GEOMETRIC FINITENESS IN NEGATIVELY PINCHED HADAMARD MANIFOLDS

MICHAEL KAPOVICH AND BEIBEI LIU

Abstract. In this paper, we generalize Bonahon's characterization of geometrically infinite torsion-free discrete subgroups of $\mathrm{PSL}(2,\mathbb{C})$ to geometrically infinite discrete torsion-free subgroups Γ of isometries of negatively pinched Hadamard manifolds X. We then generalize a theorem of Bishop to prove that every such geometrically infinite isometry subgroup Γ has a set of nonconical limit points with cardinality of continuum.

1. Introduction

The notion of geometrically finite discrete groups was originally introduced by Ahlfors in [1], for subgroups of isometries of the 3-dimensional hyperbolic space $\mathbb{H}^3$ as the finiteness condition for the number of faces of a convex fundamental polyhedron. In the same paper, Ahlfors proves that the limit set of a geometrically finite subgroup of isometries of $\mathbb{H}^3$ has either zero or full Lebesgue measure in S^2. The notion of geometric finiteness turned out to be quite fruitful in the study of Kleinian groups. Alternative definitions of geometric finiteness were later given by Marden [15], Beardon and Maskit [5], and Thurston [19]. These definitions were further extended by Bowditch [8] and Ratcliffe [18] for isometry subgroups of higher dimensional hyperbolic spaces and, a bit later, by Bowditch [9] to negatively pinched Hadamard manifolds. While the original Ahlfors' definition turned out to be too limited (when used beyond the hyperbolic 3-space), other definitions of geometric finiteness were proven to be equivalent by Bowditch in [9].

Our work is motivated by the definition of geometric finiteness due to Beadon and Maskit [5] who proved

Theorem 1.1. *A discrete isometry subgroup Γ of $\mathbb{H}^3$ is geometrically finite if and only if every limit point of Γ is either a* conical limit point *or is a* bounded parabolic fixed point*.*

This theorem was improved by Bishop in [6]:

Theorem 1.2. *A Kleinian group $\Gamma < \mathrm{Isom}(\mathbb{H}^3)$ is geometrically finite if and only if every point of $\Lambda(\Gamma)$ is either a conical limit point or a parabolic fixed point. Furthermore, if $\Gamma < \mathrm{Isom}(\mathbb{H}^3)$ is geometrically infinite, $\Lambda(\Gamma)$ contains a set of nonconical limit points with cardinality of continuum.*

The key ingredient in Bishop's proof of Theorem 1.2 is Bonahon's theorem[1] [7]:

Theorem 1.3. *A discrete torsion-free subgroup $\Gamma < \mathrm{Isom}(\mathbb{H}^3)$ is geometrically infinite if and only if there exists a sequence of closed geodesics λ_i in the manifold $M = \mathbb{H}^3/\Gamma$ which "escapes every compact subset of M," i.e., for every compact subset $K \subset M$,*

$$\mathrm{card}\left(\{i : \lambda_i \cap K \neq \emptyset\}\right) < \infty.$$

According to Bishop, Bonahon's theorem also holds for groups with torsion, but it is unclear to us if Bonahon's proof extends to cover this case, as some of Bonahon's arguments

Date: January 24, 2018.

[1]Bonahon uses this result to prove his famous theorem about tameness of hyperbolic 3-manifolds.

require one to know that every nontrivial element of Γ is either loxodromic or parabolic. However, for higher dimensional hyperbolic spaces $\mathbb{H}^n$, we extend Bonahon's proof and prove that Bonahon's theorem holds for discrete isometry subgroups with torsion, see [14].

Bowditch generalized the notion of geometric finiteness to discrete subgroups of isometries of negatively pinched Hadamard manifolds [9]. A negatively pinched Hadamard manifold is a complete, simply connected Riemannian manifold such that all sectional curvatures lie between two negative constants. From now on, we use X to denote a negatively pinched Hadamard manifold, $\partial_\infty X$ its visual (ideal) boundary, $\bar{X}$ the visual compactification $X \cup \partial_\infty X$, Γ a discrete subgroup of isometries of X, $\Lambda = \Lambda(\Gamma)$ the limit set of Γ. The convex core $Core(M)$ of $M = X/\Gamma$ is defined as the Γ-quotient of the closed convex hull of $\Lambda(\Gamma)$ in X. Recall also that a point $\xi \in \partial_\infty X$ is a *conical limit point*[2] of Γ if for every $x \in X$ and every geodesic ray l in X asymptotic to ξ, there exists a positive constant A such that the set $\Gamma(x) \cap N_A(l)$ accumulates to ξ, where $N_A(l)$ denotes the A-neighborhood of l in X. A parabolic fixed point $\xi \in \partial_\infty X$ (i.e. a fixed point of a parabolic element of Γ) is called *bounded* if

$$(\Lambda(\Gamma) - \{\xi\})/\Gamma_\xi$$

is compact. Here Γ_ξ is the stabilizer of ξ in Γ.

Bowditch [9], gave four equivalent definitions of geometric finiteness for Γ:

Theorem 1.4. *The followings are equivalent for discrete subgroups $\Gamma < \mathrm{Isom}(X)$:*

1. *The quotient space $\bar{M}(\Gamma) = (\bar{X} - \Lambda)/\Gamma$ has finitely many topological ends each of which is a "cusp".*
2. *The limit set $\Lambda(\Gamma)$ of Γ consists entirely of conical limit points and bounded parabolic fixed points.*
3. *The noncuspidal part of the convex core $Core(M)$ of $M = X/\Gamma$ is compact.*
4. *For some $\delta > 0$, the uniform δ-neighbourhood of the convex core, $N_\delta(Core(M))$, has finite volume and there is a bound on the orders of finite subgroups of Γ.*

If one of these equivalent conditions holds, the subgroup $\Gamma < \mathrm{Isom}(X)$ is said to be geometrically finite; otherwise, Γ is said to be geometrically infinite.

The main results of our paper are:

Theorem 1.5. *Suppose that $\Gamma < \mathrm{Isom}(X)$ is a torsion-free discrete subgroup. Then the followings are equivalent:*

1. *Γ is geometrically infinite.*
2. *There exists a sequence of closed geodesics $\lambda_i \subset M = X/\Gamma$ which escapes every compact subset of M.*
3. *The set of nonconical limit points of Γ has cardinality of continuum.*

Corollary 1.6. *If $\Gamma < \mathrm{Isom}(X)$ is a torsion-free discrete subgroup then Γ is geometrically finite if and only if every limit point of Γ is either a conical limit point or a parabolic fixed point.*

We have the following conjecture regarding the Hausdorff dimension of the set of nonconical limit points of any geometrically infinite torsion-free discrete subgroup $\Gamma < \mathrm{Isom}(X)$.

Conjecture 1.7. *Suppose that $\Gamma < \mathrm{Isom}(X)$ is a geometrically infinite torsion-free discrete subgroup. Then the Hausdorff dimension of the set of nonconical limit points of Γ equals the Hausdorff dimension of the limit set itself. Here, the Hausdorff dimension is defined with respect to any of the* visual metrics *on $\partial_\infty X$, see* [17].

[2]Another way is to describe conical limit points of Γ as points $\xi \in \partial_\infty X$ such that one, equivalently, every, geodesic ray $\mathbb{R}_+ \to X$ asymptotic to ξ projects to a non-proper map $\mathbb{R}_+ \to M$.

This conjecture is a theorem by Fernández and Melián [12] in the case of Fuchsian subgroups of the 1st kind, $\Gamma < \mathrm{Isom}(\mathbb{H}^2)$.

Below is an outline of the proof of Theorem 1.5. Our proof of the implication (1)⇒(2) mostly follows Bonahon's argument with the following exception: At some point of the proof Bonahon has to show that certain elements of Γ are loxodromic. For this he uses a calculation with 2×2 parabolic matrices: If g, h are parabolic elements of $\mathrm{Isom}(\mathbb{H}^3)$ generating a nonelementary subgroup then either gh or hg is non-parabolic. This argument is no longer valid for isometries of higher dimensional hyperbolic spaces, let alone Hadamard manifolds. We replace this computation with a more difficult argument showing that there exists a number $\ell = \ell(n, \kappa)$ such that for every n-dimensional Hadamard manifold X with sectional curvatures pinched between $-\kappa^2$ and -1 and for any pair of parabolic isometries $g, h \in \mathrm{Isom}(X)$ generating a nonelementary discrete subgroup, a certain word $w = w(g, h)$ of length $\le \ell$ is loxodromic (Theorem 8.5). Our proof of the implication (2)⇒(3) is similar to Bishop's but more coarse-geometric in nature. Given a sequence of closed geodesics λ_i in M escaping compact subsets, we define a family of proper piecewise geodesic paths γ_τ in M consisting of alternating geodesic arcs μ_i, ν_i, such that μ_i connects λ_i to λ_{i+1} and is orthogonal to both, while the image of ν_i is contained in the loop λ_i. If the lengths of ν_i are sufficiently long, then the path γ_τ lifts to a uniform quasigeodesic $\tilde{\gamma}_\tau$ in X, which, therefore, is uniformly close to a geodesic $\tilde{\gamma}^*_\tau$. Projecting the latter to M, we obtain a geodesic γ^*_τ uniformly close to γ_τ, which implies that the ideal point $\tilde{\gamma}^*_\tau(\infty) \in \partial_\infty X$ is a nonconical limit point of Γ. Different choices of the arcs ν_i yield distinct limit points, which, in turn implies that $\Lambda(\Gamma)$ contains a set of nonconical limit points with cardinality of continuum. The direction (3)⇒(1) is a direct corollary of Theorem 1.4.

Organization of the paper. In Section 3, we review the angle comparison theorem [9, Proposition 1.1.2] for negatively pinched Hadamard manifolds and derive some useful geometric inequalities. In Section 5, we review the notions of elementary parabolic subgroups and elementary loxodromic subgroups of isometries of negatively pinched Hadamard manifolds, [9]. In Section 6, we review the thick-thin decomposition in negatively pinched Hadamard manifolds and some properties of parabolic subgroups, [9]. In Section 7, we use the results in Section 3 to prove that certain piecewise geodesic paths in Hadamard manifolds with sectional curvatures ≤ -1 are uniform quasigeodesics. In Section 8, we explain how to produce loxodromic isometries as words $w(g, h)$ of uniformly bounded length, where g, h are parabolic isometries of X with distinct fixed points. In Section 9, we generalize Bonahon's theorem, the implication (1)⇒(2) in Theorem 1.5. In Section 10, we construct the set of nonconical limit points with cardinality of continuum and complete the proof of Theorem 1.5.

Acknowledgements. The first author was partly supported by the NSF grant DMS-16-04241 as well as by KIAS (the Korea Institute for Advanced Study) through the KIAS scholar program. Some of this work was done during his stay at KIAS and he is thankful to KIAS for its hospitality.

2. Notation

In a metric space (Y, d), we will use the notation $B(a, r)$ to denote the *open r-ball* centered at a in Y. For a subset $A \subset Y$ and a point $y \in Y$, we will denote by $d(y, A)$ the *minimal distance* from y to A, i.e.

$$d(y, A) := \inf\{d(y, a) \mid a \in A\}.$$

We use the notation $N_r(A)$ for the *closed* r-neighborhood of A in Y:

$$N_r(A) = \{y \in Y : d(y, A) \le r\}.$$

The *Hausdorff distance* $\mathrm{hd}(Q_1, Q_2)$ between two closed subsets Q_1 and Q_2 of (Y, d) is the infimum of $r \in [0, \infty)$ such that $Q_1 \subseteq N_r(Q_2)$ and $Q_2 \subseteq N_r(Q_1)$.

Throughout the paper, X will denote a negatively pinched Hadamard manifold, unless otherwise stated; we assume that all sectional curvatures of X lie between $-\kappa^2$ and -1. We let d denote the Riemannian distance function on X. We let $\mathrm{Isom}(X)$ denote the isometry group of X.

For a Hadamard manifold X, the exponential map is a diffeomorphism, in particular, X is diffeomorphic to $\mathbb{R}^n$, where n is the dimension of X. Then X can be compactified by adjoining the ideal boundary sphere $\partial_\infty X$, and we will use the notation $\bar{X} = X \cup \partial_\infty X$ for this compactification. The space $\bar{X}$ is homeomorphic to the closed n-dimensional ball.

In this paper, geodesics will be always parameterized by their arc-length; we will conflate geodesics in X with their images.

Given a closed subset $A \subseteq X$ and $x \in X$, we write

$$\mathrm{Proj}_A(x) = \{y \in A \mid d(x, y) = d(x, A)\}$$

as the projection of x to A. It consists of all points in A which are closest to x. If A is convex, then $\mathrm{Proj}_A(x)$ is a singleton.

Hadamard spaces are uniquely geodesic and we will let $xy \subset X$ denote the geodesic segment connecting $x \in X$ to $y \in X$. Similarly, given $x \in X$ and $\xi \in \partial_\infty X$ we will use the notation $x\xi$ for the unique geodesic ray emanating from x and asymptotic to ξ; for two distinct points $\xi, \eta \in \partial_\infty X$, we use the notation $\xi\eta$ to denote the unique (up to reparameterization) geodesic asymptotic to ξ and η.

Given $\xi \in \partial_\infty X$, horospheres about ξ are level sets of a Busemann function h about ξ. For details of Busemann functions, see [2, 9] (notice that Bowditch uses a nonstandard notation for Busemann functions, which are negatives of the standard Busemann functions). A set of the form $h^{-1}((-\infty, r])$ for $r \in \mathbb{R}$ is called a *horoball* about ξ. Horoballs are convex.

Given points $P_1, P_2, \cdots, P_n \in X$ we let $[P_1 P_2 \cdots P_n]$ denote the geodesic polygon in X which is the union of geodesic segments $P_i P_{i+1}$, i taken modulo n.

Given two distinct points $x, y \in X$, and a point $q \in xy$, we define the *normal hypersurface* $\mathscr{N}_q(x, y)$, i.e. the image of the normal exponential map to the segment xy at point q:

$$\mathscr{N}_q(x, y) = \exp_q(T_q^\perp(xy)),$$

where $T_q^\perp(xy) \subset T_q X$ is the orthogonal complement in the tangent space at q to the segment xy. In the special case when q is the midpoint of xy, $\mathscr{N}_q(x, y)$ is the *perpendicular bisector* of the segment xy, and we will denote it $\mathrm{Bis}(x, y)$. Similarly, we define the normal hypersurface $\mathscr{N}_q(\xi, \eta)$ for any point q in the biinfinite geodesic $\xi\eta$.

Note that if X is a real-hyperbolic space, then $\mathrm{Bis}(x, y)$ is totally geodesic and equals the set of points equidistant from x and y. For general Hadamard spaces, this is not the case. However, if X is δ-hyperbolic, then each $N_p(x, y)$ is δ-quasiconvex, see Definition 3.12.

We let δ denote the *hyperbolicity constant* of X; hence, $\delta \le \cosh^{-1}(\sqrt{2})$. We will use the notation $\mathrm{Hull}(A)$ for the *closed convex hull* of a subset $A \subset X$, i.e. the intersection of all closed convex subsets of X containing A. The notion of the closed convex hull extends to the closed subsets of $\partial_\infty X$ as follows. Given a closed subset $A \subset \partial_\infty X$, we denote by $\mathrm{Hull}(A)$ the smallest closed convex subset of X whose accumulation set in $\bar{X}$ equals A. (Note that $\mathrm{Hull}(A)$ exists as long as A contains more than one point.)

For a subset $A \subset X$ the *quasiconvex hull* $\mathrm{QHull}(A)$ of A in X is defined as the union of all geodesics connecting points of A. Similarly, for a closed subset $A \subset \partial_\infty X$, the quasiconvex hull $\mathrm{QHull}(A)$ is the union of all biinfinite geodesics asymptotic to points of A. Then $\mathrm{QHull}(A) \subset \mathrm{Hull}(A)$, unless A is a singleton in $\partial_\infty X$.

We will use the notation Γ for a discrete subgroup of isometries of X. We let $\Lambda = \Lambda(\Gamma) \subset \partial_\infty X$ denote the *limit set* of Γ, i.e. the accumulation set in $\partial_\infty X$ of one (equivalently, any) Γ-orbit in X. The group Γ acts properly discontinuously on $\bar{X} \setminus \Lambda$, [9, Proposition 3.2.6]. We obtain an orbifold with boundary

$$\bar{M} = M_c(\Gamma) = \left(\bar{X} \setminus \Lambda\right)/\Gamma.$$

If Γ is torsion-free, then $\bar{M}$ is a partial compactification of the quotient manifold $M = X/\Gamma$. We let $\pi : X \to M$ denote the covering projection.

3. Review of negatively pinched Hadamard manifolds

For any triangle $[ABC]$ in (X, d), we define a comparison triangle $[A'B'C']$ for $[ABC]$ in $(\mathbb{H}^2, d')$ as follows.

Definition 3.1. For a triangle $[ABC]$ in (X, d), let A', B', C' be 3 points in the hyperbolic plane $(\mathbb{H}^2, d')$ satisfying that $d'(A', B') = d(A, B), d'(B', C') = d(B, C)$ and $d'(C', A') = d(C, A)$. Then we call $[A'B'C']$ a *comparison triangle* for $[ABC]$.

In general, for any geodesic polygon $[P_1P_2 \cdots P_n]$ in (X, d), we define a comparison polygon $[P_1'P_2' \cdots P_n']$ for $[P_1 \cdots P_n]$ in $(\mathbb{H}^2, d')$.

Definition 3.2. For any geodesic polygon $[P_1P_2 \cdots P_n]$ in X, we pick points $P_1', \cdots, P_n'$ in $\mathbb{H}^2$ such that $[P_1'P_i'P_{i+1}']$ is a comparison triangle for $[P_1P_iP_{i+1}]$ and the triangles $[P_1'P_{i-1}'P_i']$ and $[P_1'P_i'P_{i+1}']$ lie on different sides of $P_1'P_i'$ for each $2 \leq i \leq n-1$. The geodesic polygon $[P_1'P_2' \cdots P_n']$ is called a *comparison polygon* for $[P_1P_2 \cdots P_n]$.

Remark 3.3. Such a comparison polygon $[P_1'P_2' \cdots P_n']$ is not necessarily convex and embedded. In the rest of the section, we have additional assumptions for the polygons $[P_1P_2 \cdots P_n]$. Under these assumptions, their comparison polygons in $\mathbb{H}^2$ are embedded and convex, see Corollary 3.7 and Corollary 3.9.

One important property of negatively pinched Hadamard manifolds X is the following *angle comparison* theorem [10].

Proposition 3.4. [9, Proposition 1.1.2] *For a triangle $[ABC]$ in (X, d), let $[A'B'C']$ denote a comparison triangle for $[ABC]$. Then $\angle ABC \leq \angle A'B'C', \angle BCA \leq \angle B'C'A'$ and $\angle CAB \leq \angle C'A'B'$.*

Proposition 3.4 implies some useful geometric inequalities in X:

Corollary 3.5. *Consider a triangle in X with vertices ABC so that the angles at A, B, C are α, β, γ and the sides opposite to A, B, C have lengths a, b, c, respectively. If $\gamma \geq \pi/2$, then*

$$\cosh a \sin \beta \leq 1.$$

Proof. Let $[A'B'C']$ be a comparison triangle for $[ABC]$ in $(\mathbb{H}^2, d')$. Let α', β', γ' denote the angles at A', B', C' respectively as in Figure 1. By Proposition 3.4, $d'(A', B') = c, d'(A', C') = b, d'(B', C') = a$ and $\beta' \geq \beta, \gamma' \geq \gamma \geq \pi/2$. Take the point $C'' \in A'B'$ such that $\angle B'C'C'' = \pi/2$. In the right triangle $[B'C'C'']$ in $\mathbb{H}^2$, we have $\cosh a \sin \beta' = \cos(\angle C'C''B')$, see [4, Theorem 7.11.3]. So we obtain the inequality:

$$\cosh a \sin \beta \leq \cosh a \sin \beta' \leq 1.$$

□

Remark 3.6. If $A \in \partial_\infty X$, we use a sequence of triangles in X to approximate the triangle $[ABC]$ and prove that $\cosh a \sin \beta \leq 1$ still holds by continuity.

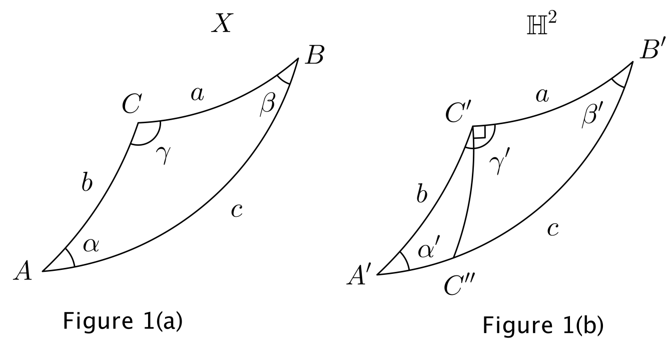

FIGURE 1

Corollary 3.7. *Let $[ABCD]$ denote a quadrilateral in X such that $\angle ABC \geq \pi/2, \angle BCD \geq \pi/2$ and $\angle CDA \geq \pi/2$ as in Figure 2(a). Then:*

(1) $\sinh(d(B,C))\sinh(d(C,D)) \leq 1$.
(2) *Suppose that $\angle BAD \geq \alpha > 0$. If $\cosh(d(A,B))\sin\alpha > 1$, then*

$$\cosh(d(C,D)) \geq \cosh(d(A,B))\sin\alpha > 1.$$

Proof. Let $[A'B'C'D']$ be a comparison quadrilateral for $[ABCD]$ in $(\mathbb{H}^2, d')$ so that $[A'B'C']$ is a comparison triangle for $[ABC]$ and $[A'C'D']$ is a comparison triangle for $[ACD]$. By Proposition 3.4, $\angle A'B'C' \geq \pi/2$, $\angle A'D'C' \geq \pi/2$ and

$$\angle B'C'D' = \angle B'C'A' + \angle A'C'D' \geq \angle BCD \geq \pi/2.$$

So $0 < \angle B'A'D' \leq \pi/2$ and $[A'B'C'D']$ is an embedded convex quadrilateral.

We first prove that $\sinh d(B,C)\sinh(d(C,D)) \leq 1$. In Figure 2(c), take the point $H \in A'B'$ such that $\angle HC'D' = \pi/2$ and take the point $G \in A'H$ such that $\angle GD'C' = \pi/2$. We claim that $\angle C'HA' \geq \pi/2$. Observe that

$$\angle C'HB' + \angle HB'C' + \angle B'C'H \leq \pi$$

$$\angle C'HA' + \angle C'HB' = \pi.$$

Thus $\angle C'HA' \geq \angle C'B'H \geq \pi/2$. We also have $d'(C',H) \geq d'(C',B')$ since

$$\frac{\sinh(d'(C',H))}{\sin(\angle C'B'H)} = \frac{\sinh(d'(C',B'))}{\sin(\angle C'HB')}.$$

Take the point $H' \in GD'$ such that $\angle C'HH' = \pi/2$. In the quadrilateral $[C'HH'D']$, $\cos(\angle HH'D') = \sinh(d'(H,C'))\sinh(d'(C',D'))$ [4, Theorem 7.17.1]. So we have

$$\begin{aligned}\sinh(d(C,D))\sinh(d(B,C)) &= \sinh(d'(C',D')\sinh(d'(B',C'))\\ &\leq \sinh(d'(C',D'))\sinh(d'(C',H))\\ &\leq 1.\end{aligned}$$

Next, we prove that if $\cosh(d(A,B))\sin\alpha > 1$, then $\cosh(d(C,D)) \geq \cosh(d(A,B))\sin\alpha$. In Figure 2(b), take the $C'' \in C'D'$ such that $\angle A'B'C'' = \pi/2$. Observe that C'' cannot be on $A'D'$. Otherwise in the right triangle $[A'B'C'']$, we have

$$\cosh(d(A,B))\sin\alpha \leq \cosh(d'(A',B'))\sin(\angle B'A'D') \leq 1$$

which is a contradiction. Let EF denote the geodesic segment which is orthogonal to $B'E$ and $A'F$. In the quadrilateral $[A'B'EF]$, $\cosh(d'(E,F)) = \cosh(d'(A',B'))\sin(\angle B'A'F)$ by hyperbolic trigonometry [4, Theorem 7.17.1]. So

$$\cosh(d(C,D)) \geq \cosh(d'(C'',D')) \geq \cosh(d'(E,F)) \geq \cosh(d(A,B))\sin\alpha.$$

□

Remark 3.8. If $A \in \partial_\infty X$ and $\angle BAD = 0$, we use quadrilaterals in X to approximate the quadrilateral $[ABCD]$ and prove that $\sinh(d(B,C))\sinh(d(C,D)) \leq 1$ by continuity.

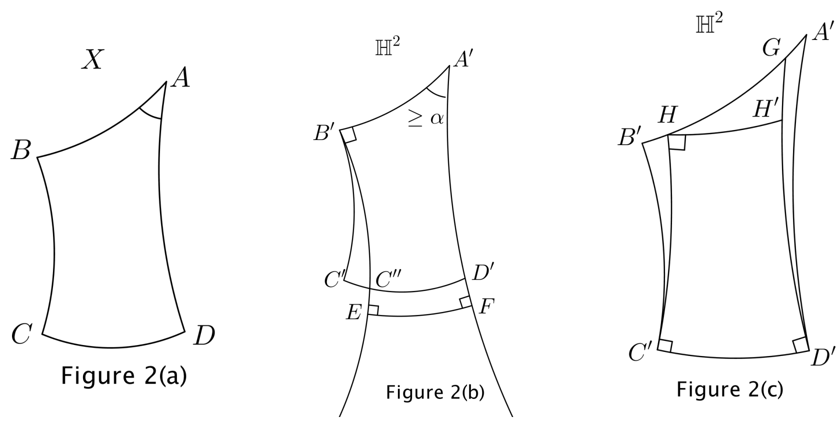

FIGURE 2

Corollary 3.9. *Let $[ABCDE]$ be a pentagon in X with each angle $\geq \pi/2$ as in Figure 3(a). Then if $d(A,B) \to \infty$, we have $d(C,D) \to \infty$.*

Proof. Let $[A'B'C'D'E']$ be a comparison pentagon for $[ABCDE]$ in $(\mathbb{H}^2, d')$ as in Figure 3. By Proposition 3.4,

$$\angle A'B'C' \geq \pi/2, \quad \angle B'C'D' \geq \pi/2, \quad \angle A'E'D' \geq \pi/2$$

and

$$\angle E'D'C' \geq \pi/2, \quad \angle B'A'E' \geq \pi/2.$$

So the pentagon $[A'B'C'D'E']$ is convex as in Figure 3.

Take the point $C'' \in C'D'$ such that $\angle A'B'C'' = \pi/2$ as in Figure 3(b). Observe that C'' cannot be in $E'D'$ if $d(A,B) \to \infty$. Otherwise we will obtain a quadrilateral $[A'B'C''E']$. By Corollary 3.7, $\sinh(d'(A',B'))\sinh(d'(A',E')) \leq 1$. This is a contradiction when $d(A,B)$ is sufficiently large. Choose a point E'' in $\mathbb{H}^2$ such that $\angle E''A'B' = \pi/2$. Then either E'' is in $E'D'$ as in Figure 3(c) or E'' is in $C'D'$ as in Figure 3(b).

If $E'' \in C'D'$, we have a quadrilateral $[A'B'C''E'']$, and $\angle C''B'A' = \angle B'A'E'' = \pi/2$. So $d'(C'',E'') \geq d'(A',B')$. If $E'' \in E'D'$, take the point $F \in C''D'$ such that $\angle FE''A' = \pi/2$ in Figure 3(c). Here F cannot be in $B'C''$. Otherwise we obtain a quadrilateral $[A'B'FE'']$ which is impossible if $d(A,B) \to \infty$ by Corollary 3.7. Observe that $d'(A',E'') \geq d'(A',E')$. Let GH denote the geodesic segment which is orthogonal to $B'G$ and $E''H$. Then $d'(C'',F) \geq d'(G,H)$. In the pentagon $[A'E''HGB']$, we have $\cosh(d'(G,H)) = \sinh(d'(A',B'))\sinh(d'(A',E''))$, see [4, Theorem 7.18.1]. So

$$\cosh(d(C,D)) \geq \cosh(d'(G,H)) \geq \sinh(d(A,B))\sinh(d'(A',E'))$$

Thus in both cases, $d(C,D) \to \infty$ if $d(A,B) \to \infty$. □

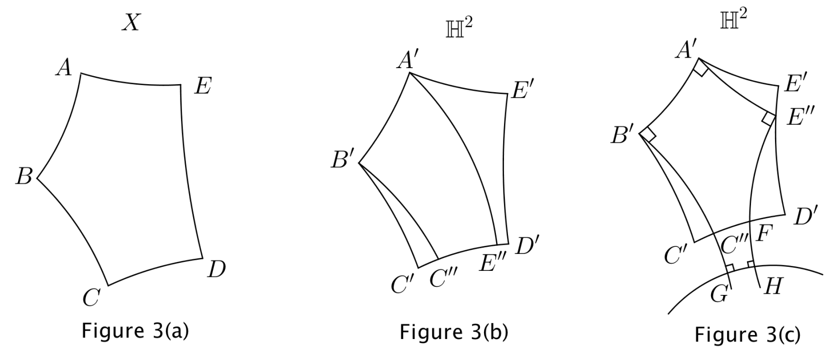

FIGURE 3

Another comparison theorem, *the CAT(-1) inequality*, can be used to derive the following proposition (see [9]):

Proposition 3.10. [9, Lemma 2.2.1] *Suppose $x_0, x_1, \cdots, x_n \in \bar{X}$ are $n+1$ points; then*

$$x_0x_n \subseteq N_\lambda(x_0x_1 \cup x_1x_2 \cup \cdots \cup x_{n-1}x_n)$$

where $\lambda = \lambda_0 \lceil \log_2 n \rceil, \lambda_0 = \cosh^{-1}(\sqrt{2})$.

Given a point $\xi \in \partial_\infty X$, for any point $y \in X$, we use a map $\rho_y : \mathbb{R}^+ \to X$ to parametrize the geodesic $y\xi$ by its arc-length. The following lemma is deduced from the $CAT(-1)$ inequality, see [9]:

Lemma 3.11. [9, Proposition 1.1.11]

(1) *Given any $y, z \in X$, the function $d(\rho_y(t), \rho_z(t))$ is monotonically decreasing in t.*
(2) *For each r, there exists a constant $R = R(r)$, such that if $y, z \in X$ lie in the same horosphere about ξ and $d(y, z) \leq r$, then $d(\rho_y(t), \rho_z(t)) \leq Re^{-t}$ for all t.*

Next we discuss convex and quasiconvex sets in X.

Definition 3.12. A subset $A \subseteq X$ is *convex* if $xy \subseteq A$ for all $x, y \in A$. A closed subset $A \subseteq X$ is λ-*quasiconvex* if $xy \subseteq N_\lambda(A)$ for all $x, y \in A$. Convex closed subsets are 0-quasiconvex.

Remark 3.13. If A is a λ-quasiconvex set, then $\mathrm{QHull}(A) \subseteq N_\lambda(A)$.

Proposition 3.14. [9, Proposition 2.5.4] *There is a function $r : \mathbb{R}_+ \to \mathbb{R}_+$ such that for every λ-quasiconvex subset $A \subseteq X$, we have*

$$\mathrm{Hull}(A) \subseteq N_{r(\lambda)}(A)$$

where the function $r(\lambda)$ only depends on κ.

Remark 3.15. Note that, by the definition of the hyperbolicity constant δ of X, the quasiconvex hull $\mathrm{QHull}(A)$ is 2δ-quasiconvex for every closed subset $A \subseteq \bar{X}$. Thus, $\mathrm{Hull}(A) \subseteq N_r(\mathrm{QHull}(A))$ for some uniform constant $r \in [0, \infty)$.

Remark 3.16. For any closed subset $A \subseteq \partial_\infty X$ with more than one point, $\partial_\infty \mathrm{Hull}(A) = A$.

Lemma 3.17. *Assume that ξ, η are distinct points in $\partial_\infty X$ and $(x_i) \subseteq X$ is a sequence of points which converges to ξ and $(y_i) \subseteq X$ is a sequence of points which converges to η. Then for every point $p \in \xi\eta \subseteq X$, $p \in N_{2\delta}(x_iy_i)$ for all sufficiently large i.*

Proof. Since (x_i) converges to ξ and (y_i) converges to η, then $d(p, x_i\xi) \to \infty$ and $d(p, y_i\eta) \to \infty$ as $i \to \infty$. By δ-hyperbolicity of X,

$$p \in N_{2\delta}(x_iy_i \cup x_i\xi \cup y_i\eta).$$

Since $d(p, x_i\xi) \to \infty$ and $d(p, y_i\eta) \to \infty$, then

$$p \in N_{2\delta}(x_iy_i)$$

for sufficiently large i. □

Remark 3.18. This lemma holds for any δ-hyperbolic geodesic metric space.

4. Escaping sequences of closed geodesics in negatively curved manifolds

In this section, X is a Hadamard manifold of negative curvature ≤ -1 with the hyperbolicity constant δ, $\Gamma < \mathrm{Isom}(X)$ is a torsion-free discrete isometry subgroup and $M = X/\Gamma$ is the quotient manifold. A sequence of subsets $A_i \subset M$ is said to *escape every compact subset of* M if for every compact $K \subset M$, the subset

$$\{i \in \mathbb{N} : A_i \cap K \neq \emptyset\}$$

is finite. Equivalently, for every $x \in M$, $d(x, A_i) \to \infty$ as $i \to \infty$.

Lemma 4.1. *Suppose that (a_i) is a sequence of closed geodesics in $M = X/\Gamma$ which escapes every compact subset of M and $x \in M$. Then, after passing to a subsequence in (a_i), there exist geodesic arcs b_i connecting a_i, a_{i+1} and orthogonal to these geodesics, such that the sequence (b_i) also escapes every compact subset of M.*

Proof. Consider a sequence of compact subsets $K_n := \bar{B}(x, 7\delta n)$ exhausting M. Without loss of generality, we may assume that $a_i \cap K_n = \emptyset$ for all $i \geq n$.

We first prove the following claim:

Claim. *For each compact subset $K \subset M$ and for each infinite subsequence $(a_i)_{i\in I}, I \subset \mathbb{N}$, there exists a further infinite subsequence, $(a_i)_{i\in J}, J \subset I$, such that for each pair of distinct elements $i, j \in J$, there exists a geodesic arc b_{ij} connecting a_i to a_j and orthogonal to both, which is disjoint from K.*

Proof. Given two closed geodesics a, a' in M, we consider the set $\pi_1(M, a, a')$ of *relative homotopy classes* of paths in M connecting a and a', where the relative homotopy is defined through paths connecting a to a'.

In each class $[b'] \in \pi_1(M, a, a')$, there exists a continuous path b which is the length minimizer in the class. By minimality of its length, b is a geodesic arc orthogonal to a and a' at its end-points.

For each compact subset $K \subset M$, there exists $m \in \mathbb{N}$ such that for all $i \in I_m := I \cap [m, \infty)$, $a_i \cap K' = \emptyset$ where $K' = N_{7\delta}(K)$. For $i \in I_m$ let c_i denote a shortest arc between a_i and K'; this geodesic arc terminates a point $x_i \in K'$. By compactness of K', the sequence $(x_i)_{i\in I_m}$ contains a convergent subsequence, $(x_i)_{i\in J}, J \subset I_m$ and, without loss of generality, we may assume that for all $i, j \in J$, $d(x_i, x_j) \leq \delta$. Let x_ix_j denote a (not necessarily unique) geodesic in M of length $\leq \delta$ connecting x_i to x_j. For each pair of indices $i, j \in J$, consider the concatenation

$$b'_{ij} = c_i * x_ix_j * c_j^{-1},$$

which defines a class $[b'_{ij}] \in \pi_1(M, a_i, a_j)$. Let $b_{ij} \in [b'_{ij}]$ be the length-minimizing geodesic arc in this relative homotopy class. Then b_{ij} is orthogonal to a_i and a_j. By δ-hyperbolicity of X,

$$b_{ij} \subseteq N_{7\delta}(a_i \cup c_i \cup c_j \cup a_j).$$

Hence, $b_{ij} \cap K = \emptyset$ for any pair of distinct indices $i, j \in J$. This proves the claim. □

We now prove the lemma. Assume inductively (by induction on N) that we have constructed an infinite subset $S_N \subset \mathbb{N}$ such that:

For the N-th element $i_N \in S_N$, for each $j > i_N, j \in S_N$, there exists a geodesic arc b_j in M connecting a_{i_N} to a_j and orthogonal to both, which is disjoint from K_{N-1}.

Using the claim, we find an infinite subset $S_{N+1} \subset S_N$ which contains the first N elements of S_N, such that for all $s, t > i_N, s, t \in S_{N+1}$, there exists a geodesic $b_{s,t}$ in M connecting a_s to a_t, orthogonal to both and disjoint from K_N.

The intersection

$$S := \bigcap_{N \in \mathbb{N}} S_N$$

equals $\{i_N : N \in \mathbb{N}\}$ and, hence, is infinite. We, therefore, obtain a subsequence $(a_i)_{i \in S}$ such that for all $i, j \in S, i < j$, there exists a geodesic b_{ij} in M connecting a_i to a_j and orthogonal to both, which is disjoint from K_{i-1}. ☐

Remark 4.2. It is important to pass a subsequence of (a_i), otherwise, the lemma is false. A counter-example is given by a geometrically infinite manifold with two distinct ends E_1 and E_2 where we have a sequence of closed geodesics a_i (escaping every compact subset of M) contained in E_1 for odd i and in E_2 for even i. Then b_i will always intersect a compact subset separating the two ends no matter what b_i we take.

5. Elementary groups of isometries

Every isometry g of X extends to a homeomorphism (still denoted by g) of $\bar{X}$. We let $\mathrm{Fix}(g)$ denote the fixed point set of $g : \bar{X} \to \bar{X}$. For a subgroup $\Gamma < \mathrm{Isom}(X)$, we use the notation

$$\mathrm{Fix}(\Gamma) := \bigcap_{g \in \Gamma} \mathrm{Fix}(g),$$

to denote the fixed point set of Γ in $\bar{X}$. Typically, this set is empty.

Isometries of X are classified as follows:

(1) g is *parabolic* if $\mathrm{Fix}(g)$ is a singleton $\{p\} \subset \partial_\infty X$. In this case, g preserves (setwise) every horosphere centered at p.
(2) g is loxodromic if $\mathrm{Fix}(g)$ consists of two distinct points $p, q \in \partial_\infty X$. The loxodromic isometry g preserves the geodesic $pq \subset X$ and acts on it as a nontrivial translation. The geodesic pq is called the *axis* A_g of g.
(3) g is elliptic if it fixes a point in X. The fixed point set of an elliptic isometry is a totally-geodesic subspace of X invariant under g. In particular, the identity map is an elliptic isometry of X.

If $g \in \mathrm{Isom}(X)$ is such that $\mathrm{Fix}(g)$ contains three distinct points $\xi, \eta, \zeta \in \partial_\infty X$, then g also fixes pointwise the convex hull $\mathrm{Hull}(\{\xi, \eta, \zeta\})$ and, hence, g is an elliptic isometry of X.

For each isometry $g \in \mathrm{Isom}(X)$ we define its translation length $l(g)$ as follows:

$$l(g) = \inf_{x \in X} d(x, g(x)).$$

Proposition 5.1. *Let $\langle g \rangle < \mathrm{Isom}(X)$ be a cyclic group generated by a loxodromic isometry g with translation length $l(g) \geq \epsilon > 0$. Let γ denote the simple closed geodesic $A_g/\langle g \rangle$ in M where $M = X/\langle g \rangle$. If $w \subseteq M$ is a piecewise-geodesic loop freely homotopic to γ which consists of r geodesic segments, then γ is contained in some C-neighborhood of the loop w where $C = \cosh^{-1}(\sqrt{2})\lceil \log_2 r \rceil + \sinh^{-1}(2/\epsilon) + 2\delta$.*

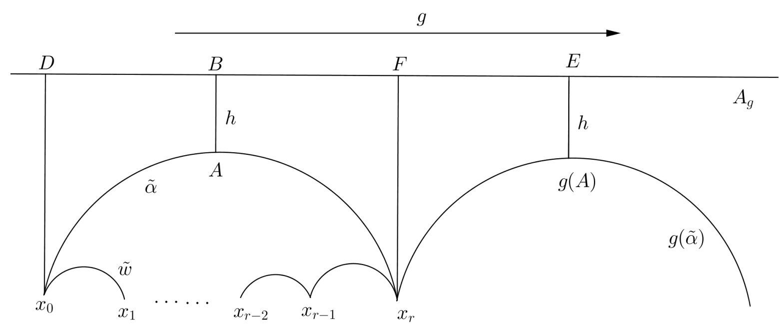

FIGURE 4

Proof. Let $x \in w$ be one of the vertices. Connect this point to itself by a geodesic segment α in M which is homotopic to w (rel $\{x\}$). The loop $w * \alpha^{-1}$ lifts to a polygonal loop $\beta \subseteq X$ with consecutive vertices $x_0, x_1, \cdots, x_r$ so that the geodesic segment $\tilde{\alpha} := x_0 x_r$ covers α. Let $\tilde{w}$ denote the union of edges of β distinct from $\tilde{\alpha}$. By Proposition 3.10, $\tilde{\alpha}$ is contained in the λ-neighborhood of the piecewise geodesic path $\tilde{w}$ where $\lambda = \cosh^{-1}(\sqrt{2})\lceil \log_2 r \rceil$. It follows that $\alpha \subseteq N_\lambda(w)$.

Let $h = d(\tilde{\alpha}, A_g)$. Choose a point $A \in \tilde{\alpha}$ which is nearest to A_g. Let $B \in A_g$ be the nearest point to A. Let $F = \mathrm{Proj}_{A_g}(x_r)$. Then we obtain a quadrilateral $[ABFx_r]$ with $\angle ABF = \angle BFx_r = \angle BAx_r = \pi/2$. By Corollary 3.7,

$$d(B, F) \leq \sinh(d(B, F)) \leq 1/\sinh(h).$$

Take the point $D \in A_g$ which is closest to x_0. By a similar argument, we have $d(B, D) \leq 1/\sinh(h)$. So $d(F, D) \leq 2/\sinh(h)$. The projection $Proj_{A_g}$ is $\langle g \rangle$-equivariant, thus F, D are identified by the isometry g. Hence

$$\epsilon \leq d(D, g(D)) = d(D, F) \leq 2/\sinh(h)$$

and $h \leq \sinh^{-1}(2/\epsilon)$.

Let $E \in A_g$ be the nearest point to $g(A)$. Then $\pi(BE)$ in $M = X/\langle g \rangle$ is the geodesic loop γ where π is the covering projection. By δ-hyperbolicity of X, BE is within the $(h + 2\delta)$-neighborhood of the lifts of α as in Figure 4. Thus γ is within the $(\sinh^{-1}(2/\epsilon) + 2\delta)$-neighborhood of α. Since α is contained in the $(\cosh^{-1}(\sqrt{2})\lceil \log_2 r \rceil)$-neighborhood of w, the loop γ is contained in the $(\cosh^{-1}(\sqrt{2})\lceil \log_2 r \rceil + \sinh^{-1}(2/\epsilon) + 2\delta)$-neighborhood of w. □

A discrete subgroup Γ of isometries of X is called *elementary* if either $\mathrm{Fix}(\Gamma) \neq \emptyset$ or if Γ preserves set-wise some bi-infinite geodesic in X. (In the latter case, Γ contains an index 2 subgroup Γ' such that $\mathrm{Fix}(\Gamma') \neq \emptyset$.) We are particularly interested in the following two types of elementary subgroups.

Definition 5.2. A discrete elementary subgroup $\Gamma < \mathrm{Isom}(X)$ is *parabolic* if it contains a parabolic isometry g and $\mathrm{Fix}(g) = \mathrm{Fix}(\Gamma) = \{p\} \subseteq \partial_\infty X$.

Remark 5.3. Such Γ preserves setwise every horosphere centered at p. Thus, every parabolic subgroup consists of parabolic and elliptic elements.

Definition 5.4. A discrete elementary subgroup $\Gamma < \mathrm{Isom}(X)$ is *loxodromic* if it contains a loxodromic element and preserves setwise its axis A.

Thus, every loxodromic subgroup Γ consists of loxodromic elements with the axis A and elliptic elements.

Consider a subgroup Γ of isometries of X. Given any subset $Q \subseteq \bar{X}$, let

$$\mathrm{stab}_\Gamma(Q) = \{\gamma \in \Gamma \mid \gamma(Q) = Q\}$$

denote the setwise stabilizer of Q.

Definition 5.5. A point $p \in \partial_\infty X$ is called a *parabolic fixed point* of a subgroup $\Gamma < \mathrm{Isom}(X)$ if $\mathrm{stab}_\Gamma(p)$ is parabolic.

Remark 5.6. If $p \in \partial_\infty X$ is a parabolic fixed point of a discrete subgroup $\Gamma < \mathrm{Isom}(X)$, then $\mathrm{stab}_\Gamma(p)$ is a maximal parabolic subgroup of Γ, see [9, Proposition 3.2.1]. Thus, we have a bijective correspondence between the Γ-orbits of parabolic fixed points of Γ and Γ-conjugacy classes of maximal parabolic subgroups of Γ.

Consider an elementary loxodromic subgroup $G < \Gamma$ with the axis β. Then $\mathrm{stab}_\Gamma(\beta)$ is a maximal loxodromic subgroup of Γ, see [9, Proposition 3.2.1].

Observe that the all isometries of finite order are elliptic and that a discrete subgroup $\Gamma < \mathrm{Isom}(X)$ cannot contain elliptic elements of infinite order. Thus, a torsion-free discrete subgroup Γ contains no elliptic elements besides the identity.

6. The thick-thin decomposition

Given $p \in X, \varepsilon > 0$, consider the set

$$\mathcal{F}_\varepsilon(p) = \{\gamma \in \mathrm{Isom}(X) \mid d(p, \gamma p) \leq \varepsilon\}.$$

Given $\Gamma < \mathrm{Isom}(X)$, let $\Gamma_\varepsilon(p) = \langle \Gamma \cap \mathcal{F}_\varepsilon(p) \rangle$ denote the subgroup generated by all elements $\gamma \in \Gamma$ which move p a distance at most ε. Define the set $T_\varepsilon(\Gamma) = \{p \in X \mid \Gamma_\varepsilon(p) \text{ is infinite}\}$. It is a closed Γ-invariant subset of X.

Proposition 6.1 (The Margulis Lemma)**.** *There is a constant $\varepsilon(n, \kappa) > 0$ such that if $\Gamma < \mathrm{Isom}(X)$ is discrete and $p \in X$, then $\Gamma_\varepsilon(p)$ is virtually nilpotent for all $\varepsilon \leq \varepsilon(n, \kappa)$. Here, $\varepsilon(n, \kappa)$ depends only on the dimension n of X and the lower curvature bound $-\kappa^2$.*

See e.g. [3].

Remark 6.2. The constant $\varepsilon(n, \kappa)$ is called the *Margulis constant*.

Lemma 6.3. *Suppose that $G < \mathrm{Isom}(X)$ is a discrete parabolic subgroup and $\varepsilon > 0$. For any $z \in T_{\varepsilon/3}(G)$, we have $B(z, \varepsilon/3) \subseteq T_\varepsilon(G)$.*

Proof. The set $\mathcal{F}_{\varepsilon/3}(z) = \{\gamma \in G | d(z, \gamma(z)) \leq \varepsilon/3\}$ generates an infinite subgroup of G since $z \in T_{\varepsilon/3}(G)$. For any element $\gamma \in \mathcal{F}_{\varepsilon/3}(z)$ and $z' \in B(z, \varepsilon/3)$, we have

$$d(z', \gamma(z')) \leq d(z, z') + d(z, \gamma(z)) + d(\gamma(z), \gamma(z')) \leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

So $\mathcal{F}_\varepsilon(z') = \{\gamma \in G | d(z', \gamma(z')) \leq \varepsilon\}$ also generates an infinite subgroup. Thus $z' \in T_\varepsilon(G_i)$ and $B(z, \varepsilon/3) \subseteq T_\varepsilon(G)$.

□

Proposition 6.4. [9, Proposition 3.5.2] *Suppose $G < \mathrm{Isom}(X)$ is a discrete parabolic subgroup with the fixed point $p \in \partial_\infty X$, and $\varepsilon > 0$. Then $T_\varepsilon(G) \cup \{p\}$ is starlike about p, i.e. for each $x \in \bar{X} \setminus \{p\}$, the intersection $xp \cap T_\varepsilon(G)$ is a ray asymptotpic to p.*

Corollary 6.5. *Suppose that $G < \mathrm{Isom}(X)$ is a discrete parabolic subgroup with the fixed point $p \in \partial_\infty X$. For every $\varepsilon > 0$, $T_\varepsilon(G)$ is a δ-quasiconvex subset of X.*

Proof. By Proposition 6.4, $T_\varepsilon(G) \cup \{p\}$ is starlike about p. Every starlike set is δ-quasiconvex, [9, Corollary 1.1.6]. Thus $T_\varepsilon(G)$ is δ-quasiconvex for every discrete parabolic subgroup $G < \mathrm{Isom}(X)$. □

Remark 6.6. According to Proposition 3.14, there exists $r \in [0, \infty)$ such that $\mathrm{Hull}(T_\varepsilon(G)) \subseteq N_r(T_\varepsilon(G))$ for any $\varepsilon > 0$ and r depends only on κ.

Lemma 6.7. *If $G < \mathrm{Isom}(X)$ is a discrete parabolic subgroup with the fixed point $p \in \partial_\infty X$, then $\partial_\infty T_\varepsilon(G) = \{p\}$.*

Proof. By Lemma 3.11(2), for any $p' \in \partial_\infty X \setminus \{p\}$, both $p'p \cap T_\varepsilon(G)$ and $X \cap (p'p \setminus T_\varepsilon(G))$ are nonempty [9, Proposition 3.5.2]. If $p' \in \partial_\infty T_\varepsilon(G)$, there exists a sequence of points $(x_i) \subseteq T_\varepsilon(G)$ which converges to p'. By Proposition 6.4, $x_i p \subseteq T_\varepsilon(G)$. Since $T_\varepsilon(G)$ is closed in X, then $p'p \subseteq T_\varepsilon(G)$ which is a contradiction. □

Proposition 6.8. *Suppose that $G < \mathrm{Isom}(X)$ is a discrete parabolic subgroup with the fixed point $p \in \partial_\infty X$. Given $r > 0$ and $x \in X$ with $d(x, \mathrm{Hull}(T_\varepsilon(G))) = r$, if (x_i) is a sequence of points on the boundary of $N_r(\mathrm{Hull}(T_\varepsilon(G)))$ and $d(x, x_i) \to \infty$, then there exists $z_i \in xx_i$ such that the sequence (z_i) converges to p and for every $\varepsilon > 0$, $z_i \in N_\delta(T_\varepsilon(G))$ for all sufficiently large i.*

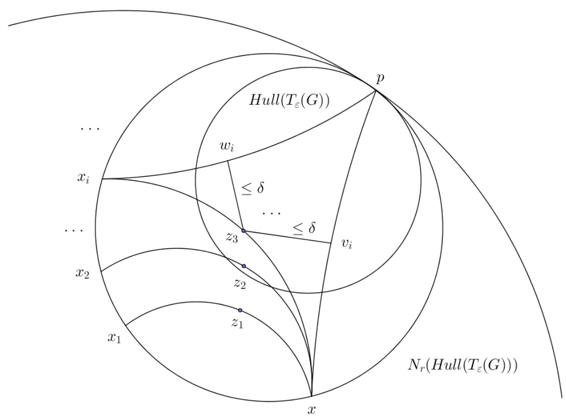

FIGURE 5

Proof. By δ-hyperbolicity of X, there exists a point $z_i \in xx_i$ such that $d(z_i, px) \le \delta$ and $d(z_i, px_i) \le \delta$. Let $w_i \in px_i$ and $v_i \in px$ be the points closest to z_i, see Figure 5. Then $d(z_i, w_i) \le \delta$, $d(z_i, v_i) \le \delta$ and, hence, $d(w_i, v_i) \le 2\delta$.

According to Lemma 6.7, the sequence (x_i) converges to the point p. Hence, any sequence of points on $x_i p$ converges to p as well; in particular, (w_i) converges to p. As $d(w_i, z_i) \le \delta$, we also obtain

$$\lim_{i \to \infty} z_i = p.$$

Since $d(z_i, v_i) \le \delta$, it suffices to show that $v_i \in T_\varepsilon(G)$ for all sufficiently large i. This follows from the fact that $d(x, v_i) \to \infty$ and that $xp \cap T_\varepsilon(G)$ is a geodesic ray asymptotic to p. □

Given $0 < \varepsilon < \varepsilon(n, \kappa)$ and a discrete subgroup Γ, the set $T_\varepsilon(\Gamma)$ is a disjoint union of the subsets of the form $T_\varepsilon(G)$, where G ranges over all maximal infinite elementary subgroups of Γ, [9, Proposition 3.5.5]. For the quotient orbifold $M = X/\Gamma$, set

$$\mathrm{thin}_\varepsilon(M) = T_\varepsilon(\Gamma)/\Gamma.$$

This closed subset is the *thin part*[3] of the quotient orbifold M. The thin part is a disjoint union of its connected components, and that each such component has the form $T_\varepsilon(G)/G$ where G ranges over all maximal infinite elementary subgroups of Γ. If $G < \Gamma$ is a maximal parabolic subgroup, $T_\varepsilon(G)/G$ is called a *Margulis cusp*. If $G < \Gamma$ is a maximal loxodromic subgroup, $T_\varepsilon(G)/G$ is called a *Margulis tube*.

The closure of the complement $M/\mathrm{thin}_\varepsilon(M)$ is the *thick part* of M, denoted by $\mathrm{thick}_\varepsilon(M)$. Let $\mathrm{cusp}_\varepsilon(M)$ denote the union of all Margulis cusps of M; it is called the *cuspidal* part of M. The closure of the complement $M\backslash \mathrm{cusp}_\varepsilon(M)$ is denoted by $\mathrm{noncusp}_\varepsilon(M)$; it is called the *noncuspidal* part of M. Observe that $\mathrm{cusp}_\varepsilon(M) \subseteq \mathrm{thin}_\varepsilon(M)$ and $\mathrm{thick}_\varepsilon(M) \subseteq \mathrm{noncusp}_\varepsilon(M)$. If M is a manifold (i.e., Γ is torsion-free), the ε-thin part is also the collection of all points $x \in M$ where the injectivity radius of M at x is no greater than $\varepsilon/2$.

7. Quasigeodesics

In this section, X is a Hadamard manifold with sectional curvatures ≤ -1. We will prove that certain concatenations of geodesics in X are uniform quasigeodesics, therefore, according to the Morse Lemma, are uniformly close to geodesics.

Lemma 7.1. *Let $\gamma = \gamma_1 * \cdots * \gamma_n \subseteq \bar{X}$ be a piecewise geodesic path from x to y where each γ_i is a geodesic. Assume that for each i, the length of γ_i is λ_i and for $1 \leq i \leq n-1$, the angle between γ_i and γ_{i+1} is α_i. If for all i, $\lambda_i \geq L > 1$ and $\cosh(L/2)\sin(\alpha_i/2) > 1$, then γ is a $(2L, 4L+1)$-quasigeodesic.*

Proof. Let $\mathrm{Bis}(x_i, x_{i+1})$ denote the perpendicular bisector of $\gamma_i = x_i x_{i+1}$ where $x_1 = x$ and $x_{n+1} = y$. If the closures in $\bar{X}$ of the bisectors $\mathrm{Bis}(x_i, x_{i+1})$ and $\mathrm{Bis}(x_{i+1}, x_{i+2})$ intersect each other, then we have the following quadrilateral $[ABCD]$ with $\angle DAB = \angle DCB = \pi/2$ as in Figure 6(a), where $B \in \bar{X}$. Connecting D, B by a geodesic segment (or a ray), we get two right triangles $[ADB]$ and $[BCD]$, and one of the angles $\angle ADB, \angle CDB$ is $\geq \alpha_i/2$. Without loss of generality, we can assume that $\angle ADB \geq \alpha_i/2$. By Corollary 3.5 and Remark 3.6, $\cosh(d(A, D)) \sin\angle ADB \leq 1$. However, we know that

$$\cosh(d(A, D)) \sin\angle ADB \geq \cosh(L/2)\sin(\alpha_i/2) > 1$$

which is a contradiction. Thus, the closures of $\mathrm{Bis}(x_i, x_{i+1})$ and $\mathrm{Bis}(x_{i+1}, x_{i+2})$ are disjoint.

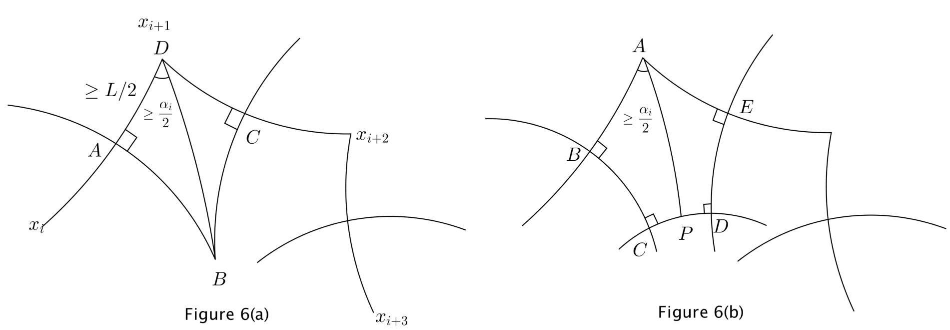

Figure 6

Let $C \in \mathrm{Bis}(x_i, x_{i+1}), D \in \mathrm{Bis}(x_{i+1}, x_{i+2})$ denote points (not necessarily unique) such that $d(C, D)$ minimizes the distance function between the points of these perpendicular bisectors. Since $CB \subset \mathrm{Bis}(x_i, x_{i+1})$, $DE \subset \mathrm{Bis}(x_{i+1}, x_{i+2})$, it follows that the segment

[3]more precisely, ε-thin part

CD is orthogonal to both CB and DE. The segment CD lies in a unique (up to reparameterization) bi-infinite geodesic $\xi\eta$. Then $A \in \mathscr{N}_P(\xi,\eta)$ for some point $P \in \xi\eta$. We claim that $P \in CD$. Otherwise, we obtain a triangle in X with two right angles which is a contradiction. So the geodesic $AP \subseteq \mathscr{N}_P(C,D)$ and AP is orthogonal to CD as in Figure 6(b). We get two quadrilaterals $[ABCP]$ and $[APDE]$. Without loss of generality, assume that $\angle BAP \geq \alpha_i/2$. By Corollary 3.7,

$$d(C,D) \geq d(C,P) \geq \cosh(L/2)\sin(\alpha_i/2) \geq 1.$$

Now we prove that the piecewise geodesic path γ is a quasigeodesic. For each i, if $d(x_i,x_{i+1}) \geq 2L$, take the point $y_{i1} \in \gamma_i$ such that $d(x_i,y_{i1}) = L$. If $L \leq d(y_{i1},x_{i+1}) < 2L$, we'll stop. Otherwise, take the point $y_{i2} \in \gamma_i$ such that $d(y_{i1},y_{i2}) = L$. If $d(y_{i2},x_{i+1}) \geq 2L$, we continue the process until we get y_{ij} such that $L \leq d(y_{ij},x_{i+1}) < 2L$. So we get a piecewise geodesic path $\gamma = \gamma'_1 * \cdots * \gamma'_{n'}$ satisfying the properties that the hyperbolic length of each geodesic arc γ'_i is no less than L and less than $2L$, and adjacent geodesic arcs γ'_i and γ'_{i+1} meet at an angle either α_i or π as in Figure 7(a).

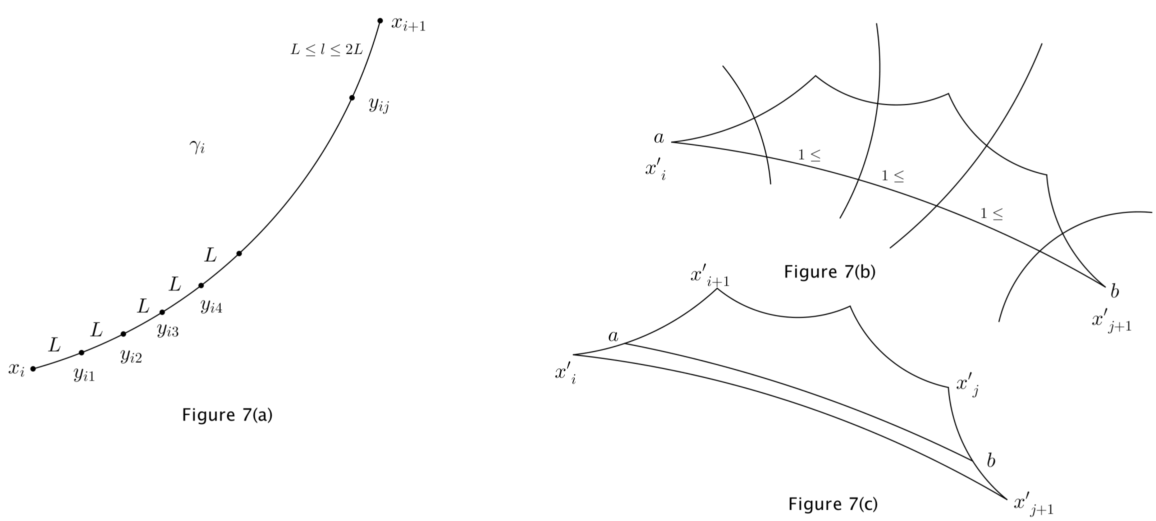

FIGURE 7

In order to prove that γ is a quasigeodesic, it suffices to show that there exist constants λ and ϵ such that

$$\frac{1}{\lambda} length(\gamma|_{[t_a,t_b]}) - \epsilon \leq d(a,b) \leq \lambda \cdot length(\gamma|_{[t_a,t_b]}) + \epsilon$$

for any two points $a, b \in \gamma$ where $\gamma(t_a) = a$ and $\gamma(t_b) = b$. Suppose that a, b are both endpoints of some geodesic arcs γ'_i, γ'_j as in Figure 7(b). The bisectors of the geodesic segments in γ divide ab into several pieces, and each piece has hyperbolic length at least 1 except for the first piece and the last piece. So $d(a,b) > j - i$, while

$$(j-i+1)L \leq length(\gamma|_{[t_a,t_b]}) < 2(j-i+1)L.$$

Let $\lambda = 2L$ and $\epsilon \geq 1$. We have

$$\frac{1}{\lambda} length(\gamma|_{[t_a,t_b]}) - \epsilon \leq d(a,b).$$

If at least one of a, b is not the endpoint of any geodesic arc γ'_i, without loss of generality, assume that a lies in the interior of some geodesic arc $\gamma'_i = x'_i x'_{i+1}$, and $b \in \gamma'_j = x'_j x'_{j+1}$ as in Figure 7(c). Then we have

$$d(x'_i, x'_{j+1}) < d(a,b) + 4L$$

since $d(x'_i, a) < 2L$ and $d(b, x'_{j+1}) < 2L$. By the previous argument, we have the following inequalities

$$\frac{1}{\lambda} length(\gamma|_{[t_a,t_b]}) - \epsilon \leq \frac{1}{\lambda} length(\gamma|_{[t_i,t_{j+1}]}) - \epsilon \leq d(x'_i, x'_{j+1}) \leq d(a,b) + 4L.$$

where $\gamma(t_i) = x'_i$ and $\gamma(t_{j+1}) = x'_{j+1}$. Thus let $\lambda = 2L$ and $\epsilon = 4L+1$. For any two points $a, b \in \gamma$, we have

$$\frac{1}{\lambda} length(\gamma|_{[t_a,t_b]}) - \epsilon \leq d(a,b) \leq \lambda \cdot length(\gamma|_{[t_a,t_b]}) + \epsilon.$$

Therefore γ is a $(2L, 4L+1)$-quasigeodesic.

□

Proposition 7.2. *Given $\theta > 0$, there exist constants $C, L < \infty$ such that the following holds. Suppose that $\gamma = \gamma_1 * \cdots * \gamma_n \subseteq \bar{X}$ is a piecewise geodesic path from x to y. Assume that each geodesic arc γ_i has length at least L, and adjacent geodesic arcs meet at an angle $\geq \theta$. Then the Hausdorff distance between the path γ and the geodesic xy is no greater than C. Here C, L depend only on κ and θ.*

Proof. We can choose $L > 0$ such that $\cosh(L/2)\sin(\theta/2) > 1$. By Lemma 6.1, the piecewise geodesic path γ is a $(2L, 4L+1)$-quasigeodesic. So there is a constant $C = C(2L, 4L+1)$ such that the Hausdorff distance between the piecewise geodesic path γ and the geodesic xy is no greater than C [11, Lemma 9.38, Lemma 9.80].

□

Proposition 7.3 (Generalized version)**.** *Given $\theta, \varepsilon > 0$, there exist constants $C, L < \infty$ such that the following holds. Suppose that $\gamma = \gamma_1 * \cdots * \gamma_n \subseteq \bar{X}$ is a piecewise geodesic path from x to y such that:*

(1) *Each geodesic arc γ_j has length either at least ε or at least L.*
(2) *If γ_j has length $< L$, then the adjacent geodesic arcs γ_{j-1} and γ_{j+1} have lengths at least L and γ_j meets γ_{j-1} and γ_{j+1} at angles $\geq \pi/2$.*
(3) *Other adjacent geodesic arcs meet at an angle $\geq \theta$.*

Then the Hausdorff distance between γ and the geodesic xy is no greater than C. Here L and C depend only on θ, ε and κ.

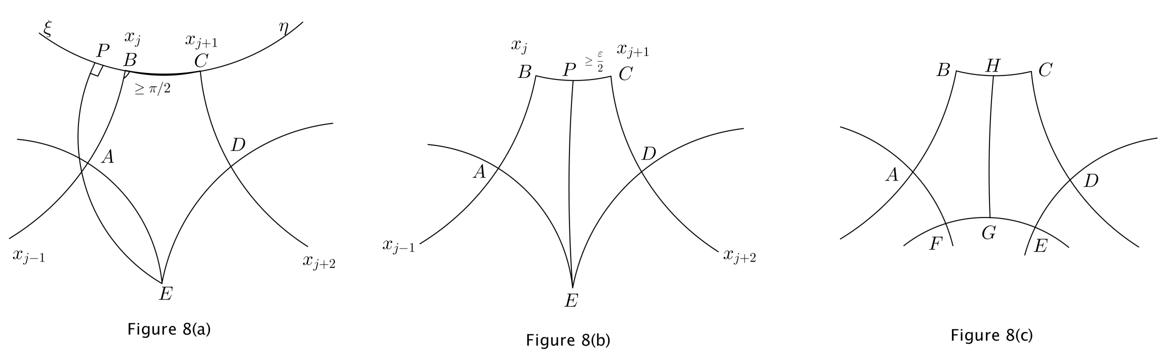

FIGURE 8

Proof. Assume that γ_j has length $< L$ for some j. Let $\mathrm{Bis}(x_{j-1}, x_j)$ and $\mathrm{Bis}(x_{j+1}, x_{j+2})$ be the perpendicular bisectors of γ_{j-1} and γ_{j+1} as in Figure 8. We claim that the closures of these bisectors in $\bar{X}$ do not intersect each other. If they intersect, consider the pentagon $[ABCDE]$ where $E \in \bar{X}$. The geodesic segment BC lies in a unique (up to reparameterization) bi-infinite geodesic $\xi\eta$. Then $E \in \mathscr{N}_P(\xi, \eta)$ for some point $P \in \xi\eta$.

Assume that $P \in B\xi$. Consider the quadrilateral $[EPCD]$ in Figure 8(a). Then $d(C, P) \geq \varepsilon$. By Corollary 3.7 and Remark 3.8, $\sinh(d(C, P)) \sinh(d(C, D)) \leq 1$. Therefore,

$$\sinh(L/2) \leq \sinh(d(C, D)) \leq 1/\sinh(\varepsilon)$$

which is a contradiction if we choose $L > 2 \operatorname{arcsinh}(1/\sinh(\varepsilon))$. If $P \in C\eta$, we consider the quadrilateral $[EPBA]$ and use a similar argument to get a contradiction. So $P \in BC$ and we have two quadrilaterals $[AEPB]$ and $[EPCD]$ as in Figure 8(b). Since $d(B, C) \geq \varepsilon$, one of $d(B, C)$ and $d(C, P)$ has length at least $\varepsilon/2$. Without loss of generality, assume that $d(C, P) \geq \varepsilon/2$. In the quadrilateral $[EPCD]$, $\sinh(d(C, P)) \sinh(d(C, D)) \leq 1$ by Corollary 3.7 and Remark 3.8. Therefore,

$$\sinh(L/2) \leq \sinh(d(C, D)) \leq 1/\sinh(\varepsilon/2).$$

This is a contradiction for $L > 2 \operatorname{arcsinh}(1/\sinh(\varepsilon/2))$. Thus the closures of $\mathrm{Bis}(x_{j-1}, x_j)$ and $\mathrm{Bis}(x_{j+1}, x_{j+2})$ do not intersect each other by choosing $L > 2 \operatorname{arcsinh}(1/\sinh(\varepsilon/2))$.

Let $E \in \mathrm{Bis}(x_{j-1}, x_j), F \in \mathrm{Bis}(x_{j+1}, x_{j+2})$ denote points (not necessarily unique) such that $d(E, F)$ minimizes the distance function between the points of these perpendicular bisectors. The segment EF is orthogonal to both AF and DE, see Figure 8(c). Consider the hexagon $[ABCDEF]$. The segment EF lies in a unique (up to reparameterization) bi-infinite geodesic $\zeta\theta$. Then the midpoint H of the geodesic segment BC lies in $\mathscr{N}_G(\zeta, \theta)$ for some point $G \in \zeta\theta$. We claim that $G \in EF$. Otherwise, we obtain a triangle in X with two right angles which is contradiction. So $HG \subseteq \mathscr{N}_G(E, F)$ and HG is orthogonal to EF at G as in Figure 8(c). Without loss of generality, assume that $\angle BHG \geq \pi/2$. Consider the pentagon $[ABHGF]$. By Corollary 3.9, $d(F, G) \to \infty$ as $d(A, B) \to \infty$. For each positive constant α, we can choose sufficiently large $L < \infty$ such that $d(E, F) \geq \alpha$. By a similar argument as in the proof of Proposition 7.2, we can show that γ is a uniform quasigeodesic and there exists a constant C such that the Hausdorff distance $\mathrm{hd}(\gamma, xy)$ is no greater than C by the Morse Lemma [11, Lemma 9.38, Lemma 9.80].

□

8. Loxodromic products

In order to prove our generalization of Bonahon's theorem, we need to construct a loxodromic element with uniformly bounded word length in $\langle f, g\rangle$ where f, g are two parabolic isometries generating a discrete nonelementary subgroup of $\mathrm{Isom}(X)$.

Lemma 8.1. [16, Theorem Σ_m] *Let $F = \{A_1, A_2, \cdots, A_m\}$ be a family of open subsets of an n-dimensional topological space X. If for every subfamily F' of size j where $1 \leq j \leq n+2$, the intersection $\cap F'$ is nonempty and contractible, then the intersection $\cap F \neq \emptyset$.*

Proof. This lemma is a special case of the topological Helly theorem [16]. Here we give another proof of the lemma. Suppose k is the smallest integer such that there exists a subfamily $F' = \{A_{i(1)}, A_{i(2)}, \cdots, A_{i(k)}\}$ with size k with empty intersection $\cap F' = \emptyset$. By the assumption, $k \geq n+3$. Then

$$U := \bigcup_{1 \leq j \leq k} A_{i(j)}$$

is homotopy equivalent to the nerve $N(F')$ [13, Corollary 4G.3], which, in turn, is homotopy equivalent to S^{k-2}. Then $H_{k-2}(S^{k-2}) \cong H_{k-2}(U) \cong \mathbb{Z}$ which is a contradiction since $k-2 \geq n+1$ and X has dimension n.

□

Proposition 8.2. *Let X be a δ-hyperbolic n-dimensional Hadamard space. Suppose that $B_1, \cdots, B_k$ are convex subsets of X such that $B_i \cap B_j \neq \emptyset$ for all i and j. Then there is a point $x \in X$ such that $d(x, B_i) \leq n\delta$ for all $i = 1, ..., k$.*

Proof. For $k = 1, 2$, the lemma is clearly true.

We first claim that for each $3 \leq k \leq n+2$, there exists a point $x \in X$ such that $d(x, B_i) \leq (k-2)\delta$. We prove the claim by induction on k. When $k = 3$, pick points $x_{ij} \in B_i \cap B_j$, $i \neq j$. Then $x_{ij}x_{il} \subset B_i$ for all i, j, l. Since X is δ-hyperbolic, there exists a point $x \in X$ within distance $\leq \delta$ from all three sides of the geodesic triangle $[x_{12}x_{23}x_{31}]$. Hence,

$$d(x, B_i) \leq \delta, i = 1, 2, 3$$

as well.

Assume that the claim holds for $k-1$. Set $B'_i = N_\delta(B_i)$ and $C_i = B'_i \cap B_1$ where $i \in \{2, 3, \cdots, k\}$. By convexity of the distance function on X, each B'_i is still convex in X and, hence, is a Hadamard space. Furthermore, each B'_i is again δ-hyperbolic.

We claim that $C_i \cap C_j \neq \emptyset$ for all $i, j \in \{2, 3, \cdots, k\}$. By the nonemptyness assumption, there exist points $x_{1i} \in B_1 \cap B_i \neq \emptyset, x_{1j} \in B_1 \cap B_j \neq \emptyset$ and $x_{ij} \in B_i \cap B_j \neq \emptyset$. By δ-hyperbolicity of X, there exists a point $y \in x_{1i}x_{1j}$ such that $d(y, x_{1i}x_{ij}) \leq \delta$, $d(y, x_{2j}x_{ij}) \leq \delta$.

Therefore, $y \in B_1 \cap N_\delta(B_i) \cap N_\delta(B_j) = C_i \cap C_j$. By the induction hypothesis, there exists a point $x' \in X$ such that $d(x', C_i) \leq (k-3)\delta$ for each $i \in \{2, 3, \cdots, k\}$. Thus,

$$d(x', B_i) \leq (k-2)\delta, i \in \{1, 2, \cdots, k\}$$

as required.

For $k > n+2$, set $U_i = N_{n\delta}(B_i)$. Then by the claim, we know that for any subfamily of $\{U_i\}$ of size j where $1 \leq j \leq n+2$, its intersection is nonempty and the intersection is contractible since it is convex. By Lemma 8.1, the intersection of the family $\{U_i\}$ is also nonempty. Let x be a point in this intersection. Then $d(x, B_i) \leq n\delta$ for all $i \in \{1, 2, \cdots, k\}$.

□

Proposition 8.3. *There exists a function $k : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{N}$ with the following property. Let $g_1, g_2, \cdots, g_k$ be parabolic elements in a discrete subgroup $\Gamma < \mathrm{Isom}(X)$. For each g_i let $G_i < \Gamma$ be the unique maximal parabolic subgroup containing g_i, i.e. $G_i = \mathrm{stab}_\Gamma(p_i)$, where $p_i \in \partial_\infty X$ is the fixed point of g_i. Suppose that*

$$T_\varepsilon(G_i) \cap T_\varepsilon(G_j) = \emptyset$$

for all $i \neq j$. Then, whenever $k \geq k(D, \varepsilon)$, there exists a pair of indices i, j with

$$d(T_\varepsilon(G_i), T_\varepsilon(G_j)) > D.$$

Proof. For each i, $\mathrm{Hull}(T_\varepsilon(G_i))$ is convex and by Remark 6.6, $\mathrm{Hull}(T_\varepsilon(G_i)) \subseteq N_r(T_\varepsilon(G_i))$, for some uniform constant $r = r(\kappa)$. Suppose that $g_1, g_2, \cdots, g_k$ and D are such that for all i and j,

$$d(T_\varepsilon(G_i), T_\varepsilon(G_j)) \leq D.$$

Then $d(\mathrm{Hull}(T_\varepsilon(G_i)), \mathrm{Hull}(T_\varepsilon(G_j))) \leq D$.

Our goal is to get a uniform upper bound on k. Consider the $D/2$-neighborhoods $N_{D/2}(\mathrm{Hull}(T_\varepsilon(G_i)))$. They are convex in X and have nonempty pairwise intersections. Thus, by Proposition 8.2, there is a point $x \in X$ such that

$$d(x, T_\varepsilon(G_i)) \leq R_1 := n\delta + \frac{D}{2} + r, i = 1, ..., k.$$

Then

$$T_\varepsilon(G_i) \cap B(x, R_1) \neq \emptyset, i = 1, ..., k.$$

Next, we claim that there exists $R_2 \geq 0$, depending only on ε, such that

$$T_\varepsilon(G_i) \subseteq N_{R_2}(T_{\varepsilon/3}(G_i)).$$

Choose any point $y \in T_\varepsilon(G_i)$ and let $\rho_i : [0, \infty) \to X$ be the ray yp_i. By Lemma 3.11, there exists an absolute constant $R = R(\epsilon)$ such that

$$d(\rho_i(t), g(\rho_i(t))) \leq Re^{-t}$$

whenever $g \in G_i$ is a parabolic (or elliptic) isometry such that

$$d(y, g(y)) \leq \varepsilon.$$

Let $t = \max\{\ln(3R/\varepsilon), 0\}$. Then $d(\rho_i(t), g(\rho_i(t))) \leq \varepsilon/3$. So $T_\varepsilon(G_i) \subseteq N_t(T_{\varepsilon/3}(G_i))$ for any i. Let $R_2 = t$. By the argument above, $B(x, R_1 + R_2) \cap T_{\varepsilon/3}(G_i) \neq \emptyset$ for any i. Assume that $z_i \in B(x, R_1 + R_2) \cap T_{\varepsilon/3}(G_i)$. Then $B(z_i, \varepsilon/3) \subseteq B(x, R_3)$ where $R_3 = R_1 + R_2 + \varepsilon/3$. By Lemma 6.3, $B(z_i, \varepsilon/3) \subseteq B(x, R_3) \cap T_\varepsilon(G_i)$. Since $T_\varepsilon(G_i)$ and $T_\varepsilon(G_j)$ have empty intersection for all $i \neq j$, $B(z_i, \varepsilon/3)$ and $B(z_j, \varepsilon/3)$ are disjoint. Let $V(r, n)$ be the volume of the uniform r-ball in $\mathbb{H}^n$. Then for each i, $B(z_i, \varepsilon/3)$ has volume at least $V(\varepsilon/3, n)$ [9, Proposition 1.1.12]. The volume of $B(x, R_3)$ is at most $V(\kappa R_3, n)/\kappa^n$, see [9, Propostion 1.2.4]. Let $k(D, \varepsilon) = \dfrac{V(\kappa R_3, n)/\kappa^n}{V(\varepsilon/3, n)} + 1$. If $k \geq k(D, \varepsilon)$, we obtain a contradiction. Thus for $k \geq k(D, \varepsilon)$, there exist i and j such that $d(T_\varepsilon(G_i), T_\varepsilon(G_j)) > D$.

□

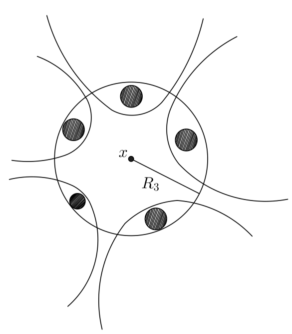

FIGURE 9

Proposition 8.4. *Suppose that g_1, g_2 are two parabolic elements. There exists a constant L which only depends on ε, κ such that if $d(T_\varepsilon(g_1), T_\varepsilon(g_2)) > L$, then $h = g_2 g_1$ is loxodromic.*

Proof. Let $B_i = T_\varepsilon(g_i)$, so $d(B_1, B_2) > L$. Consider the orbits of B_1 and B_2 under the action of the cyclic group generated by $g_2 g_1$ as in Figure 10. Let $x_0 \in B_1, y_0 \in B_2$ denote points such that $d(x_0, y_0)$ minimizes the distance function between points of B_1 and B_2. For positive integers $m > 0$, we let

$$x_{2m-1} = (g_2 g_1)^{m-1} g_2(x_0), \quad x_{2m} = (g_2 g_1)^m (x_0)$$

and

$$y_{2m-1} = (g_2g_1)^{m-1}g_2(y_0), \quad y_{2m} = (g_2g_1)^m(y_0).$$

Similarly, for negative integers $m < 0$, we let

$$x_{2m+1} = (g_2g_1)^{m+1}g_1^{-1}(x_0), \quad x_{2m} = (g_2g_1)^m(x_0)$$

and

$$y_{2m+1} = (g_2g_1)^{m+1}g_1^{-1}(y_0), \quad y_{2m} = (g_2g_1)^m(y_0).$$

We construct a sequence of piecewise geodesic paths $\{\gamma_m\}$ where $\gamma_m = x_{-2m}y_{-2m} * y_{-2m}y_{-2m+1} \cdots * x_0y_0 * y_0y_1 * y_1x_1 \cdots * x_{2m}y_{2m}$ for any positive integer m. Observe that $d(x_i, y_i) = d(B_1, B_2) > L$ and $d(x_{2i-1}, x_{2i}) = \varepsilon, d(y_{2i}, y_{2i+1}) = \varepsilon$ for any integer i. By convexity of B_1, B_2, the angle between any adjacent geodesic arcs in γ_m is at least $\pi/2$. Let γ denote the limit of the sequence (γ_m). By Proposition 7.3, there exists a constant $L > 0$ such that the piecewise geodesic path $\gamma : \mathbb{R} \to X$ is unbounded and is a uniform quasigeodesic invariant under the action of h. By the Morse Lemma [11, Lemma 9.38, Lemma 9.80], the Hausdorff distance between γ and the complete geodesic which connects the endpoints of γ is bounded by a uniformly constant C. So g_2g_1 fixes the endpoints of γ and acts on the complete geodesic as a translation. Thus g_2g_1 is loxodromic.

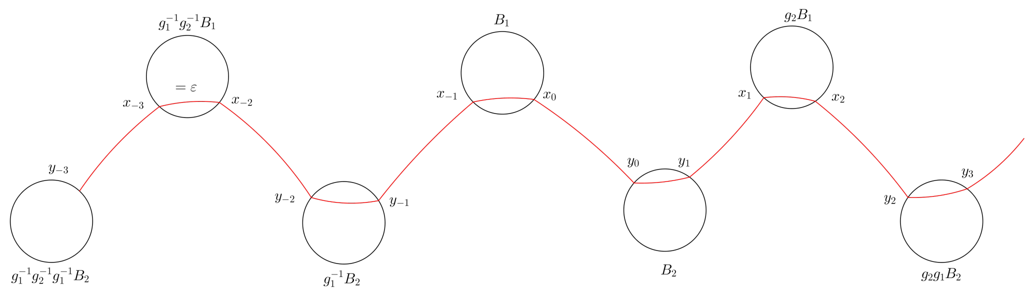

FIGURE 10

□

Theorem 8.5. *Suppose that g_1, g_2 are two parabolic elements with different fixed points. Then there exists a word $w \in \langle g_1, g_2\rangle$ such that $l(w) \leq 4k(L, \varepsilon) + 2$ and w is loxodromic where $l(w)$ denotes the length of the word and $k(L, \varepsilon)$ is the function in Proposition 8.3, $0 < \varepsilon < \varepsilon(n, \kappa)$ and L is the constant in Proposition 8.4.*

Proof. Let p_i denote the fixed point of the parabolic element g_i where $i = 1$ or 2.

Assume that any element in $\langle g_1, g_2\rangle$ with word length no greater than $2k(L, \varepsilon)+1$ is parabolic. Otherwise, there exists a loxodromic element $w \in \langle g_1, g_2\rangle$ with length $\leq 4k(L, \varepsilon) + 2$.

Consider the parabolic elements $g_2^i g_1 g_2^{-i} \in \langle g_1, g_2\rangle$, $0 \leq i \leq k(L, \varepsilon)$. The fixed point of each $g_2^i g_1 g_2^{-i}$ is $g_2^i(p_1)$. We claim that the points $g_2^i(p_1)$ and $g_2^j(p_1)$ are distinct for $i \neq j$. If not, $g_2^i(p_1) = g_2^j(p_1)$ for some $i > j$. Then $g_2^{i-j}(p_1) = p_1$, and, thus, g_2^{i-j} has two distinct fixed points p_1 and p_2. This is a contradiction since any parabolic element has only one fixed point. Thus, $g_2^i g_1 g_2^{-i}$ are parabolic elements with distinct fixed points for all $0 \leq i \leq k(L, \varepsilon)$. Since $0 < \varepsilon < \varepsilon(n, \kappa)$, $T_\varepsilon(g_2^i g_1 g_2^{-i}), T_\varepsilon(g_2^j g_1 g_2^{-j})$ are disjoint for any pair of indices i, j [9]. By Proposition 8.3, there exist $0 \leq i, j \leq k(L, \varepsilon)$ such that $d(T_\varepsilon(g_2^i g_1 g_2^{-i}), T_\varepsilon(g_2^j g_1 g_2^{-j})) > L$. By Proposition 8.4, the element $g_2^j g_1 g_2^{i-j} g_1 g_2^{-i} \in \langle g_1, g_2\rangle$ is loxodromic, and its word length is no greater than $4k(L, \varepsilon)+2$. Thus we can find a word $w \in \langle g_1, g_2\rangle$ such that $l(w) \leq 4k(L, \varepsilon)+2$ and w is loxodromic.

□

9. A generalization of Bonahon's theorem

In this section, we use the construction in Section 8 to generalize Bonahon's theorem for any torsion-free discrete subgroup $\Gamma < \mathrm{Isom}(X)$ where X is a negatively pinched Hadamard manifold.

Lemma 9.1. *For every* $\tilde{x} \in \mathrm{Hull}(\Lambda(\Gamma))$,

$$\mathrm{hd}(\mathrm{QHull}(\Gamma\tilde{x}), \mathrm{QHull}(\Lambda(\Gamma))) < \infty$$

Proof. By the assumption that $\tilde{x} \in \mathrm{Hull}(\Lambda(\Gamma))$ and Remark 3.15, there exists a universal constant $r_1 = r(\kappa) \in [0, \infty)$ such that

$$\mathrm{QHull}(\Gamma\tilde{x}) \subseteq \mathrm{Hull}(\Lambda(\Gamma)) \subseteq N_{r_1}(\mathrm{QHull}(\Lambda(\Gamma)))$$

Next, we want to prove that there exists a constant $r_2 \in [0, \infty)$ such that $\mathrm{QHull}(\Lambda(\Gamma)) \subseteq N_{r_2}(\mathrm{QHull}(\Gamma\tilde{x}))$.

Pick any point $p \in \mathrm{QHull}(\Lambda(\Gamma))$. Then p lies on some geodesic $\xi\eta$ where $\xi, \eta \in \Lambda(\Gamma)$ are distinct points. Since ξ and η are in the limit set, there exist sequences of elements (f_i) and (g_i) in Γ such that the sequence $(f_i(\tilde{x}))$ converges to ξ and the sequence $(g_i(\tilde{x}))$ converges to η. By Lemma 3.17, $p \in N_{2\delta}(f_i(\tilde{x})g_i(\tilde{x}))$ for all sufficiently large i. Let $r = \max\{r_1, 2\delta\}$. Then $\mathrm{hd}(\mathrm{QHull}(\Gamma\tilde{x}), \mathrm{QHull}(\Lambda(\Gamma))) = r < \infty$. $\square$

Remark 9.2. Let $\gamma_i = f_i(\tilde{x})g_i(\tilde{x})$. Then there exists a sequence of points (p_i) such that $p_i \in \gamma_i$ and the sequence (p_i) converges to p.

If $\Gamma < \mathrm{Isom}(X)$ is geometrically infinite, then

$$Core(M) \cap \mathrm{noncusp}_\varepsilon(M)$$

is noncompact, [9]. By Lemma 9.1, $(\mathrm{QHull}(\Gamma\tilde{x})/\Gamma) \cap \mathrm{noncusp}_\varepsilon(M)$ is unbounded.

Now we generalize Bonahon's theorem for any geometrically infinite torsion-free discrete subgroup $\Gamma < \mathrm{Isom}(X)$:

Proof of the implication $(1) \Rightarrow (2)$ **in Theorem 1.5:** If there exists a sequence of closed geodesics $\beta_i \subseteq M$ whose lengths go to 0 as $i \to \infty$, then the sequence (β_i) escapes every compact subset of M. From now on, we assume that there exists a constant $\epsilon > 0$ such that the length $l(\beta) \geq \epsilon$ for any closed geodesic β in M.

Recall that a Margulis cusp is denoted by $T_\varepsilon(G)/G$ where $G < \Gamma$ is a maximal parabolic subgroup. There exists a universal constant $r \in [0, \infty)$ such that $\mathrm{Hull}(T_\varepsilon(G)) \subseteq N_r(T_\varepsilon(G))$ for any maximal parabolic subgroup G (see Section 5). Let $B(G) = N_{2+4\delta}(\mathrm{Hull}(T_\varepsilon(G)))$. Let M^o be the union of all subsets $B(G)/\Gamma$ where G ranges over all maximal parabolic subgroups of Γ. We let M^c denote the closure of $\mathrm{Core}(M) \setminus M^o$. Since Γ is geometrically infinite, the noncuspidal part of the convex core $\mathrm{Core}(M) \setminus \mathrm{cusp}_\varepsilon(M)$ is unbounded by Theorem 1.4. Then M^c is also unbounded since $M^o \subseteq N_{r+2+4\delta}(\mathrm{cusp}_\varepsilon(M))$.

Fix a point $x \in M^c$. Let $C_n = B(x, nR) = \{y \in M^c \mid d(x, y) \leq nR\}$ where $R = r + 2 + 4\delta + \varepsilon$. Let $\tilde{x}$ be a lift of x in X. By Lemma 9.1 $(\mathrm{QHull}(\Gamma\tilde{x})/\Gamma) \cap M^c$ is unbounded. For every C_n, there exists a sequence of geodesic loops (γ_i) connecting x to itself in $\mathrm{Core}(M)$ such that the Hausdorff distance $\mathrm{hd}(\gamma_i \cap M^c, C_n) \to \infty$ as $i \to \infty$. Let $y_i \in \gamma_i \cap M^c$ be such that $d(y_i, C_n)$ is maximal on $\gamma_i \cap M^c$. We pick a component α_i of $\gamma_i \cap M^c$ such that $y_i \in \alpha_i$. Let $\boldsymbol{\delta} C_n$ denote the relative boundary $\partial C_n \setminus \partial M^c_{cusp}$ of C_n where $M^c_{cusp} = M^o \cap \mathrm{Core}(M)$. Consider the sequence of geodesic arcs (α_i).

After passing to a subsequence in (α_i), one of the following three cases occurs:

Case (a): Each α_i has both endpoints x'_i and x''_i on ∂M^c_{cusp} as in Figure 11(a). By construction, there exist y'_i and y''_i in the cuspidal part such that $d(x'_i, y'_i) \leq r_1, d(y'_i, y''_i) \leq r_1$ where $r_1 = 2 + 4\delta + r$. Then we find short nontrivial geodesic loops α'_i, α''_i contained in the cuspidal part $\mathrm{cusp}_\varepsilon(M)$ such that α'_i connects y'_i to itself and α''_i connects y''_i to itself and the lengths $l(\alpha'_i) \leq \varepsilon, l(\alpha''_i) \leq \varepsilon$. Let

$$w' = x'_i y'_i * \alpha'_i * y'_i x'_i \in \Omega(M, x'_i)$$

and

$$w'' = \alpha_i * x''_i y''_i * \alpha''_i * y''_i x''_i * \alpha_i^{-1} \in \Omega(M, x'_i)$$

where $\Omega(M, x'_i)$ denotes the loop space of M. Observe that $w' \cap C_{n-1} = \emptyset$ and $w'' \cap C_{n-1} = \emptyset$.

Let g', g'' denote the elements of $\Gamma = \pi_1(M, x'_i)$ represented by w' and w'' respectively. By the construction, g' and g'' are both parabolic. We claim that g' and g'' have different fixed points. Otherwise, $g, g'' \in G'$ where $G' < \Gamma$ is some maximal parabolic subgroup. Then $y'_i, y''_i \in T_\varepsilon(G')/\Gamma$ and $x'_i, x''_i \in B(G')/\Gamma$. Since $\mathrm{Hull}(T_\varepsilon(G'))$ is convex, $B(G') = N_{2+4\delta}(\mathrm{Hull}(T_\varepsilon(G')))$ is also convex by convexity of the distance function. So $x'_i x''_i \subseteq B(G')/\Gamma$. However, $x'_i x''_i$ lies outside of $B(G')/\Gamma$ by construction which is a contradiction.

By Theorem 8.5, there exists a loxordomic element $\omega_n \in \langle g', g'' \rangle < \Gamma = \pi_1(M, x'_i)$ with the word length uniformly bounded by a constant C. Let w_n be a concatenation of w'_i, w''_i and their reverses which represents ω_n. Then the number of geodesic arcs in w_n is uniformly bounded by $5C$. The piecewise geodesic loop w_n is freely homotopic to a closed geodesic w_n^* in M; hence, by Proposition 5.1, w_n^* is contained in some D-neighborhood of the loop w_n where $D = \cosh^{-1}(\sqrt{2})\lceil \log_2 5C \rceil + \sinh^{-1}(2/\epsilon) + 2\delta$. Thus $d(x, w_n^*) \geq (n-1)R - D$.

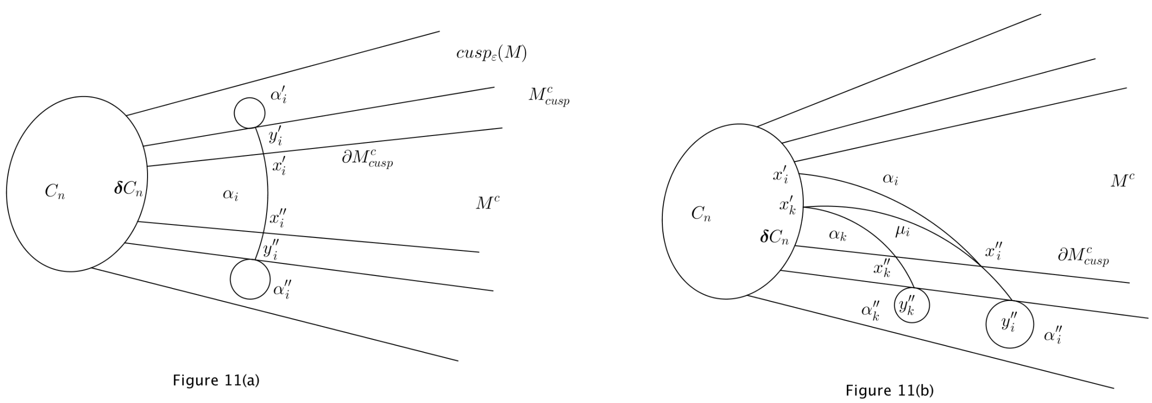

Figure 11

Case (b): For each i, the geodesic arc α_i connects $x'_i \in \boldsymbol{\delta} C_n$ to $x''_i \in \partial M^c_{cusp}$, as in Figure 11(b). For each x''_i, there exists a point $y''_i \in \mathrm{cusp}_\varepsilon(M)$ such that $d(x''_i, y''_i) \leq r_1$ and a short nontrivial geodesic loop α''_i contained in the cuspidal part which connects y''_i to itself and has length $l(\alpha''_i) \leq \varepsilon$. Since $\boldsymbol{\delta} C_n$ is compact, after passing to a further subsequence in (α_i), there exists $k \in \mathbb{N}$ such that for all $i \geq k$, $d(x'_i, x'_k) \leq 1$ and less than the injectivity radius of M at x'_k. Hence, there exists a unique shortest geodesic $x'_k x'_i$ in the manifold M. Let $\mu_i = x'_k x''_i$ denote the geodesic arc homotopic to the concatenation $x'_k x'_i * x'_i x''_i$ rel. $\{x'_i, x''_i\}$. Then, by δ-hyperbolicity of X, the geodesic $\mu_i = x'_k x''_i$ is contained in the $(1+\delta)$-neighborhood of α_i.

Let

$$w'_k = \alpha_k * x''_k y''_k * \alpha''_k * y''_k x''_k * \alpha_k^{-1} \in \Omega(M, x'_k)$$

and

$$w'_i = \mu_i * x''_i y''_i * \alpha''_i * y''_i x''_i * (\mu_i)^{-1} \in \Omega(M, x'_k)$$

for all $i > k$. By the construction, $w_i' \cap C_{n-1} = \emptyset$ for each $i \geq k$.

Let g_i denote the element of $\Gamma = \pi_1(M, x_k')$ represented by w_i', $i \geq k$. Then each g_i is parabolic. We claim that there exists a pair of indices $i, j \geq k$ such that g_i and g_j have distinct fixed points. Otherwise, assume that all parabolic elements g_i have the same fixed point p. Then $x_i'' \in B(G')/\Gamma$ for any $i \geq k$ where $G' = \mathrm{Stab}_\Gamma(p)$.

Since $\mu_i \cup \alpha_k$ is in the $(1+\delta)$-neighborhood of M^c, by δ-hyperbolicity of X we have that $x_k''x_i''$ is in $(1+2\delta)$-neighborhood of M^c for every $i > k$. By the definition of M^c, it follows that

$$x_k''x_i'' \cap N_\delta(\mathrm{Hull}(T_\varepsilon(G')))/\Gamma = \emptyset.$$

By the construction, the length $l(\alpha_i) \to \infty$ as $i \to \infty$. Hence, the length $l(\mu_i) \to \infty$ and the length $l(x_k''x_i'') \to \infty$ as $i \to \infty$. By Lemma 6.8, there exists points $z_i \in x_k''x_i''$ such that $z_i \in N_\delta(T_\varepsilon(G'))/\Gamma$ for sufficiently large i. Therefore,

$$x_k''x_i'' \cap N_\delta(\mathrm{Hull}(T_\varepsilon(G')))/\Gamma \neq \emptyset,$$

which is a contradiction.

We conclude that for some $i, j \geq k$, the parabolic elements g_i, g_j of Γ have distinct fixed points and, hence, generate a nonelementary subgroup of $\mathrm{Isom}(X)$. By Theorem 8.5, there exists a loxodromic element $\omega_n \in \langle g_i, g_j \rangle$ with the word length uniformly bounded by a constant C. By a similar argument as in Case (a), we obtain a closed geodesic w_n^* in M such that $d(x, w_n^*) \geq (n-1)R - D$.

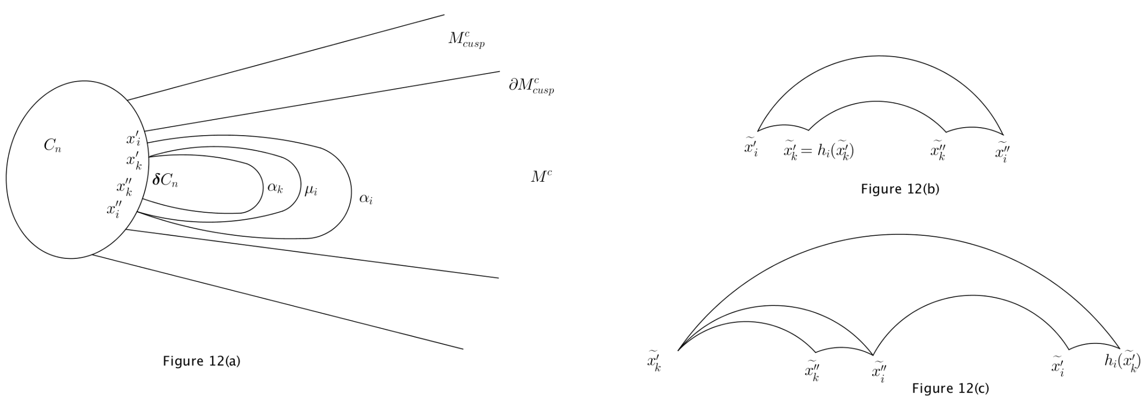

FIGURE 12

Case (c): We assume that for each i, the geodesic arc α_i connects $x_i' \in \boldsymbol{\delta}C_n$ to $x_i'' \in \boldsymbol{\delta}C_n$. The argument is similar to the one in Case (b). Since $\boldsymbol{\delta}C_n$ is compact, after passing to a further subsequence in (α_i), there exists $k \in \mathbb{N}$ such that for all $i \geq k$, $d(x_i', x_k') \leq 1$, $d(x_i'', x_k'') \leq 1$ and there are unique shortest geodesics $x_k'x_i'$ and $x_k''x_i''$. For each $i > k$ we define a geodesic $\mu_i = x_k'x_i''$ as in Case (b), see Figure 12(a). Then, by δ-hyperbolicity of X, each μ_i is in the $(\delta+1)$-neighborhood of α_i. Let $v_i = \alpha_k * x_k''x_i'' * (\mu_i)^{-1} \in \Omega(M, x_k')$ for $i > k$. By the construction $v_i \cap C_{n-1} = \emptyset$.

Let h_i denote the element in $\Gamma = \pi_1(M, x_k')$ represented by v_i. If h_i is loxodromic for some $i > k$, there exists a closed geodesic w_n^* contained in the D-neighborhood of v_i, cf. Case (a). In this situation, $d(x, w_n^*) \geq (n-1)R - D$.

Assume, therefore, that h_i are not loxodromic for all $i > k$.

We first claim that h_i is not the identity for all sufficiently large i. Let $\widetilde{x_k'}$ be a lift of x_k' in X. Pick points $\widetilde{x_k''}, \widetilde{x_i''}, \widetilde{x_i'}$ and $h_i(\widetilde{x_k'})$ in X such that $\widetilde{x_k'}\widetilde{x_k''}$ is a lift of α_k, $\widetilde{x_k''}\widetilde{x_i''}$ is a lift of $x''_k x''_i$, $\widetilde{x'_i}\widetilde{x''_i}$ is a lift of α_i and $\widetilde{x'_i}h_i(\widetilde{x'_k})$ is a lift of $x'_i x'_k$ as in Figure 12(b) and Figure 12(c). If $h_i = 1$, then $h_i(\widetilde{x'_k}) = \widetilde{x'_k}$ and $d(\widetilde{x'_i}, \widetilde{x''_i}) \leq 2 + d(\widetilde{x'_k}, \widetilde{x''_k})$ as in Figure 12(b). By construction, the length $l(\alpha_i) \to \infty$ as $i \to \infty$, so $d(\widetilde{x'_i}, \widetilde{x''_i}) \to \infty$. Thus for sufficiently large i, $h_i(\widetilde{x'_k}) \neq \widetilde{x'_k}$.

Now we assume that h_i are parabolic for all $i > k'$ where $k' > k$ is a sufficiently large number. We claim that there exists a pair of indices $i, j > k'$ such that h_i and h_j have distinct fixed points. Otherwise, all the parabolic elements h_i have the same fixed point p for $i > k'$. By the δ-hyperbolicity of X, $\widetilde{x'_k}h_i(\widetilde{x'_k}) \subseteq N_{3\delta+2}(\widetilde{x'_k}\widetilde{x''_k} \cup \widetilde{x''_i}\widetilde{x'_i})$. Since α_k and α_i lie outside of $B(G')/\Gamma$ where $G' = \mathrm{Stab}_\Gamma(p)$, $\widetilde{x'_k}h_i(\widetilde{x'_k})$ lies outside of $N_\delta(\mathrm{Hull}(T_\varepsilon(G')))$. Let $r_3 = d(\widetilde{x'_k}, \mathrm{Hull}(T_\varepsilon(G')))$. Then $d(h_i(\widetilde{x'_k}), \mathrm{Hull}(T_\varepsilon(G'))) = r_3$.

By the construction, the length $l(\alpha_i) \to \infty$ as $i \to \infty$. Then the length $l(\widetilde{x'_k}h_i(\widetilde{x'_k})) \to \infty$ as well. Observe that the points $\widetilde{x'_k}$ and $h_i(\widetilde{x'_k})$ lie on the boundary of $N_{r_3}(\mathrm{Hull}(T_\varepsilon(G)))$ for all $i > k'$. By Lemma 6.8, there exist points $\widetilde{z_i} \in \widetilde{x'_k}h_i(\widetilde{x'_k})$ such that $\widetilde{z_i} \in N_\delta(T_\varepsilon(G'))$ for sufficiently large i, which is a contradiction. Hence, for some $i > k', j > k'$, parabolic isometries h_i and h_j have distinct fixed points.

By Theorem 8.5, there exists a loxodromic element $\omega_n \in \langle h_i, h_j \rangle$ of the word length bounded by a uniform constant C. By a similar argument as in Case (a), there exists a closed geodesic w_n^* such that $d(x, w_n^*) \geq (n-1)R - D$.

Thus in all cases, for each n, the manifold M contains a closed geodesic w_n^* such that $d(x, w_n^*) \geq (n-1)R - D$. The sequence of closed geodesics $\{w_n^*\}$, therefore, escapes every compact subset of M. □

10. Continuum of nonconical limit points

In this section, using the generalized Bonahon theorem in Section 9, for each geometrically infinite discrete torsion-free subgroup $\Gamma < \mathrm{Isom}(X)$ we find a set of nonconical limit points with cardinality of continuum. This set of nonconical limit points is used to prove Theorem 1.5.

Theorem 10.1. *If $\Gamma < \mathrm{Isom}(X)$ is a geometrically infinite discrete torsion-free isometry subgroup, then the set of nonconical limit points of Γ has cardinality of continuum.*

Proof. The proof is inspired by Bishop's construction of nonconical limit points of geometrically infinite Kleinian groups in the 3-dimensional hyperbolic space $\mathbb{H}^3$ [6, Theorem 1.1]. Let $\pi : X \to M = X/\Gamma$ denote the covering projection. Pick a point $\tilde{x} \in X$ and let $x := \pi(\tilde{x})$. If Γ is geometrically infinite, by the generalized Bonahon's theorem in Section 9, there exists a sequence of oriented closed geodesics (λ_i) in M which escapes every compact subset of M, i.e.

$$\lim_{i\to\infty} d(x, \lambda_i) = \infty.$$

Let L be the constant as in Proposition 7.2 when $\theta = \pi/2$. Without loss of generality, we assume that $d(x, \lambda_1) \geq L$ and the minimal distance between any consecutive pair of geodesics λ_i, λ_{i+1} is at least L. For each i, let l_i denote the length of the closed geodesic λ_i and let m_i be a positive integer such that $m_i l_i > L$.

We then pass to a subsequence in (λ_i) as in Lemma 4.1 (retaining the notation (λ_i) for the subsequence) such that there exists a sequence of geodesic arcs $\mu_i := x_i^+ x_{i+1}^-$ meeting λ_i, λ_{i+1} orthogonally at its end-points, such that

$$\lim_{i\to\infty} d(x, \mu_i) = \infty.$$

Let D_i denote the length of the shortest positively oriented arc of λ_i connecting x_i^- to x_i^+. We let μ_0 denote the shortest geodesic in M connecting x to x_1^-.

We next construct a family of piecewise geodesic paths γ_τ in M starting at x such that the geodesic pieces of γ_τ are the arcs μ_i above and arcs ν_i whose images are contained in λ_i and have the same orientation: Each ν_i wraps around λ_i certain number of times and connects x_i^- to x_i^+. More formally, we define a map $\mathscr{P} : \mathbb{N}^\infty \to P(M)$ where $\mathbb{N}^\infty$ is the set of sequences of positive integers and $P(M)$ is the space of paths in M as follows:

$$\mathscr{P} : \tau = (t_1, t_2, \cdots, t_i, \cdots) \mapsto \gamma_\tau = \mu_0 * \nu_1 * \mu_1 * \nu_2 * \mu_2 * \cdots * \nu_i * \mu_i * \cdots$$

where the image of the geodesic arc ν_i is contained in λ_i and has length

$$l(\nu_i) = t_i m_i l_i + D_i.$$

Observe that for $i \geq 1$, the arc μ_i connects λ_i and λ_{i+1} and is orthogonal to both, with length $l(\mu_i) \geq L$ and ν_i starts at x_i^- and ends at x_i^+ with length $l(\nu_i) \geq L$.

For each γ_τ, we have a canonical lift $\tilde{\gamma}_\tau$ in X, which is a path starting at $\tilde{x}$. We will use the notation $\tilde{\mu}_i, \tilde{\nu}_i$ for the lifts of the subarcs μ_i, ν_i respectively, see Figure 13(a, b). By the construction, each γ_τ has the following properties:

(1) Each geodesic piece of $\tilde{\gamma}_\tau$ has length at least L.
(2) Adjacent geodesic segments of $\tilde{\gamma}_\tau$ make the angle equal to $\pi/2$ at their common endpoint.
(3) The path $\gamma_\tau : [0, \infty) \to M$ is a proper map.

By Proposition 7.2, $\tilde{\gamma}_\tau$ is a $(2L, 4L+1)$-quasigeodesic. Hence, there exists a limit

$$\lim_{t \to \infty} \tilde{\gamma}_\tau(t) = \tilde{\gamma}_\tau(\infty) \in \partial_\infty X,$$

such that the Hausdorff distance between $\tilde{\gamma}_\tau$ and $x\tilde{\gamma}_\tau(\infty)$ is bounded by a uniform constant C, depending only on L and κ.

We claim that each $\tilde{\gamma}_\tau(\infty)$ is a nonconical limit point. Observe that $\tilde{\gamma}_\tau(\infty)$ is a limit of loxodromic fixed points, so $\tilde{\gamma}_\tau(\infty) \in \Lambda(\Gamma)$. Let γ_τ^* be the projection of $x\tilde{\gamma}_\tau(\infty)$ under π. Then the image of γ_τ^* is uniformly close to γ_τ. Since γ_τ is a proper path in M, so is γ_τ^*. Hence, $\tilde{\gamma}_\tau(\infty)$ is a nonconical limit point of Γ.

We claim that the set of nonconical limit points $\tilde{\gamma}_\tau(\infty)$, $\tau \in \mathbb{N}^\infty$, has the cardinality of continuum. It suffices to prove that the map

$$\mathscr{P}_\infty : \tau \mapsto \tilde{\gamma}_\tau(\infty)$$

is injective.

Let $\tau = (t_1, t_2, \cdots, t_i)$ and $\tau' = (t_1', t_2', \cdots, t_i', \cdots)$ be two distinct sequences of positive integers. Let n be the smallest positive integer such that $t_n \neq t_n'$. Then the paths $\tilde{\gamma}_\tau, \tilde{\gamma}_{\tau'}$ can be written as concatenations

$$\tilde{\alpha}_\tau \star \tilde{\nu}_n * \tilde{\beta}_\tau, \quad \tilde{\alpha}_\tau \star \tilde{\nu}_n' * \tilde{\beta}_{\tau'},$$

where $\tilde{\alpha}_\tau$ is the common initial subpath

$$\tilde{\mu}_0 * \tilde{\nu}_1 * \tilde{\mu}_1 * \tilde{\nu}_2 * \tilde{\mu}_2 * \cdots * \tilde{\nu}_{n-1} * \tilde{\mu}_{n-1}.$$

The geodesic segments $\tilde{\nu}_n, \tilde{\nu}_n'$ have the form

$$\tilde{\nu}_n = \tilde{x}_n^- \tilde{x}_n^+,$$

$$\tilde{\nu}_n' = \tilde{x}_n^- \tilde{x'}_n^+.$$

Consider the bi-infinite piecewise geodesic path

$$\sigma := \tilde{\beta}_\tau^{-1} \star \tilde{x}_n^+ \tilde{x'}_n^+ \star \tilde{\beta}_{\tau'}$$

in X. Each geodesic piece of the path has length at least L and adjacent geodesic segments of the path are orthogonal to each other. By Proposition 7.2, σ is a complete $(2L, 4L+1)$-quasigeodesic and, hence, it is backward/forward asymptotic to distinct points in $\partial_\infty X$. These points in $\partial_\infty X$ are respectively $\tilde{\gamma}_\tau(\infty)$ and $\tilde{\gamma}_{\tau'}(\infty)$. Hence, the map $\mathscr{P}_\infty$ is injective. We conclude that the endpoints of the piecewise geodesic paths $\tilde{\gamma}_\tau$ yield a set of nonconical limit points of Γ which has the cardinality of continuum.

Remark 10.2. This proof is a simplification of Bishop's argument in [6], since, unlike [6], we have orthogonality of the consecutive segments in each γ_τ.

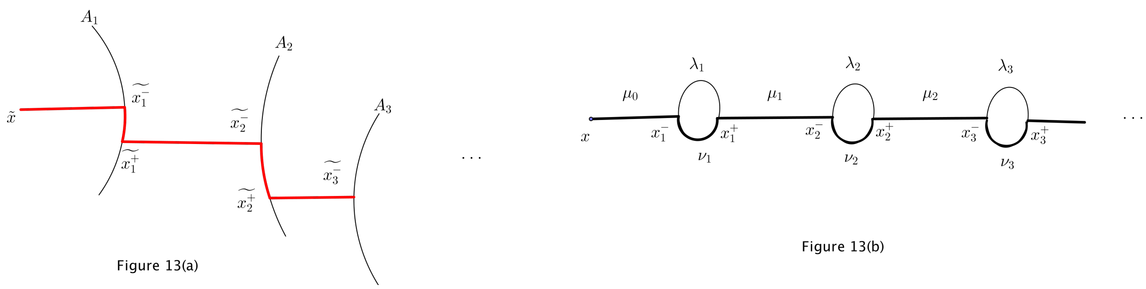

FIGURE 13. Here A_i denotes a geodesic in X covering the loop λ_i, $i \in \mathbb{N}$.

Here is an alternative way to see that the image of $\mathscr{P}_\infty$ has the cardinality of continuum. Let $\mathfrak{G}_b$ be the set consisting of all infinite piecewise geodesic paths $\tilde{\gamma}_\tau$, $\tau \in \mathbb{N}^\infty$.

As above, for each $n \in \mathbb{N}$, we represent $\tilde{\gamma}_\tau$ as the concatenation,

$$\tilde{\alpha}_\tau \star \tilde{\nu}_n \star \tilde{\beta}_\tau, \tilde{\nu}_n = \tilde{x}_n^- \tilde{x}_n^+.$$

We define a new piecewise geodesic path $\tilde{\gamma}_{\tau,n}$ by replacing $\tilde{\nu}_n \star \tilde{\beta}_\tau$ with the unique geodesic ray $\tilde{x}_n^- \xi_n$ containing $\tilde{\nu}_n$:

$$\tilde{\gamma}_{\tau,n} := \tilde{\alpha}_\tau \star \tilde{x}_n^- \xi_n.$$

Let $\mathfrak{G}_a$ denote the set of such paths $\tilde{\gamma}_{\tau,n}, \tau \in \mathbb{N}^\infty, n \in \mathbb{N}$. As usual, we parameterize all paths by their arclength. We obtain a subset $\mathfrak{G} = \mathfrak{G}_a \cup \mathfrak{G}_b$ in the space of paths $P(X)$ equipped with the topology of uniform convergence on compacts. It is clear that the subset $\mathfrak{G}_b$ is dense in $\mathfrak{G}$: For $\tau = (t_i)$,

$$\tilde{\gamma}_{\tau,n} = \lim_{k\to\infty} \mathscr{P}(t_1, ..., t_{n-1}, k, t_{n+1}, ...). \tag{10.1}$$

Similarly,

$$\tilde{\gamma}_\tau = \lim_{n\to\infty} \tilde{\gamma}_{\tau,n}. \tag{10.2}$$

Lemma 10.3. *$\mathfrak{G}$ is closed in $P(X)$.*

Proof. By the denseness of $\mathfrak{G}_b$ in $\mathfrak{G}$, it suffices to show that every sequence in $\mathfrak{G}_b$, after extraction, converges to an element of $\mathfrak{G}$. We equip $\mathbb{N}^\infty$ with the product topology; then the map

$$\mathscr{P} : \mathbb{N}^\infty \to \mathfrak{G}_b$$

is continuous. The image of the product of finite subintervals in $\mathbb{N}$ under $\mathscr{P}$ is then compact. Therefore, consider a sequence $\tau_j = (t_{ij}) \in \mathbb{N}^\infty$ for which there exists the smallest integer n such that

$$\sup\{t_{nj}, j \in \mathbb{N}\} = \infty.$$

After extraction, we may assume that the first $n-1$ coordinates of this sequence are constant, equal $(t_1, ..., t_{n-1})$ and that

$$\lim_{j\to\infty} t_{nj} = \infty.$$

Then

$$\lim_{j\to\infty} \mathscr{P}(\tau_j) = \gamma_{\tau,n} \in \mathfrak{G}_a. \quad \square$$

Each path $\alpha \in \mathfrak{G}$ is a $(2L, 4L+1)$-quasigeodesic. Since the image of the geodesic ray $\alpha^* = \tilde{x}\alpha(\infty)$ is uniformly close to that of α, it follows that the map $\alpha \mapsto \alpha(\infty)$ is continuous. Hence, the set of limit points

$$\mathfrak{G}(\infty) = \{\alpha(\infty) : \alpha \in \mathfrak{G}\}$$

is closed, hence, compact.

Next, we show that $\mathfrak{G}(\infty)$ is perfect, i.e. has no isolated points. For each $\alpha = \tilde{\gamma}_{\tau,n} \in \mathfrak{G}_a$, the ideal point $\alpha(\infty)$ is a loxodromic fixed point (it is one of the ideal endpoints of a geodesic in X projecting to the closed geodesic λ_n in M). At the same time, according to (10.1), $\alpha(\infty)$ is the limit of nonconical limit points $\beta_k(\infty)$ for some sequence $\beta_k \in \mathfrak{G}_b$. Hence, $\alpha(\infty)$ is not an isolated point of $\mathfrak{G}(\infty)$. Similarly, for every $\tau \in \mathbb{N}^\infty$, in view of (10.2), the nonconical limit point $\tilde{\gamma}_\tau(\infty)$ is the limit of conical limit points $\tilde{\gamma}_{\tau,n}(\infty)$. Hence, $\tilde{\gamma}_\tau(\infty)$ is not isolated in $\mathfrak{G}(\infty)$ either. Thus $\mathfrak{G}(\infty)$ also has no isolated points. Therefore, $\mathfrak{G}(\infty)$ is a nonempty compact metrizable perfect space, hence, has the cardinality of continuum. By the construction, $\mathfrak{G}_a(\infty)$ is countable, and, therefore, $\mathfrak{G}_b(\infty)$ has the cardinality of continuum. $\square$

Proof of Theorem 1.5: The implication (1) $\Rightarrow$ (2) (a generalization of Bonahon's theorem) is the main result of Section 9. The implication (2) $\Rightarrow$ (3) is the content of Theorem 10.1. It remains to prove that (3) $\Rightarrow$ (1). If Γ is geometrically finite, by Theorem 1.4 $\Lambda(\Gamma)$ consists of conical limit points and bounded parabolic fixed points. Since Γ is discrete, it is at most countable; therefore, the set of fixed points of parabolic elements of Γ is again at most countable. If $\Lambda(\Gamma)$ contains a subset of nonconical limit points of cardinality of continuum, we can find a point in the limit set which is neither a conical limit point nor a parabolic fixed point. It follows that Γ is geometrically infinite. $\square$

Proof of Corollary 1.6: If Γ is geometrically finite, by Theorem 1.4, $\Lambda(\Gamma)$ consists of conical limit points and bounded parabolic fixed points. Now we prove that if $\Lambda(\Gamma)$ consists of conical limit points and parabolic fixed points, then Γ is geometrically finite. Suppose that Γ is geometrically infinite. By Theorem 1.5, there is a set of nonconical limit points with cardinality of continuum. Since the set of parabolic fixed points is at most countable, there exists a limit point in $\Lambda(\Gamma)$ which is neither a conical limit point nor a parabolic fixed point. This contradicts to our assumption. Hence, Γ is geometrically finite. $\square$

References

[1] L. V. Ahlfors, *Fundamental polyhedrons and limit point sets of Kleinian groups.* Proc. Nat. Acad. Sci. U.S.A. 55 (1966), 251–254.

[2] W. Ballmann, "Lectures on spaces of nonpositive curvature." With an appendix by Misha Brin. DMV Seminar, vol. 25, Birkhäuser Verlag, Basel, 1995.

[3] W. Ballmann, M. Gromov and V. Schroeder, "Manifolds of nonpositive curvature." Progr. Math. 61, Birkhäuser, Boston, 1985.

[4] A. Beardon, "The geometry of discrete groups." Springer-Verlag, Berlin-New York 1983.

[5] A. Beardon and B. Maskit, *Limit sets of Kleinian groups and finite sided fundamental polyhedra.* Acta Math. 132(1974), 1–12.

[6] C. J. Bishop, *On a theorem of Beardon and Maskit.* Annales Academiae Scientiarum Fennicae, Mathematica 21 (1996) 383–388.

[7] F. Bonahon, *Bouts des varietes hyperboliques de dimension 3*. Ann. Math., 124 (1986) 71–158.
[8] B. H. Bowditch, *Geometrical finiteness for hyperbolic groups*. J. Funct. Anal. 113 (1993), 245–317.
[9] B. H. Bowditch, *Geometrical finiteness with variable negative curvature*. Duke Math. J. 77 (1995), no. 1, 229–274.
[10] J. Cheeger and D. G. Ebin, " Comparison Theorem in Riemannian Geometry." North-Holland Math. Lib. 9, North-Holland, Amsterdam, 1975.
[11] C. Druţu and M. Kapovich, "Geometric group theory.", To appear in AMS series Colloquium Publications, 2018.
[12] J. L. Fernández and M. V. Melián, *Escaping geodesics of Riemannian surfaces*, Acta Math. 187 (2001), no. 2, 213–236.
[13] A.Hatcher, "Algebraic Topology." Cambridge University Press, 2002.
[14] M. Kapovich, B. Liu, *Geometrically infinite discrete isometry subgroups with torsion in* $\mathrm{Isom}(\mathbb{H}^n)$, in preparation.
[15] A. Marden, *The geometry of finitely generated Kleinian groups*. Ann. of Math. (2) 99 (1974), 383–462.
[16] L. Montejano, *A new topological Helly Theorem and some transversal results*. Discrete Comput. Geom. 52 (2014), no. 2, 390–398.
[17] F. Paulin, *On the critical exponent of a discrete group of hyperbolic isometries*. Differential Geometry and its Application 7 (1997), 231–236.
[18] J. Ratcliffe, "Foundations of hyperbolic manifolds." Graduate Texts in mathematics, 149 (2nd ed.), Berlin, New York: Springer-Verlag.
[19] W. P. Thurston, "The geometry and topology of 3-manifolds." Chapter 4, revised notes, 1982.

M.K.: Department of Mathematics, UC Davis, One Shields Avenue, Davis CA 95616, USA

KIAS, 85 Hoegiro, Dongdaemun-gu, Seoul 130-722, South Korea
E-mail address: kapovich@math.ucdavis.edu

B.L.: Department of Mathematics, UC Davis, One Shields Avenue, Davis CA 95616, USA
E-mail address: bxliu@math.ucdavis.edu