BRANCHED COVERS BETWEEN SPHERES AND POLYGONAL INEQUALITIES IN SIMPLICIAL TREES

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We consider 2-dimensional spheres Σ equipped with metrics with conical singularities (with the angles $\alpha_1 = 2\pi\theta_1, ..., \alpha_n = 2\pi\theta_n$ and points $x_1, ..., x_n$) and curvature $+1$ away from the singular points. Every such metric has a (multivalued) developing map

$$\Sigma' := \Sigma - \{x_1, ..., x_n\} \to S^2$$

and the associated holonomy representation $\pi_1(\Sigma') \to SU(2)$. Singular metrics on spheres with conical singularities were studied by many authors, in particular, see [9, 2, 7, 8], but the problem of necessary and sufficient conditions for existence of metrics with the given set of angles is open. In the case when the holonomy representation is nontrivial, some nontrivial restrictions on the cone angles (taken modulo 2π) come from *triangle inequalities* for n-gons in $SU(2)$, see [8].

The cases when all the angles are nonintegral is completely analyzed in the combination of [8] and [1]. More precisely, setting $b_i := \theta_i - 1, \mathbf{b} = (b_1, ..., b_n)$, it is proven in [8] that the conditions:

1. $\sum_{i=1}^n b_i > -2$ (the Gauss-Bonnet condition)
2. $dist_{\ell_1}(\mathbf{b}, \mathbb{Z}^n_{odd}) \geq 1$ (the spherical polygonal inequalities)

are necessary for the existence of a singular metric on S^2 with the prescribed cone angles, while (1) and the condition (2) with the strict inequality is also sufficient for the existence of such a metric. (Here $\mathbb{Z}^n_{odd}$ is the subset in the integer lattice consisting of vectors with odd ℓ_1-norm.)

Furthermore, it is proven in [1] that (for $n \geq 3$) there are no metrics on S^2 in the equality case in (2) provided that all the angles are "non-integer" (more precisely, when $\theta_i \notin \mathbb{Z}$ fo all i).

In this note we restrict to the "complementary" special case when all the cone angles α_i are "integer", i.e. $\alpha_i = 2\pi\theta_i$, $\theta_i \in \mathbb{N}$. Every such metric has trivial holonomy and, hence, a single-valued developing map $\Sigma' \to S^2$ which extends to a branched cover $\Sigma \to S^2$ with ramification points at the singular points of the metric, i.e. points where $\theta_i \geq 2$. Conversely, every branched cover $S^2 \to S^2$ determines a singular spherical metric with integer conical singularities via pull-back of the standard metric on S^2.

The novelty in this case, as we will see below, is the appearance of polygonal inequalities in simplicial trees which turn out to be necessary and sufficient for the existence of a singular metric.

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Let $f : \Sigma \to S^2$ be a degree d branched cover from a closed connected surface to the sphere. Assume that $x_1, ..., x_n \in \Sigma$ are the ramification points and $y_1, ..., y_n$ (possibly, some equal) their images under f, the branch-points. By perturbing f slightly we can always achieve that f is *simple*, i.e. all the points y_i are distinct, which we will assume from now on. At each x_i the map f has the local degree $d_i \le d$, where $d_i = \theta_i$ in the notation used earlier. We have $b_i := d_i - 1 \ge 1$, and set $b := d - 1$. We will refer to the vector $\mathbf{b} = (b_1, ..., b_n)$ as the *ramification datum* of f.

Then (assuming that f is simple) the Riemann-Hurwitz formula reads:

$$\chi(\Sigma) = d\chi(S^2) - \sum_{i=1}^{n} b_i.$$

If $\Sigma \cong S^2$, then the formula becomes

$$2b = 2(d-1) = \sum_{i=1}^{n} b_i.$$

In particular, $\sum_{i=1}^{n} b_i = 2b$ is even (the "integrality condition" I) and $b_i \le \frac{1}{2}\sum_{i=j}^{n} b_j$ (the "polygons inequality" P). This simple observation is well-known.

Remark 1. *The integrality condition in our setting is equivalent to the condition*

$$dist_{\ell_1}(\mathbf{b}, \mathbb{Z}^n_{odd}) = 1$$

discussed above.

The following interpretation of the conditions I and P seems to be new in the context of branched covers. Interpreting b_i's as side-lengths of a polygon, the iterated triangle inequalities are necessary and sufficient conditions for existence of a Euclidean n-gon with the given side-lengths, see e.g. [6]. From the Euclidean viewpoint, the integrality condition may appear strange. Consider instead n-gons in simplicial trees T, where every edge has unit length and each vertex has valence ≥ 3. We say that an n-gon in T is *simplicial* if its vertices belong to the vertex set of T. Then the condition (I & P) is necessary and sufficient for the existence of simplicial polygons with the side-lengths $b_1, ..., b_n$ in T. See Lemma 9.3 in [3] for a simple self-contained proof and [5] for a much more general statement in the case of polygons in Euclidean buildings. This interpretation will be used in our proof below.

We note also that the condition (I & P) has yet another interesting interpretation, which we, however, will not be using: One can interpret the integers b_i as the *highest weights* of irreducible finite-dimensional representations $V(b_i)$ of the group $SL(2, \mathbb{C})$ (equivalently, $SU(2)$). Then (I & P) is necessary and sufficient for existence of nonzero $SL(2, \mathbb{C})$-invariant vectors in the n-fold tensor product

$$V(b_1) \otimes ... \otimes V(b_n),$$

see e.g. [4]. It is unclear if this connection to the theory of polygons in Euclidean buildings and to the combinatorial representation theory is a mere coincidence, or has some deeper meaning.

Ignoring the integrality condition, we also observe that the condition P is necessary and sufficient for the existence of a weighted configuration $(z_1, ..., z_n)$ on $\mathbb{C}P^1$ (with the weights $b_1, ..., b_n$) which is *semistable* (in Mumford's sense) with respect to the action of $PSL(2, \mathbb{C})$, cf [6, 5, 4]. Again, it is unclear how to relate this directly to the existence of branched covers between 2-spheres.

Theorem 2. *The condition (I & P) is necessary and sufficient for the existence of a branched-cover $S^2 \to S^2$ with the given ramification datum $\mathbf{b} = (b_1, ..., b_n)$.*

Proof. If $n = 2$ the conditions amount to the requirement that $b_1 = b_2$ and the existence is given by the map $z \mapsto z^d$ $(d = b_1 = b_2 + 1)$ of the Riemann sphere.

We first prove this claim for $n = 3$. One can derive this from Eremenko's paper [2], but we will give a direct proof by induction on $\max(b_1, b_2, b_3)$ for the sake of completeness. For the induction argument it is convenient to allow some b_i's to vanish, hence, the first case is that of the triple $(0, 0, 0)$ in which case $f = id$, the identity map.

Lemma 3. *The statement of the theorem holds when $n = 3$.*

Proof. We will be constructing maps of spheres by defining singular metrics on the 2-sphere Σ with the cone angles of the form $2\pi(b_i + 1)$. The singular points x_1, x_2, x_3 of the sphere will be located on the equator of the standard sphere and we will be adding "lunes" to the singular metrics with the tips at x_1, x_2, x_3 and the angles $2\pi, 2\pi$. For each singular metric constructed this way, there will be smooth geodesic arcs connecting x_i, x_j which map injectivielty under the developing map, which allows us to add lunes inductively.

Assume that a singular metric exists for all triples (b'_1, b'_2, b'_3) with $b'_1 + b'_2 + b'_3 \le \beta$. Consider now a triple (b_1, b_2, b_3) satisfying the conditions (I & P), such that $b_1 + b_2 + b_3 = \beta + 1$. Assume that

$$b_1 \le b_2 \le b_3.$$

We claim that the triple $(b'_1, b'_2, b'_3) := (b_1, b_2 - 1, b_3 - 1)$ again satisfies (I & P). The integrality part is clear. It is also clear that $b'_2 \le b_1 + b'_3$ and $b'_3 \le b_1 + b'_2$. If $b_1 \le b_2 < b_3$ then the inequality $b_1 \le b'_2 + b'_3$ follows from $b_1 \le b'_3$. If $a = b_1 = b_2 = b_3 > 1$ then $a \le 2(a - 1) = b'_2 + b'_3$ unless $a = 1$. But the triple $(1, 1, 1)$ is ruled out by the integrality condition. We then realize the triple (b_1, b_2, b_3) by a singular spherical metric by taking the singular sphere Σ realizing the triple (b'_1, b'_2, b'_3) and adding to it the lune with the tips x_2, x_3. $\square$

We now proceed to the general case $n \ge 4$. Given an n-tuple $\mathbf{b} = (b_1, ..., b_n)$ satisfying (I & P), we find a simplicial n-gon $A_1...A_n$ in a tree T with the side-lengths $|A_iA_{i+1}| = b_i$, i is taken modulo n. We then draw a diagonal in this polygon, say, A_1A_{n-1}. Setting $a := |A_1A_{n-1}|$ we obtain two new tuples

$$(b_1, ..., b_{n-2}, a), (a, b_{n-1}, b_n)$$

both satisfying I & P. By the induction assumption, both tuples correspond to singular spheres

$$\Sigma_1 := \Sigma(x_1, ..., x_{n-2}, z_1), \Sigma_2 := \Sigma(x_{n+1}, x_n, z_2).$$

Let $f_i : \Sigma_i \to S^2, i = 1, 2$, be the corresponding *simple* branched covers. By postcomposing these covers with suitable Moebius transformations, we can assume that $\pi/2$-balls B_1, B_2 in Σ_1, Σ_2 centered at z_1, z_2 contain no singular points other than their centers and are bounded by regular geodesics of lengths $2\pi(a + 1)$. We now cut out the interiors of the balls B_1, B_2 from the respective surfaces and glue $\Sigma_1 - int(B_1), \Sigma_2 - int(B_2)$ isometrically along their boundaries. The resulting singular sphere has the cone angles

$$2\pi(b_1 + 1), ..., 2\pi(b_{n-2} + 1), 2\pi(b_{n-1} + 1), 2\pi(b_n + 1)$$

at the singular points $x_1, ..., x_n$ and no singularities otherwise. □

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