Lie algebra $so(4, 1)$ splits as $so(3, 1) \oplus \mathfrak{m}$ so that τ acts as 1 on $so(3, 1)$ and -1 on $\mathfrak{m}$. The splitting is orthogonal with respect to the Killing form on $so(4, 1)$, thus it is invariant under the adjoint action of $so(3, 1)$. It follows that for any $\xi, \eta \in \mathfrak{m}$,

$$[\xi, \eta] \in so(3, 1). \tag{72}$$

We recall that the first obstruction to the integrability of infinitesimal deformations $\zeta \in H^1(\Gamma, so(4, 1))$ is the cup-product $[\zeta, \zeta]$. The first cohomology group $H^1(\Gamma, so(4, 1))$ splits as

$$H^1(\Gamma, so(3, 1)) \oplus H^1(\Gamma, \mathfrak{m})$$

and the first summand is equal to zero according to Calabi–Weil rigidity theorem. Thus for any class $\zeta \in H^1(\Gamma, so(4,1))$ we can choose a representative $\tilde{\zeta} \in Z^1(\Gamma, \mathfrak{m})$.

We owe the following argument to Gregg Zuckerman.

PROPOSITION 11.1. *The cup-product*

$$[\ ,\ ]: H^1(\Gamma, so(4,1)) \otimes H^1(\Gamma, so(4,1)) \to H^2(\Gamma, so(4,1)) \tag{73}$$

is identically zero.

Proof. For classes $\xi_1, \xi_2 \in H^1(\Gamma, so(4,1))$ we choose representatives $\tilde{\xi}_1, \tilde{\xi}_2 \in Z^1(\Gamma, \mathfrak{m})$. The cup-product $[\xi_1, \xi_2]$ is represented by the 2-cocycle on Γ:

$$o(x, y) = [\tilde{\xi}_1(x), ad_x \tilde{\xi}_2(y)] \tag{74}$$

where $[\ ,\ ]$ is the Lie bracket on $so(4,1)$. Then eq. (72) implies that $o(x, y) \in Z^2(\Gamma, s_0(3,1))$. However, according to the Calabi–Weil rigidity and Poincaré duality we get $H^2(\Gamma, so(3,1)) = 0$. □

Theorems 10.7 and 10.8 imply that vanishing of the cup-product alone is not *a priori* enough to guarantee smoothness of the variety $V(\Gamma, SO(4,1))$ near $[\rho]$. However, we do not know any example when the identity representation $[\rho]$ actually is a singular point. Results of [8] imply that such pathological examples do not exist in the class of reflection groups.

12. REMARKS ON MECHANICAL LINKAGES

Our examples of mechanical linkages were motivated by a construction due to Connelly [4] of a rigid mechanical linkage in $\mathbb{R}^2$, which is not rigid at first and second order. Unfortunately, the infinitesimal deformation of second order constructed by Connelly can be extended to a deformation of third order and we cannot use his construction to prove Theorem 10.7. More generally, for each positive integer n Connelly constructs a locally rigid mechanical linkage in $\mathbb{R}^2$ which admits a nontrivial infinitesimal deformation of order n. This construction works for $\mathbb{S}^2$ as well but to construct a representation of a Coxeter group one needs rationality conditions for lengths of edges which are difficult to arrange. Note that the books on mechanical engineering [1, 17] contain lots of examples of mechanical linkages which can draw quite complicated algebraic curves.

We recall the classical results of Kempe [11] that for any planar compact real algebraic curve C there exists a finite collection of mechanical linkages in $\mathbb{R}^2$ which can draw C "piece-by-piece". To apply this theorem to the construction of Coxeter groups with arbitrarily complicated singularities of representation varieties one has to solve the same rationality problem.

Question 12.1. Suppose that V is an affine variety in $\mathbb{R}^n$. Is it true that there exists a co-compact lattice $\Gamma \subset SO(3,1)$, compact Lie group $\mathbf{G}$ and a representation $\rho: \Gamma \to \mathbf{G}$ such that the germ $(Hom(\Gamma, \mathbf{G}), \rho)$ is analytically isomorphic to the germ $(V \times \mathbb{R}^m, 0)$ for some m?

We will address this problem in another paper [10].

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