Sullivan's structural stability of expanding group actions

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Abstract

In his 1985 paper Sullivan sketched a proof of his structural stability theorem for group actions satisfying certain expansion-hyperbolicity axioms. We generalize the theorem by weakening these axioms substantially, while adding more details to Sullivan's original proof. We then present a number of examples satisfying Sullivan's axioms, such as Anosov subgroups of Lie groups as well as hyperbolic and non-hyperbolic groups acting on metric spaces.

Contents

1 Introduction **2**

2 Notation and preliminaries **5**
- 2.1 Topological dynamics . . . 6
- 2.2 Coarse geometry . . . 6

3 Sullivan's structural stability theorem **9**
- 3.1 S-expansion condition . . . 9
- 3.2 Expansion enables encoding . . . 11
- 3.3 S-hyperbolicity condition . . . 16
- 3.4 S-hyperbolicity implies structural stability . . . 18
- 3.5 Toy examples . . . 19

4 Proof of Sullivan's theorem **20**
- 4.1 Specifying small perturbations ρ' . . . 21
- 4.2 Definition of ϕ . . . 22
- 4.3 ϕ is well-defined . . . 24
- 4.4 ϕ is equivariant . . . 26
- 4.5 ϕ is continuous . . . 27
- 4.6 ϕ is injective . . . 28
- 4.7 ϕ depends continuously on ρ' . . . 28
- 4.8 ρ' is S-expanding . . . 29
- 4.9 ρ' is again uniformly S-hyperbolic . . . 29

5 Actions of hyperbolic groups **30**
5.1 S-expansion implies S-fellow-traveling . . . 30
5.2 S-hyperbolic actions of hyperbolic groups . . . 31

6 Anosov subgroups **33**
6.1 Definition . . . 33
6.2 S-hyperbolicity and stability for Anosov subgroups . . . 35
6.3 Historical remarks on stability . . . 37

7 Examples **37**
7.1 S-expansion does not imply S-hyperbolicity . . . 37
7.2 S-hyperbolic actions of hyperbolic groups . . . 38
7.3 S-hyperbolic actions of non-hyperbolic groups . . . 41
7.4 Embedding into Lie group actions on homogeneous manifolds . . . 42
7.5 Algebraically stable but not convex-cocompact . . . 44

1 Introduction

In [Sul85, §9. Theorem II] Sullivan stated his structural stability theorem for group actions satisfying certain expansion-hyperbolicity axioms but presented only a sketch of a proof.

Our goal in this paper is two-fold. First, we provide a detailed proof of Sullivan's theorem, while weakening his expansion-hyperbolicity axioms so far as to retain the same conclusion; we use the name *S-hyperbolic actions* for group actions satisfying the weakened axioms. Second, we establish some basic properties of S-hyperbolic actions and explore examples of such actions. Examples include actions of some non-hyperbolic groups, as well as actions of hyperbolic groups with invariant subsets either homeomorphic to Gromov boundaries (such as Anosov actions) or, more generally, admitting equivariant continuous quasi-open maps to Gromov boundaries.

Let us summarize the contents of Sullivan's paper [Sul85] in order to see the context where his structural stability theorem appears. Let $\Gamma < \mathrm{PSL}(2,\mathbb{C})$ be a finitely generated, non-solvable, non-rigid, non-relatively-compact and torsion-free group of conformal transformations of the Riemann sphere $\mathsf{P}^1(\mathbb{C})$. Sullivan showed that the following are equivalent:

(1) the subgroup $\Gamma < \mathrm{PSL}(2,\mathbb{C})$ is convex-cocompact;
(2) the Γ-action on the limit set $\Lambda \subset \mathsf{P}^1(\mathbb{C})$ satisfies the expansion-hyperbolicity axioms;
(3) this action is structurally stable in the sense of C^1-dynamics;
(4) the subgroup $\Gamma < \mathrm{PSL}(2,\mathbb{C})$ is algebraically stable.

Here $\Gamma < \mathrm{PSL}(2,\mathbb{C})$ is said to be *structurally stable in the sense of C^1-dynamics* if, for every representation $\rho : \Gamma \to \rho(\Gamma) = \Gamma' < \mathsf{Diff}^1(\mathsf{P}^1(\mathbb{C}))$ sufficiently close to the identity embedding, there exists a Γ'-invariant compact subset $\Lambda' \subset \mathsf{P}^1(\mathbb{C})$ and an equivariant homeomorphism $\phi : \Lambda \to \Lambda'$ also close to the identity embedding, and it is said to be *algebraically stable* if all representations $\Gamma \to \mathrm{PSL}(2,\mathbb{C})$ sufficiently close to the identity embedding are injective.

Without defining the expansion-hyperbolicity axioms for now, we note that the implication ($4 \Rightarrow 1$) is the main result (Theorem A) of the paper [Sul85]. The implications ($1 \Rightarrow 2$) and ($2 \Rightarrow 3$) are his Theorems I and II, respectively, and ($3 \Rightarrow 4$) is immediate. For groups with torsion the implication ($4 \Rightarrow 1$) is false (see Example 7.12) but other implications still hold. For the rest of the paper we will allow groups with torsion.

It is Sullivan's Theorem II that we refer to as Sullivan's structural stability theorem in the present paper. In fact, its statement is more general than the implication ($2 \Rightarrow 3$) above.

Theorem ([Sul85, Theorem II]). *Consider a group action $\Gamma \to \mathrm{Diff}^1(M)$ on a Riemannian manifold M with a compact invariant subset $\Lambda \subset M$. If the action satisfies the expansion-hyperbolicity axioms, then it is structurally stable in the sense of C^1-dynamics.*

In the first part of the paper, we generalize this theorem as follows. We adopt Sullivan's remark that the theorem generalizes to actions on metric spaces. So we consider a continuous action of a finitely generated group Γ on a metric space M with an invariant compact subset $\Lambda \subset M$, no point of which is isolated in M; the subset Λ plays the role of a "limit set" of Γ. We denote such an action by

$$\Gamma \to \mathrm{Homeo}(M; \Lambda),$$

topologize the set of such actions accordingly, and talk about structural stability *in the sense of Lipschitz dynamics* (see Section 3.4). We also weaken Sullivan's original expansion-hyperbolicity axioms to what we call S-expansion and S-hyperbolicity conditions (Definitions 3.3 and 3.21), respectively; by definition, the latter implies the former.

1.1 Theorem (Theorem 3.27(1)). *If an action $\Gamma \to \mathrm{Homeo}(M; \Lambda)$ is S-hyperbolic, then the action is structurally stable in the sense of Lipschitz dynamics.*

See Theorem 3.27 for the full statement. The proof of this theorem takes the whole Section 4, where we basically follow Sullivan's idea of proof, while filling in the greater details he sometimes left.

In the second part of the paper, we establish some basic properties of S-expanding and S-hyperbolic actions, and exhibit various examples of such actions.

In Sections 3.1 and 3.2 we discuss the S-expansion condition and explore the implications of the key Lemma 3.15. For instance, we obtain:

1.2 Theorem (Theorem 3.18). *If $\Gamma \to \mathrm{Homeo}(M; \Lambda)$ is an S-expanding action, then no point of Λ is a wandering point of the action.*

In Section 5 we focus on the case when the group Γ is hyperbolic and establish the following two basic results. By definition, a map between topological spaces is *nowhere constant* (resp. *quasi-open*) if the image of every non-empty open subset is not a singleton (resp. has non-empty interior).

1.3 Theorem (Definition 3.23 and Theorem 5.7). *Let Γ be a non-elementary hyperbolic group. If $\Gamma \to \mathrm{Homeo}(M; \Lambda)$ is an S-hyperbolic action, then there exists an equivariant continuous* coding map *$\pi : \Lambda \to \partial_\infty \Gamma$ to the Gromov boundary of Γ; the map π restricts to a quasi-open map on each minimal non-empty closed Γ-invariant subset $\Lambda_\mu \subset \Lambda$.*

In general, even for hyperbolic groups, S-expansion does not imply S-hyperbolicity. Nevertheless, we prove

1.4 Theorem (Theorem 5.1)**.** *Let* Γ *be a non-elementary hyperbolic group. Suppose that* $\Gamma \to \mathrm{Homeo}(M;\Lambda)$ *is an S-expanding action, for which there exists an equivariant continuous nowhere constant map* $f:\Lambda\to\partial_\infty\Gamma$*. Then the action is S-hyperbolic and the map* f *equals the coding map* π *(of the previous theorem).*

Convex-cocompact subgroups of rank one semi-simple Lie groups are classical objects with a very rich theory. As a natural analogue of convex-cocompact subgroups, Labourie [Lab06] introduced the notion of *Anosov subgroups* of higher rank semi-simple Lie groups G, and his definition was further developed by Guichard and Wienhard [GW12] to include all hyperbolic groups. Subsequently, Kapovich, Leeb and Porti [KLP17] provided new characterizations of Anosov subgroups investigating their properties from many different perspectives.

In Section 6, building upon the work in [KLP17] and Theorem 1.4, we give a (yet another) new characterization of Anosov subgroups in terms of S-expanding actions (Theorem 6.3). This characterization shows, among other things, that the action of any Anosov group on its flag limit set in the partial flag manifold G/P is S-hyperbolic. Thus, thanks to Theorem 1.1, we obtain the stability of Anosov groups in a broader context of group actions on metric spaces than those in [GW12, Theorem 5.13] and [KLP14, Theorems 1.11 and 7.36]. Based on this we also obtain an alternative proof for the openness of Anosov property in the representation variety (Corollary 6.5).

In Section 7, we present a number of examples of S-hyperbolic actions. Convex-cocompact Kleinian groups and, more generally, Anosov subgroups provide examples of S-hyperbolic actions of hyperbolic groups Γ, for which the invariant subsets Λ are equivariantly homeomorphic to the Gromov boundary $\partial_\infty\Gamma$ (via the coding map π). In contrast, in Section 7.2, we explore examples of S-hyperbolic actions of hyperbolic groups where the coding maps π are increasingly more complicated. The map π can be a covering map (Examples 7.3 and 7.4), it can be open but fail to be a local homeomorphism (Example 7.5), and it can even fail to be an open map (Example 7.6). On the other hand, in Section 7.3, we give examples of S-hyperbolic actions of *non-hyperbolic* groups: for instance, the direct product of hyperbolic groups admits an S-hyperbolic action (Example 7.8).

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2 Notation and preliminaries

The identity element of an abstract group will be denoted by e. We will use the following notation for the sets of non-negative integers and natural numbers:

$$\mathbb{N}_0 = \{0, 1, 2, \ldots\} \quad \text{and} \quad \mathbb{N} = \{1, 2, 3, \ldots\}.$$

We will follow the Bourbaki convention that neighborhoods of a point a (resp. a subset A) in a topological space X need not be open but are only required to contain an open subset which, in turn, contains a (resp. A). In particular, a topological space X is locally compact if and only if every point in X admits a neighborhood basis consisting of compact subsets of X.

A topological space is called *perfect* if it has no isolated points and has cardinality ≥ 2.

A map between topological spaces is *nowhere constant* if the image of every open subset is not a singleton. A map is said to be *open* if it sends open sets to open sets. A map $f : X \to Y$ is *open at a point* $x \in X$ if it sends every neighborhood of x to a neighborhood of $f(x)$. We let O_f denote the subset of X consisting of points where f is open. Thus, a map f is open if and only if $O_f = X$.

A map $f : X \to Y$ is said to be *quasi-open* (or *quasi-interior*) if for every subset $A \subset X$ with non-empty interior, the image $f(A)$ has non-empty interior in Y. If $f : X \to Y$ is a continuous map between locally compact metrizable spaces then it is quasi-open if and only if the subset $O_f \subset X$ is comeagre (that is, its complement is a countable union of nowhere dense subsets). For instance, the map $\mathbb{R} \to \mathbb{R}, x \mapsto x^2$, is quasi-open but not open. A more interesting example of a (non-open) quasi-open map is a *Cantor function* $f : C \to [0, 1]$, which is a continuous surjective monotonic function from a Cantor set $C \subset \mathbb{R}$. It has the property that $x_1 < x_2$ implies $f(x_1) < f(x_2)$ unless x_1, x_2 are boundary points of a component of $\mathbb{R} - C$. Thus, $C - O_f$ is the countable subset consisting of boundary points of components of $\mathbb{R} - C$.

Let (X, d) be a metric space. Given $x \in X$ and $r > 0$, the open (resp. closed) r-ball centered at x is denoted by $B_r(x)$ (resp. $\overline{B}_r(x)$). Given a subset $\Lambda \subset X$, its open (resp. closed) r-neighborhood is denoted by $N_r(\Lambda)$ (resp. $\overline{N}_r(\Lambda)$). A *Lebesgue number* of an open cover $\mathcal{U}$ of Λ is defined to be a number $\Delta > 0$ such that, for every $x \in \Lambda$, the Δ-ball $B_\Delta(x)$ is contained in some member of $\mathcal{U}$; we denote

$$\Delta_{\mathcal{U}} = \sup\{\Delta \mid \Delta \text{ is a Lebesgue number of } \mathcal{U}\}.$$

For a subset $U \subset X$ and $r > 0$ we define

$$U^r = \{x \in U \mid B_r(x) \subset U\} \subset U.$$

A sequence of subsets $W_i \subset X$ is said to be *exponentially shrinking* if the diameters of these subsets converge to zero exponentially fast, that is, there exist constants $A, C > 0$ such that

$$\operatorname{diam}(W_i) \leq A\, e^{-Ci}$$

for all i.

If X is a Riemannian manifold and Φ is a diffeomorphism of X, the *expansion factor* of Φ at $x \in X$ is defined as

$$\epsilon(\Phi, x) = \inf_{0 \neq v \in T_x X} \frac{\|D_x \Phi(v)\|}{\|v\|}. \tag{2.1}$$

We now present some dynamical and geometric preliminaries to be used later. For more details we refer the readers to [BH99] and [DK18].

2.1 Topological dynamics

A continuous action $\Gamma \times Z \to Z$ of a topological group on a topological space is *minimal* if Z contains no proper closed Γ-invariant subsets or, equivalently, if every Γ-orbit is dense in Z. A point $z \in Z$ is a *wandering point* for an action $\Gamma \times Z \to Z$ if there exists a neighborhood U of z such that $gU \cap U = \emptyset$ for all but finitely many $g \in \Gamma$. If the space Z is metrizable, then a point $z \in Z$ is *not* a wandering point if and only if there exist a sequence (g_n) of distinct elements in Γ and a sequence (z_n) in Z converging to z such that $g_n z_n \to z$. For further discussion of dynamical relations between points under group actions, see [KL18, §4.3].

A continuous action $\Gamma \times Z \to Z$ of a discrete group Γ on a compact metrizable topological space Z is a *convergence action* if the product action of Γ on Z^3 restricts to a properly discontinuous action on

$$T(Z) = \{(z_1, z_2, z_3) \in Z^3 \mid \operatorname{card}\{z_1, z_2, z_3\} = 3\}.$$

Equivalently, a continuous action of a discrete group is a convergence action if every sequence (g_i) contains a subsequence (g_{i_j}) which is either constant or converges to a point $z_+ \in Z$ uniformly on compacts in $Z - \{z_-\}$ for some $z_- \in Z$; see [Bow98b, Proposition 7.1]. In this situation, the inverse sequence $(g_{i_j}^{-1})$ converges to z_- uniformly on compacts in $Z - \{z_+\}$. The set of such limit points z_+ is the *limit set* Λ of the action of Γ; this is a closed Γ-invariant subset of Z. Observe that a convergence action need not be faithful but it necessarily has finite kernel, provided that $T(Z) \neq \emptyset$. A convergence action on Z is called *uniform* if it is cocompact on $T(Z)$.

Item (1) of the following theorem can be found in [Tuk94, Theorem 2S]; for item (2) see [Tuk98, Theorem 1A] for instance.

2.2 Theorem. *Suppose $\Gamma \times Z \to Z$ is a convergence action with limit set Λ such that* $\operatorname{card}(\Lambda) \geq 3$. *Then*

(1) *Λ is perfect and the action is minimal on Λ.*

(2) *If the action is uniform and Z is perfect, then $Z = \Lambda$.*

2.2 Coarse geometry

A metric space (X, d) is *proper* if the closed ball $\overline{B}_r(x)$ is compact for every $x \in X$ and every $r > 0$. Note that proper metric spaces are complete. A metric space (X, d) is called a *geodesic space* if every pair of points $x, y \in X$ can be joined by a geodesic segment xy, that is, an isometric embedding of an interval into X joining x to y.

2.3 Definition (Quasi-geodesic)**.** Let I be an interval of $\mathbb{R}$ (or its intersection with $\mathbb{Z}$) and (X, d) a metric space. A map $c : I \to X$ is called an (A, C)-*quasi-geodesic* with constants $A \geq 1$ and $C \geq 0$ if for all $t, t' \in I$,

$$\frac{1}{A}|t - t'| - C \leq d(c(t), c(t')) \leq A|t - t'| + C.$$

2.4 Definition (Hyperbolic space)**.** Let $\delta \geq 0$. A geodesic space X is said to be δ-*hyperbolic* if for any geodesic triangle in X, each side of the triangle is contained in the closed δ-neighborhood of the union of the other two sides. A geodesic space is said to be *hyperbolic* if it is δ-hyperbolic for some $\delta \geq 0$.

Let X be a proper δ-hyperbolic space. Two geodesic rays $\mathbb{R}_{\geq 0} \to X$ are said to be *asymptotic* if the Hausdorff distance between their images is finite. Being asymptotic is an equivalence relation on the set of geodesic rays. The set of equivalence classes of geodesic rays in X is called the *visual boundary* of X and denoted by $\partial_\infty X$. In view of the Morse lemma for hyperbolic spaces (see [BH99, Theorem III.H.1.7] or [DK18, Lemma 11.105] for example), one can also define $\partial_\infty X$ as the set of equivalence classes of quasi-geodesic rays $\mathbb{R}_{\geq 0} \to X$. We will use the notation $x\xi$ for a geodesic ray in X emanating from x and representing the point $\xi \in \partial_\infty X$.

Fix $k > 2\delta$ and let $c_0 : \mathbb{R}_{\geq 0} \to X$ be a geodesic ray representing $\xi \in \partial_\infty X$ with $c_0(0) = x$. A topology on $\partial_\infty X$ is given by setting the basis of neighborhoods of ξ to be the collection $\{V_n(\xi)\}_{n \in \mathbb{N}}$, where $V_n(\xi)$ is the set of equivalence classes of geodesic rays c such that $c(0) = x$ and $d(c(n), c_0(n)) < k$. This topology extends to the *visual compactification* of X

$$\overline{X} := X \cup \partial_\infty X,$$

which is a compact metrizable space. We refer to [BH99, III.H.3.6] for details.

Let $x, y, z \in X$. The *Gromov product* of y and z with respect to x is defined by

$$(y \cdot z)_x := \frac{1}{2}(d(x, y) + d(x, z) - d(y, z)).$$

The Gromov product is extended to $X \cup \partial_\infty X$ by

$$(y \cdot z)_x := \sup \liminf_{i,j \to \infty} (y_i \cdot z_j)_x,$$

where the supremum is taken over all sequences (y_i) and (z_j) in X such that $\lim y_i = y$ and $\lim z_j = z$.

2.5 Definition (Visual metric)**.** Let X be a hyperbolic space with base point $x \in X$. A metric d_a on $\partial_\infty X$ is called a *visual metric* with parameter $a > 1$ if there exist constants $k_1, k_2 > 0$ such that

$$k_1 a^{-(\xi \cdot \xi')_x} \leq d_a(\xi, \xi') \leq k_2 a^{-(\xi \cdot \xi')_x}$$

for all $\xi, \xi' \in \partial_\infty X$.

For every $a > 1$ sufficiently close to 1, a proper hyperbolic space admits a visual metric d_a which induces the same topology as the topology on $\partial_\infty X$ described above. We refer to [BH99, Chapter III.H.3] for more details on constructing visual metrics.

In the rest of the section we discuss hyperbolic groups and the relation to convergence actions.

2.6 Definition (Hyperbolic group)**.** A finitely generated group Γ is *word-hyperbolic* (or simply *hyperbolic*) if its Cayley graph with respect to a finite generating set of Γ is a hyperbolic metric space. A hyperbolic group is called *elementary* if it contains a cyclic subgroup of finite index and *non-elementary* otherwise.

The *Gromov boundary* $\partial_\infty\Gamma$ of a hyperbolic group Γ is defined as the visual boundary of a Cayley graph X of Γ. The closure of $\Gamma \subset X$ in the visual compactification $\overline{X}$ equals $\Gamma \cup \partial_\infty\Gamma$ and is denoted $\overline{\Gamma}$; it is the *visual compactification of* Γ.

Every hyperbolic group Γ acts on its visual compactification $\overline{\Gamma}$ by homeomorphisms. This action is a convergence action; see [Tuk94, Theorem 3.A] and [Fre95]. If a sequence (c_i) in Γ represents a quasi-geodesic ray within bounded distance from a geodesic ray $g\mu$ ($g \in \Gamma$, $\mu \in \partial_\infty\Gamma$), then this sequence, regarded as a sequence of maps $\overline{\Gamma} \to \overline{\Gamma}$, converges to μ uniformly on compacts in $\overline{\Gamma} - \{\mu'\}$ for some $\mu' \in \partial_\infty\Gamma$. We will use the following consequence of this property later in the proof of Theorem 5.1:

2.7 Lemma. *Suppose that $c : \mathbb{N}_0 \to \Gamma$, $i \mapsto c_i$, is an (A, C)-quasi-geodesic ray in Γ such that*

- *the word length of c_0 is ≤ 1, and*
- *there exists a subsequence (c_{i_j}) converging to a point $\xi \in \partial_\infty\Gamma$ pointwise on a subset $S \subset \partial_\infty\Gamma$ with* $\mathrm{card}(S) \geq 2$.

Then the image $c(\mathbb{N}_0)$ is D-Hausdorff close to a geodesic $e\xi$ in the Cayley graph X of Γ, where D depends only on (A, C) and the hyperbolicity constant of X.

Proof. Since the word length of c_0 is ≤ 1, the Morse lemma for hyperbolic groups implies that there is a geodesic ray $e\mu$ ($\mu \in \partial_\infty\Gamma$) starting at the identity $e \in \Gamma$ such that the Hausdorff distance between the image $c(\mathbb{N}_0)$ and the ray $e\mu$ in X is bounded above by a uniform constant $D > 0$ depending only on (A, C) and X.

By the above property, the sequence (c_i) converges to μ uniformly on compacts in $\overline{\Gamma} - \{\mu'\}$ for some $\mu' \in \partial_\infty\Gamma$. On the other hand, since $\mathrm{card}(S) \geq 2$, there is a point $\nu \in S$ distinct from μ' such that the subsequence (c_{i_j}) converges to ξ on $\{\nu\} \subset \partial_\infty\Gamma$. Therefore, we must have $\mu = \xi$ and $e\mu = e\xi$. $\square$

Furthermore, the action of Γ on $\partial_\infty\Gamma$ is a uniform convergence action. In particular, if Γ is non-elementary then this action has finite kernel (the unique maximal finite normal subgroup of Γ), is minimal, and $\partial_\infty\Gamma$ is a perfect topological space; compare Theorem 2.2. We refer to [DK18, Lemma 11.130] for more details.

Conversely, Bowditch [Bow98b] gave a topological characterization of hyperbolic groups and their Gromov boundaries as uniform convergence actions $\Gamma \times Z \to Z$ of discrete groups on perfect metrizable topological spaces:

2.8 Theorem (Bowditch)**.** *Suppose that Z is a compact perfect metrizable space of cardinality ≥ 2 and $\Gamma \times Z \to Z$ is a continuous action of a discrete group, which is a uniform convergence action. Then Γ is a non-elementary hyperbolic group and Z is equivariantly homeomorphic to the Gromov boundary $\partial_\infty \Gamma$.*

3 Sullivan's structural stability theorem

Throughout Sections 3 and 4, we let (M, d) be a proper metric space and suppose a discrete group Γ acts continuously on M with a non-empty invariant compact subset $\Lambda \subset M$, no point of which is isolated in M. That is, there is a homomorphism $\rho : \Gamma \to \mathrm{Homeo}(M)$ such that $\rho(\Gamma)(\Lambda) = \Lambda$. Henceforth, we shall simply write such an action as

$$\rho : \Gamma \to \mathrm{Homeo}(M; \Lambda)$$

or as $\rho : \Gamma \to \mathrm{Homeo}(\Lambda)$ when $M = \Lambda$.

3.1 Remark. Note that we do not assume faithfulness of the action of Γ on Λ and even on M.

In this situation, one considers two conditions on ρ, which we call *S-expansion* and *S-hyperbolicity* conditions, respectively. The letter S stands for Sullivan, although our conditions are weaker than his expansion-hyperbolicity axioms.

In the present section, we define the S-expansion condition, and draw the key Lemma 3.15 as well as its various consequences (Sections 3.1-3.2). We then define the S-hyperbolicity condition (Section 3.3) and make a precise statement of Sullivan's structural stability theorem (Section 3.4).

3.1 S-expansion condition

In order to define the S-expansion condition we need a little preparation.

Let f be a homeomorphism of M. Given $\lambda > 1$ and $U \subset M$, we say f is (λ, U)-*expanding* (or λ-*expanding on* U) if

$$d(f(x), f(y)) \geq \lambda \cdot d(x, y)$$

for all $x, y \in U$. In this case, we also say U is a (λ, f)-*expanding subset.* Note that f is (λ, U)-expanding if and only if f^{-1} is $(\lambda^{-1}, f(U))$-*contracting*, that is,

$$d(f^{-1}(x), f^{-1}(y)) \leq \frac{1}{\lambda} \cdot d(x, y)$$

for all $x, y \in f(U)$.

Given $\Delta > 0$, a (λ, U)-expanding homeomorphism f is said to be $(\lambda, U; \Delta)$-*expanding* if

$$B_{\lambda\eta}(f(x)) \subset f(B_\eta(x)) \ \text{ whenever } B_\eta(x) \subset U \text{ and } \eta \leq \Delta.$$

Clearly, if f is $(\lambda, U; \Delta)$-expanding then it is also $(\lambda, U; \Delta')$-expanding for every $\Delta' \le \Delta$. If M is a geodesic metric space then every (λ, U)-expanding homeomorphism is also $(\lambda, U; \Delta)$-expanding for every Δ. This implication does not hold for general metric spaces. However, we note the following fact:

3.2 Lemma. *Suppose that f is (λ, U)-expanding, where U is a bounded open subset of M. Then for every $\Delta > 0$ there exists $\Delta' = \Delta'_U > 0$ such that f is $(\lambda, U'; \Delta')$-expanding with $U' := \operatorname{int} U^\Delta \subset U$.*

Proof. Since $f(U^\Delta)$ is compact, we have $\delta := d(f(U^\Delta), M - f(U)) > 0$. Now we let $\Delta' := \lambda^{-1} \cdot \min\{\Delta, \delta\}$. If $\eta \le \Delta'$ and $B_\eta(x) \subset U'$, in particular, $f(x) \in f(U^\Delta)$, then we have $B_{\lambda\eta}(f(x)) \subset B_\delta(f(x)) \subset f(U)$ and hence $f^{-1}[B_{\lambda\eta}(f(x))] \subset B_\eta(x)$, since f^{-1} is $(1/\lambda, f(U))$-contracting. Therefore, we conclude that $B_{\lambda\eta}(f(x)) \subset f(B_\eta(x))$. □

We are now ready to define the S-expansion condition.

3.3 Definition (S-expansion)**.** An action $\rho : \Gamma \to \mathrm{Homeo}(M; \Lambda)$ is said to be *S-expanding (at Λ)* if there exist

- a finite index set $\mathcal{I}$,
- a cover $\mathcal{U} = \{U_\alpha \subset M \mid \alpha \in \mathcal{I}\}$ of Λ by open (and possibly empty) subsets U_α,
- a map $s : \mathcal{I} \to \Gamma$, $\alpha \mapsto s_\alpha$,
- and positive real numbers $L \ge \lambda > 1$ and $\Delta \le \Delta_{\mathcal{U}}$

such that, for every $\alpha \in \mathcal{I}$, the map $\rho(s_\alpha^{-1})$ is

(i) L-Lipschitz on $N_\Delta(\Lambda)$, and
(ii) $(\lambda, U_\alpha; \Delta)$-expanding,

and that the image $\Sigma := \{s_\alpha \mid \alpha \in \mathcal{I}\} \subset \Gamma$ of the map s is symmetric and generates the group Γ.

In this case, the data $(\mathcal{I}, \mathcal{U}, \Sigma, \Delta, L, \lambda)$ (or, occasionally, any subset thereof) will be referred to as the *S-expansion data* of ρ.

If $\rho : \Gamma \to \mathrm{Homeo}(M; \Lambda)$ is an S-expanding action, let $|\cdot|_\Sigma$ (resp. d_Σ) denote the word length (resp. the word metric) on the group Γ with respect to the generating set Σ from Definition 3.3. Then the L-Lipschitz property (i) implies that

$$\text{the map } \rho(g) \text{ is } L^k\text{-Lipschitz on } N_{\Delta/L^{k-1}}(\Lambda) \tag{3.4}$$

for every $g \in \Gamma$ with $|g|_\Sigma = k \in \mathbb{N}$.

3.5 Remark. A few more remarks are in order.

(a) S-expanding actions appear naturally in the context of Anosov actions on flag manifolds [KLP17, Definition 3.1] and hyperbolic group actions on their Gromov boundaries equipped with visual metrics [Coo93]. See Sections 5 and 6 for further discussion.

(b) The symmetry of the generating set Σ means that $s \in \Sigma$ if and only if $s^{-1} \in \Sigma$. This implies that $\rho(s_\alpha)$ is L-bi-Lipschitz on $N_{\Delta/L}$ for all $\alpha \in \mathcal{I}$.

(c) Suppose that M is a Riemannian manifold and the action ρ is by C^1-diffeomorphisms. Then, ρ is S-expanding provided that for every $x \in \Lambda$ there exist $g \in \Gamma$ such that $\epsilon(\rho(g), x) > 1$ (see (2.1)). Indeed, by compactness of Λ, there exists a finite cover $\mathcal{U} = \{U_\alpha \mid \alpha \in \mathcal{I}\}$ of Λ and a collection Σ of group elements $s_\alpha \in \Gamma$ such that each $\rho(s_\alpha^{-1})$ is $(\lambda, U_\alpha; \Delta_{\mathcal{U}})$-expanding. By adding, if necessary, extra generators to Σ with empty expanding subsets, we obtain the required symmetric generating set of Γ.

(d) If $U_\alpha = \emptyset$ for some $\alpha \in \mathcal{I}$ then the condition (ii) is vacuous for this α. Otherwise, it implies that the inverse $\rho(s_\alpha)$ is $(\lambda^{-1}, \rho(s_\alpha^{-1})[U_\alpha])$-contracting.

(e) The condition (ii) can be relaxed to the mere (λ, U_α)-expanding condition. Namely, we may first modify the cover $\mathcal{U} = \{U_\alpha \mid \alpha \in \mathcal{I}\}$ so that U_α are all bounded. Then, in view of Lemma 3.2, we can modify it further to $\mathcal{U}' = \{U'_\alpha \mid \alpha \in \mathcal{I}\}$, where $U'_\alpha := \operatorname{int} U_\alpha^\Delta$ as in the lemma. For each $\alpha \in \mathcal{I}$ we also let Δ'_{U_α} denote the number Δ'_U given by the lemma, and set

$$\Delta' := \min\{\Delta_{\mathcal{U}'}, \Delta'_{U_\alpha} \mid \alpha \in \mathcal{I}\}.$$

After such modification $\mathcal{U}'$ is still an open cover of Λ and the maps $\rho(s_\alpha^{-1})$ are $(\lambda, U'_\alpha; \Delta')$-expanding.

(f) The map $s : \mathcal{I} \to \Sigma \subset \Gamma$ is not necessarily injective: the ρ-image of an element of Γ can have several expansion subsets. See Examples 3.29, 7.3 and 7.4.

(g) Clearly, the properties (i) and (ii) also hold on the closures $\overline{N_\Delta}(\Lambda)$ and $\overline{U_\alpha}$, respectively.

3.2 Expansion enables encoding

Our goal here is to draw the key Lemma 3.15 for S-expanding actions and then explore its various implications.

We begin by noting that the S-expansion condition (Definition 3.3) enables us to encode points of Λ by sequences in the finite index set $\mathcal{I}$.

To see this, let $\rho : \Gamma \to \mathrm{Homeo}(M; \Lambda)$ be an S-expanding action with data $(\mathcal{I}, \mathcal{U}, \Sigma, \Delta, L, \lambda)$. We first fix a real number η such that

$$0 < \eta \leq \Delta.$$

Since $\Delta \leq \Delta_{\mathcal{U}}$, every η-ball $B_\eta(x)$ centered at $x \in \Lambda$ is contained in some member of $\mathcal{U}$. Now, to each $x \in \Lambda$ we assign a pair (α, p) of sequences

$$\alpha : \mathbb{N}_0 \to \mathcal{I},\ i \mapsto \alpha(i)$$
$$p : \mathbb{N}_0 \to \Lambda,\ i \mapsto p_i$$

as follows. We set

$$p_0 = x.$$

Let $\alpha(0) \in \mathcal{I}$ be an *arbitrary* element and set $p_1 = \rho(s_{\alpha(0)}^{-1})(p_0)$. Now, for $i \in \mathbb{N}$, choose $\alpha(i)$ inductively so that

$$B_\eta(p_i) \subset U_{\alpha(i)}$$

and then set

$$p_{i+1} = \rho(s_{\alpha(i)}^{-1})(p_i).$$

Note that the sequence α and $p_0 = x$ determine the sequence p.

3.6 Definition (Codes)**.** Suppose an action $\rho : \Gamma \to \mathrm{Homeo}(M; \Lambda)$ is S-expanding with data $(\mathcal{I}, \mathcal{U}, \Sigma, \Delta)$. Let $0 < \eta \leq \Delta$ and $x \in \Lambda$. The sequence

$$\alpha : \mathbb{N}_0 \to \mathcal{I}$$

(or the pair (α, p) of sequences) constructed as above is called an η-*code* for x. (Here, η is for the use of η-balls.) It is said to be *special* if $\alpha(0)$ satisfies $B_\eta(x) \subset U_{\alpha(0)}$. We denote by $\mathrm{Code}_\rho^{x,\eta}$ the set of all η-codes for $x \in \Lambda$.

3.7 Remark. That we do not require $\alpha(0)$ to satisfy $B_\eta(x) \subset U_{\alpha(0)}$ in general is Sullivan's trick, which will be useful in Section 4.4.

(a) The requirement $B_\eta(p_i) \subset U_{\alpha(i)}$ implies $U_{\alpha(i)} \neq \emptyset$ for $i \in \mathbb{N}$. Thus only for the initial value $\alpha(0)$ of codes α can we possibly have $U_{\alpha(0)} = \emptyset$.

(b) Since $\Delta \leq \Delta_{\mathcal{U}}$, we have $\mathrm{Code}_\rho^{x,\eta} \neq \emptyset$ for all $x \in \Lambda$ and $0 < \eta \leq \Delta$. Moreover, if $\eta \leq \delta \leq \Delta$ then $\mathrm{Code}_\rho^{x,\delta} \subset \mathrm{Code}_\rho^{x,\eta}$. Indeed, if (α, p) is a δ-code for x, then $B_\eta(p_i) \subset B_\delta(p_i) \subset U_{\alpha(i)}$ for all $i \in \mathbb{N}$, which implies that (α, p) is an η-code for x as well.

Now we claim that, for every $x \in \Lambda$, each η-code for x gives rise to a nested sequence of neighborhoods of x whose diameters tend to 0 exponentially fast. See Lemma 3.15 below.

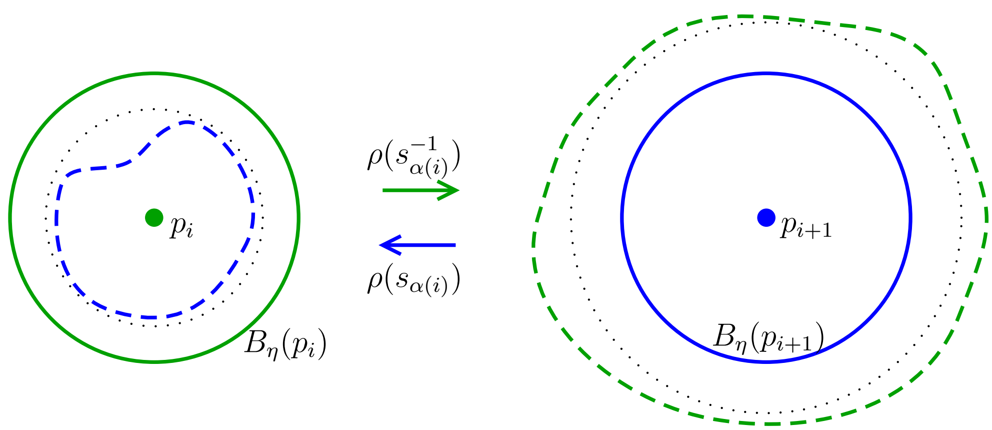

Figure 1: Actions of $\rho(s_{\alpha(i)}^{-1})$ and $\rho(s_{\alpha(i)})$ for $i \in \mathbb{N}$.

In order to prove the claim, suppose (α, p) is an η-code for $x \in \Lambda$ and let $i \in \mathbb{N}$. Since $B_\eta(p_i) \subset U_{\alpha(i)}$ and the homeomorphism $\rho(s_{\alpha(i)}^{-1})$ is $(\lambda, U_{\alpha(i)}; \Delta)$-expanding with $\rho(s_{\alpha(i)}^{-1})(p_i) = p_{i+1}$, we have by definition

$$B_{\lambda\eta}(p_{i+1}) \subset \rho(s_{\alpha(i)}^{-1})[B_\eta(p_i)] \subset \rho(s_{\alpha(i)}^{-1})[U_{\alpha(i)}]. \tag{3.8}$$

On the other hand, the inverse map $\rho(s_{\alpha(i)})$ is $(\lambda^{-1}, \rho(s_{\alpha(i)}^{-1})[U_{\alpha(i)}])$-contracting as we saw in Remark 3.5(d). Since $B_\eta(p_{i+1}) \subset \rho(s_{\alpha(i)}^{-1})[U_{\alpha(i)}]$, we obtain

$$\rho(s_{\alpha(i)})[B_\eta(p_{i+1})] \subset B_{\eta/\lambda}(p_i) \subset B_\eta(p_i). \tag{3.9}$$

See Figure 1. By a similar reasoning, we inductively obtain

$$\begin{aligned} \rho(s_{\alpha(i-1)}s_{\alpha(i)})[B_\eta(p_{i+1})] &\subset B_{\eta/\lambda^2}(p_{i-1}), \\ &\vdots \\ \rho(s_{\alpha(1)} \cdots s_{\alpha(i-1)}s_{\alpha(i)})[B_\eta(p_{i+1})] &\subset B_{\eta/\lambda^i}(p_1). \end{aligned}$$

Lastly, the map $\rho(s_{\alpha(0)})$ is L-Lipschitz on $B_{\eta/\lambda^i}(p_1) \subset N_\Delta(\Lambda)$ by Definition 3.3(i). Thus we see that

$$\rho(s_{\alpha(0)}s_{\alpha(1)} \cdots s_{\alpha(i-1)}s_{\alpha(i)})[B_\eta(p_{i+1})] \subset B_{L\eta/\lambda^i}(p_0). \tag{3.10}$$

3.11 Definition (Rays)**.** Let $\rho : \Gamma \to \mathrm{Homeo}(M; \Lambda)$ be an S-expanding action. Given an η-code α for $x \in \Lambda$, the *ray* associated to α (or simply the α*-ray*) is a sequence $c^\alpha : \mathbb{N}_0 \to \Gamma$ defined by

$$c_i^\alpha = s_{\alpha(0)}s_{\alpha(1)} \cdots s_{\alpha(i)}.$$

We denote by $\mathrm{Ray}_\rho^{x,\eta}(\Gamma)$ the set of all rays c^α associated to $\alpha \in \mathrm{Code}_\rho^{x,\eta}$. Thus we may interpret c as a map

$$c : \mathrm{Code}_\rho^{x,\eta} \to \mathrm{Ray}_\rho^{x,\eta}(\Gamma), \quad \alpha \mapsto c^\alpha.$$

3.12 Remark. (a) The initial point $c_0^\alpha = s_{\alpha(0)}$ of c^α is an element of Σ.

(b) Every ray c^α defines an edge-path in the Cayley graph of Γ (with respect to the generating set Σ). Note that each word c_i^α is reduced, since an appearance of $s_\alpha^{-1}s_\alpha$ would imply that the composite map $\rho(s_\alpha^{-1})\rho(s_\alpha)$, which is the identity, is λ^2-expanding on some non-empty open subset of M.

With this definition of α-ray c^α, we first note that

$$\rho(c_i^\alpha)(p_{i+1}) = p_0 = x \tag{3.13}$$

for all $i \in \mathbb{N}_0$. Then the inclusion (3.10) can be written as

$$x \in \rho(c_i^\alpha)[B_\eta(p_{i+1})] \subset B_{L\eta/\lambda^i}(x)$$

for all $i \in \mathbb{N}_0$. Moreover, from (3.9) we have

$$\begin{aligned} \rho(c_i^\alpha)[B_\eta(p_{i+1})] &= \rho(s_{\alpha(0)} \cdots s_{\alpha(i-1)}s_{\alpha(i)})[B_\eta(p_{i+1})] \\ &\subset \rho(s_{\alpha(0)} \cdots s_{\alpha(i-1)})[B_\eta(p_i)] \\ &= \rho(c_{i-1}^\alpha)[B_\eta(p_i)] \end{aligned} \tag{3.14}$$

for all $i \in \mathbb{N}$.

Let us summarize what we have proved thus far.

3.15 Lemma. *Let $\rho : \Gamma \to \mathrm{Homeo}(M;\Lambda)$ be an S-expanding action with data $(\mathcal{I}, \mathcal{U}, \Sigma, \Delta, L, \lambda)$ and let $0 < \eta \leq \Delta$. If (α, p) is an η-code for $x \in \Lambda$, then the sequence of neighborhoods of x*

$$x \in \rho(c_i^\alpha)[B_\eta(p_{i+1})] \quad (i \in \mathbb{N}_0)$$

is nested and exponentially shrinking. More precisely, we have

(i) $\rho(c_i^\alpha)[B_\eta(p_{i+1})] \subset \rho(c_{i-1}^\alpha)[B_\eta(p_i)]$ *for $i \in \mathbb{N}$, and*
(ii) $\rho(c_i^\alpha)[B_\eta(p_{i+1})] \subset B_{L\eta/\lambda^i}(x)$ *for all $i \in \mathbb{N}_0$.*

Consequently, we have the equality

$$\{x\} = \bigcap_{i=0}^{\infty} \rho(c_i^\alpha)[B_\eta(p_{i+1})].$$

The map $\rho(c_i^\alpha)^{-1}$ is λ^i/L-expanding on the neighborhood $\rho(c_i^\alpha)[B_\eta(p_{i+1})]$ of x for all $i \in \mathbb{N}_0$, and the map

$$\rho(c_i^\alpha)^{-1}\rho(c_j^\alpha) = \rho(s_{\alpha(i)}^{-1} s_{\alpha(i-1)}^{-1} \cdots s_{\alpha(j+1)}^{-1})$$

is λ^{i-j}-expanding on $\rho(c_j^\alpha)^{-1}\big[\rho(c_i^\alpha)[B_\eta(p_{i+1})]\big]$ for all $0 \leq j < i$.

A number of corollaries will follow.

We say an action $\rho : \Gamma \to \mathrm{Homeo}(M;\Lambda)$ has an *expansivity constant* $\epsilon > 0$ if for every distinct pair of points $x, y \in \Lambda$ there exists an element $g \in \Gamma$ such that $d(\rho(g)(x), \rho(g)(y)) \geq \epsilon$. Compare [CP93, Proposition 2.2.4].

3.16 Corollary. *If an action $\rho : \Gamma \to \mathrm{Homeo}(M;\Lambda)$ is S-expanding with data $(\mathcal{U}, \Delta)$, then Δ is an expansivity constant of this action.*

Proof. Let $x, y \in \Lambda$ be distinct and consider a Δ-code (α, p) for x. By Lemma 3.15, the sequence of neighborhoods of x

$$\rho(c_i^\alpha)[B_\Delta(p_{i+1})] \quad (i \in \mathbb{N}_0)$$

is nested and exponentially shrinking. Thus there exists an $n \in \mathbb{N}_0$ such that

$$y \notin \rho(c_n^\alpha)[B_\Delta(p_{n+1})],$$
$$\text{that is,} \quad \rho(c_n^\alpha)^{-1}(y) \notin B_\Delta(p_{n+1}) = B_\Delta(\rho(c_n^\alpha)^{-1}(x)).$$

Therefore, $d(\rho(c_n^\alpha)^{-1}(x), \rho(c_n^\alpha)^{-1}(y)) \geq \Delta$ as desired. □

The following corollary will be crucial for Definition 3.23 below, which in turn plays an essential role when we discuss actions of hyperbolic groups in Section 5. The corollary is also the reason why we need the assumption that no point of Λ is isolated in M.

3.17 Corollary. *Let $\rho : \Gamma \to \mathrm{Homeo}(M;\Lambda)$ be an S-expanding action with data $(\Sigma, \Delta, L, \lambda)$ and let $\eta \in (0, \Delta]$. Then the rays $c^\alpha \in \mathrm{Ray}_\rho^{x,\eta}(\Gamma)$ are (A, C)-quasi-geodesic rays in (Γ, d_Σ) with A and C depending only on L and λ.*

Proof. Let (α, p) be an η-code for $x \in \Lambda$. Since no point of Λ is isolated in M, for each $i \in \mathbb{N}_0$ we can choose a point y_i such that

$$x \neq y_i \in \rho(c_i^\alpha)[B_\eta(p_{i+1})] \cap N_{\Delta/L^i}(\Lambda).$$

Let $i, j \in \mathbb{N}_0$ be such that $0 \leq j < i$ and set $r_{ij} := (c_i^\alpha)^{-1} c_j^\alpha$. By (the last statement of) Lemma 3.15, the map $\rho(r_{ij})$ is λ^{i-j}-expanding on $\rho(c_j^\alpha)^{-1}\big[\rho(c_i^\alpha)[B_\eta(p_{i+1})]\big] \supset \{\rho(c_j^\alpha)^{-1}(y_i), \rho(c_j^\alpha)^{-1}(x)\}$, hence

$$d(\rho(r_{ij})(y_{ij}), \rho(r_{ij})(x_j)) \geq \lambda^{i-j} d(y_{ij}, x_j),$$

where we set $y_{ij} := \rho(c_j^\alpha)^{-1}(y_i)$ and $x_j := \rho(c_j^\alpha)^{-1}(x)$. On the other hand, we have

$$L^{|r_{ij}|_\Sigma} d(y_{ij}, x_j) \geq d(\rho(r_{ij})(y_{ij}), \rho(r_{ij})(x_j))$$

from (3.4), since $x_j \in \Lambda$ while $y_{ij} \in N_{\Delta/L^{i-j-1}}(\Lambda)$ and $|r_{ij}|_\Sigma \leq i - j$. From these two inequalities we obtain

$$d_\Sigma(c_i^\alpha, c_j^\alpha) = |r_{ij}|_\Sigma \geq \frac{\log \lambda}{\log L} \cdot (i - j).$$

Therefore, the α-ray c^α is a uniform quasi-geodesic ray. $\square$

Another consequence of the existence of codes concerns the dynamics of the action of Γ on Λ. The action of Γ on Λ need not be minimal in general even if Γ is a non-elementary hyperbolic group (see Example 7.4). Nevertheless, the action of Γ on Λ has no *wandering points*:

3.18 Theorem. *Let $\rho : \Gamma \to \mathrm{Homeo}(M;\Lambda)$ be an S-expanding action with data (Δ). For $x \in \Lambda$ and $\eta \in (0, \Delta]$, consider a ray $c^\alpha \in \mathrm{Ray}_\rho^{x,\eta}(\Gamma)$ associated to an η-code (α, p) for x. Then*

(1) *there exist a subsequence (g_j) of (c_i^α) and a point $q \in \Lambda$ such that $(\rho(g_j))$ converges to x uniformly on $B_{\eta/2}(q)$;*

(2) *there exists an infinite sequence (h_j) in Γ such that*

$$\lim_{j\to\infty} \rho(h_j)(x) = x.$$

Proof. (1) By the compactness of Λ, the sequence $p = (p_i)$ contains a subsequence (p_{i_j+1}) converging to some point $q \in \Lambda$. The point q is then covered by infinitely many balls $B_{\eta/2}(p_{i_j+1})$, $j \in \mathbb{N}$. By Lemma 3.15 the corresponding elements $g_j := c_{i_j}^\alpha \in \Gamma$ will send $B_{\eta/2}(q) \subset B_\eta(p_{i_j+1})$ to a subset $\rho(g_j)[B_{\eta/2}(q)]$ of diameter at most $L\eta/\lambda^{i_j}$ containing x. From this we conclude that the sequence $(\rho(g_j))$ converges to x uniformly on $B_{\eta/2}(q)$.

(2) On the other hand, since, in particular, $p_{i_1+1} \in B_{\eta/2}(q)$, we obtain

$$\lim_{j\to\infty} \rho(c^\alpha_{i_j})(p_{i_1+1}) = x.$$

Since $\rho(c^\alpha_{i_1})^{-1}(x) = p_{i_1+1}$, it follows that for $h_j := c^\alpha_{i_j}(c^\alpha_{i_1})^{-1}$ we have

$$\lim_{j\to\infty} \rho(h_j)(x) = x. \qquad \square$$

3.19 Remark. The idea of *Markov coding* of limit points of actions of finitely generated groups Γ by sequences in Γ is rather standard in *symbolic dynamics* and goes back to Nielsen, Hedlund and Morse; we refer the reader to the paper by Series [Ser81] for references and historical discussion. In the setting of hyperbolic groups this was introduced in Gromov's paper [Gro87, §.8] and discussed in more detail in the book by Coornaert and Papadopoulos [CP93]. Section 8.5.Y of Gromov's paper discusses a relation to Sullivan's stability theorem.

3.3 S-hyperbolicity condition

We continue the discussion from the previous section. In order to define the S-hyperbolicity condition we need to introduce an equivalence relation $\sim_N$ on the set $\mathrm{Ray}^{x,\eta}_\rho(\Gamma)$ of rays in Γ associated to η-codes for $x \in \Lambda$; recall Definitions 3.6 and 3.11.

3.20 Definition (N-equivalence)**.** Suppose $\rho : \Gamma \to \mathrm{Homeo}(M;\Lambda)$ is an S-expanding action with data (Σ, Δ) and let $0 < \eta \le \Delta$.

(a) For each $N \in \mathbb{N}$ we define the relation $\approx_N$ on $\mathrm{Ray}^{x,\eta}_\rho(\Gamma)$ by declaring that $c^\alpha \approx_N c^\beta$ if there exist infinite subsets P, $Q \subset \mathbb{N}_0$ such that the subsets

$$c^\alpha(P) = \{c^\alpha_i \mid i \in P\}, \quad c^\beta(Q) = \{c^\beta_i \mid i \in Q\}$$

of (Γ, d_Σ) are within Hausdorff distance N from each other. The *N-equivalence*, denoted by $\sim_N$, is the equivalence relation on $\mathrm{Ray}^{x,\eta}_\rho(\Gamma)$ generated by the relation $\approx_N$. In other words, we write $c^\alpha \sim_N c^\beta$ if there is a finite chain of “interpolating” rays $c^\alpha = c^{\gamma_1}, c^{\gamma_2}, \ldots, c^{\gamma_n} = c^\beta$ in $\mathrm{Ray}^{x,\eta}(\Gamma)$ such that

$$c^{\gamma_1} \approx_N c^{\gamma_2} \approx_N \cdots \approx_N c^{\gamma_n}.$$

(b) The rays c^α and c^β are said to *N-fellow-travel* if their images $c^\alpha(\mathbb{N}_0)$ and $c^\beta(\mathbb{N}_0)$ are within Hausdorff distance N from each other.

Observe that N-fellow-traveling rays are N-equivalent.

We are now ready to define the S-hyperbolicity condition.

3.21 Definition (S-hyperbolicity)**.** Let $\rho : \Gamma \to \mathrm{Homeo}(M;\Lambda)$ be an S-expanding action with data $(\mathcal{I}, \mathcal{U}, \Sigma, \Delta, L, \lambda)$.

(a) The action ρ is said to be *S-hyperbolic* (resp. *S-fellow-traveling*) if there exist a constant $\delta \in (0, \Delta)$ and an integer $N \geq 1$ such that, for every $x \in \Lambda$, all rays in $\mathrm{Ray}_\rho^{x,\delta}(\Gamma)$ are N-equivalent (resp. N-fellow-traveling).
(b) The action ρ is said to be *uniformly S-hyperbolic* (resp. *uniformly S-fellow-traveling*) if there exists a constant $\delta \in (0, \Delta)$ such that for each $\eta \in (0, \delta]$ there is an integer $N(\eta) \geq 1$ such that, for every $x \in \Lambda$, all rays in $\mathrm{Ray}_\rho^{x,\eta}(\Gamma)$ are $N(\eta)$-equivalent (resp. $N(\eta)$-fellow-traveling).

We shall always assume that the S-expansion data $(\mathcal{I}, \mathcal{U}, \Sigma, \Delta, L, \lambda)$ is understood implicitly, and simply refer to the pair (δ, N) or $(\delta, N(\eta))$ as the *(uniform) S-hyperbolicity data* of ρ; the same for the (uniform) S-fellow-traveling property.

Clearly, uniform S-hyperbolicity implies S-hyperbolicity. Also note that S-fellow-traveling implies S-hyperbolicity, since N-fellow-traveling rays are N-equivalent. All examples of S-hyperbolic actions we present in this paper are actually S-fellow-traveling. In Section 7.1 we exhibit S-expanding actions which fail to be S-hyperbolic.

3.22 Remark. Further remarks on the S-hyperbolicity condition:
(a) Section 4.3 below is the only step in the proof of Sullivan's structural stability theorem (Theorem 3.27) where the S-hyperbolicity condition comes in and plays a crucial role.
(b) Our definition of S-hyperbolicity is weaker than Sullivan's original definition in two aspects. One is that, while we use δ-balls with $\delta > 0$ in the requirement $B_\delta(p_i) \subset U_{\alpha(i)}$ of δ-codes α, he uses "0-balls" requiring only $p_i \in U_{\alpha(i)}$ in order to construct "0-codes" α, and asks two different rays thus obtained be within a uniform bounded distance. Since all δ-codes (with $\delta > 0$) are 0-codes, Sullivan's original condition is much stronger in this sense. The other difference is that we require only N-equivalence of rays while Sullivan required the fellow-traveling property.
(c) One can relax the S-hyperbolicity condition further by changing the equivalence relation $\sim_N$ and allowing interpolating rays associated with codes for expansion data different from $(\mathcal{I}, \mathcal{U}, \Sigma, \Delta, L, \lambda)$. While the S-fellow-traveling condition is well-suited for actions of hyperbolic groups, the relaxed version allows for actions of groups such as uniform lattices in higher rank semisimple Lie groups. We will discuss this in more detail elsewhere.

In view of Corollary 3.17, two N-equivalent rays in a hyperbolic group N'-fellow-travel, since two quasi-geodesics in a hyperbolic space X which are Hausdorff-close on unbounded subsets define the same point in $\partial_\infty X$. Thus the S-hyperbolicity condition enables us to define the following map when Γ is a hyperbolic group:

3.23 Definition (Coding map)**.** Let Γ be a hyperbolic group and $\rho : \Gamma \to \mathrm{Homeo}(M; \Lambda)$ an S-hyperbolic action with data (δ, N). The *coding map*

$$\pi : \Lambda \to \partial_\infty \Gamma, \ x \mapsto \pi(x)$$

of ρ is defined as follows: the value $\pi(x)$ of $x \in \Lambda$ is the equivalence class in $\partial_\infty \Gamma$ (in the sense of Section 2.2) of a ray $c^\alpha \in \mathrm{Ray}_\rho^{x,\delta}(\Gamma)$.

The map π is clearly equivariant. In Theorem 5.7 we will prove that π is a continuous surjective map.

3.4 S-hyperbolicity implies structural stability

Suppose $\rho : \Gamma \to \mathrm{Homeo}(M; \Lambda)$ is an S-expanding action. Following Sullivan's remark (at the very end of his paper [Sul85]) we would like to talk about small perturbations of ρ which are still S-expanding.

For this purpose, we equip $\mathrm{Homeo}(M)$ with what we call the *compact-open Lipschitz topology.* A neighborhood basis of $f \in \mathrm{Homeo}(M)$ in this topology is of the form $U(f; K, \epsilon)$ for a compact $K \subset M$ and $\epsilon > 0$; it consists of all $g \in \mathrm{Homeo}(M)$ that are *ϵ-close to f on K*:

$$d_{\mathrm{Lip},K}(f, g) := \sup_{x\in K} d(f(x), g(x)) + \sup_{\substack{x,y\in K\\ x\neq y}} \left| \frac{d(f(x), f(y))}{d(x, y)} - \frac{d(g(x), g(y))}{d(x, y)} \right| < \epsilon. \tag{3.24}$$

3.25 Remark. As we explained in the introduction, Sullivan actually considers in his paper [Sul85, §9] the smooth case when M is a Riemannian manifold with the Riemannian distance function d and the actions are by C^1-diffeomorphisms. In this case $\rho(s) \in \mathrm{Diff}^1(M)$ is said to be λ-expanding on U if $\|D_x\rho(s)(v)\| \geq \lambda\|v\|$ for all $x \in U$ and $v \in T_xM$, and the compact-open Lipschitz topology on $\mathrm{Diff}^1(M)$ is nothing but the compact-open C^1-topology. The proof of his structural stability theorem does not become particularly simpler in the smooth case, so we work with continuous actions on metric spaces in the present paper.

Suppose $\rho : \Gamma \to \mathrm{Homeo}(M; \Lambda)$ is an S-expanding action (Definition 3.3). Given a compact K and $\epsilon > 0$, we say a homomorphism $\rho' : \Gamma \to \mathrm{Homeo}(M)$ is a *(K, ϵ)-perturbation of ρ* if $\rho'(s_\alpha)$ is ϵ-close to $\rho(s_\alpha)$ on K for all $\alpha \in \mathcal{I}$. The set of all (K, ϵ)-perturbations of ρ will be denoted by

$$U(\rho; K, \epsilon). \tag{3.26}$$

Accordingly, we topologize $\mathsf{Hom}(\Gamma, \mathrm{Homeo}(M))$ via the topology of "algebraic convergence" by identifying it, via the map $\rho \mapsto (\rho(s_\alpha))_{\alpha\in\mathcal{I}}$, with a subset of $[\mathrm{Homeo}(M)]^{\mathcal{I}}$ equipped with the subspace topology. Then the subset $U(\rho; K, \epsilon)$ of $\mathsf{Hom}(\Gamma, \mathrm{Homeo}(M))$ is an open neighborhood of ρ. Note that, when the ambient space M itself is compact, we can set $K = M$ and simply talk about ϵ-perturbations.

Now we are able to state Sullivan's structural stability theorem for S-hyperbolic group actions [Sul85, Theorem II].

3.27 Theorem (Sullivan)**.** *If an action $\rho : \Gamma \to \mathrm{Homeo}(M; \Lambda)$ is S-hyperbolic with data (δ, N), then the following hold.*

(1) *The action ρ is structurally stable in the sense of Lipschitz dynamics. In other words, there exist a compact set $K \supset \Lambda$ and a constant $\epsilon = \epsilon(\delta, N) > 0$ such that for every*

$$\rho' \in U(\rho; K, \epsilon)$$

there exist a ρ'-invariant compact subset $\Lambda' \subset M$ and an equivariant homeomorphism

$$\phi : \Lambda \to \Lambda',$$

that is, $\rho'(g) \circ \phi = \phi \circ \rho(g)$ on Λ for all $g \in \Gamma$.

(2) *The map* $U(\rho; K, \epsilon) \to C^0(\Lambda, M)$*,* $\rho' \mapsto \phi$ *is continuous at* ρ*.*

(3) *Every action* $\rho' \in U(\rho; K, \epsilon)$ *is S-expanding.*

If an action $\rho : \Gamma \to \mathrm{Homeo}(M; \Lambda)$ *is uniformly S-hyperbolic then, in addition to the statements above, the following is true as well.*

(4) *Every action* $\rho' \in U(\rho; K, \epsilon)$ *is again uniformly S-hyperbolic.*

A proof will be given in the next section. As an immediate consequence of Theorem 3.27(1), we have:

3.28 Corollary. *An S-hyperbolic action* $\rho : \Gamma \to \mathrm{Homeo}(M; \Lambda)$ *is algebraically stable in the following sense: for every* $\rho' \in U(\rho; K, \epsilon)$*, the kernel of the* ρ'*-action on* Λ' *equals the kernel of the* ρ*-action on* Λ*.*

We note, however, that faithfulness of the ρ-action on M does not imply faithfulness of nearby actions; see Example 7.7.

3.5 Toy examples

The most basic example of an S-hyperbolic action is a cyclic hyperbolic group of Möbius transformations acting on the unit circle S^1. Namely, we consider the Poincaré (conformal) disk model of $\mathbb{H}^2$ in $\mathbb{C} = \mathbb{R}^2$ and endow $M = S^1 = \partial_\infty \mathbb{H}^2$ with the induced Euclidean metric. Let $\gamma \in \mathrm{Isom}(\mathbb{H}^2)$ be a hyperbolic element and $\Gamma = \langle \gamma \rangle \cong \mathbb{Z}$. The *limit set* of the group Γ is $\Lambda = \{\lambda_-, \lambda_+\}$, with λ_+ the attractive and λ_- the repulsive fixed points of γ in M. Expanding subsets U_α, U_β for γ, γ^{-1} are sufficiently small arcs containing λ_-, λ_+, respectively.

Explicit expanding subsets can be found by considering the isometric circles I_γ and $I_{\gamma^{-1}}$ of γ and γ^{-1}, respectively. See Figure 2(Left). The arc of I_γ in $\mathbb{H}^2$ is a complete geodesic which is the perpendicular bisector of the points o and $\gamma^{-1}(o)$, where o denotes the Euclidean center of the Poincaré disk. Then we obtain an (λ, γ)-expanding subset U_α (with $\lambda > 1$) by cutting down slightly the open arc of $S^1 = \partial_\infty \mathbb{H}^2$ inside I_γ. For more details on isometric circles (or spheres) and their relation to Ford and Dirichlet fundamental domains, we refer to [Mas88, IV.G] for example. See also the discussion in the beginning of Section 7.2.

3.29 Example (k-fold non-trivial covering)**.** A more interesting example is obtained by taking a degree $k > 1$ covering $p : S^1 \to S^1$ of the above example. See Figure 2(Right) for the case of $k = 3$. The preimage of $\Lambda = \{\lambda_-, \lambda_+\}$ consists of $2k$ points and we can lift γ to a diffeomorphism $\widetilde{\gamma} : S^1 \to S^1$ fixing all these points. Let $\widetilde{\rho} : \Gamma = \langle \gamma \rangle \cong \mathbb{Z} \to \mathrm{Diff}(S^1)$ be the homomorphism sending the generator γ of Γ to $\widetilde{\gamma}$. The preimages $p^{-1}(U_\alpha)$, $p^{-1}(U_\beta)$ break into connected components

$$\widetilde{U}_{\alpha_i},\ \widetilde{U}_{\beta_i} \quad (i = 1, \ldots, k)$$

and the mappings $\widetilde{\gamma}$ and $\widetilde{\gamma}^{-1}$ act as expanding maps on each of these components. Therefore, we set

$$\mathcal{I} = \{\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k\}$$

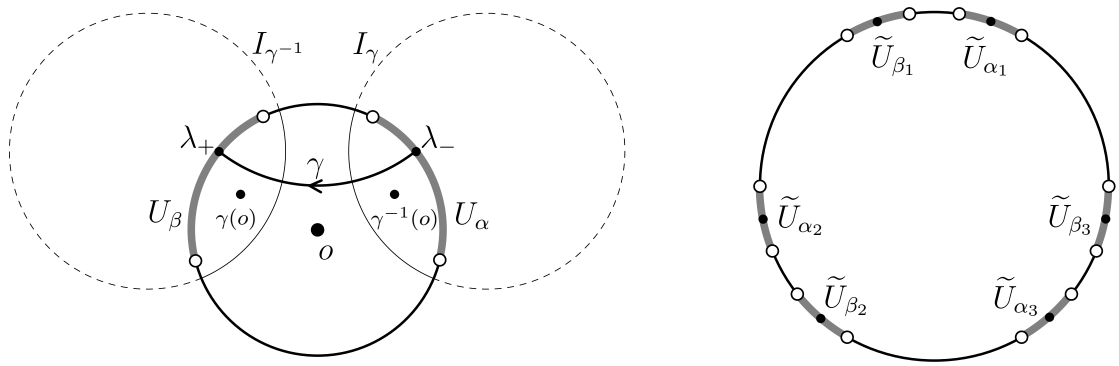

Figure 2: Expanding arcs for γ and γ^{-1} are colored gray in both examples. (Left) A hyperbolic transformation γ of $\mathbb{H}^2$. (Right) A covering of degree 3.

and define the map $s : \mathcal{I} \to \Sigma$,

$$\alpha_i \mapsto s_{\alpha_i} = \gamma, \quad \beta_i \mapsto s_{\beta_i} = \gamma^{-1}$$

from this index set to the generating set $\Sigma = \{\gamma, \gamma^{-1}\}$ of Γ. Then $\widetilde{U}_{\alpha_i}$, $\widetilde{U}_{\beta_i}$ will be expanding subsets for the actions $\widetilde{\rho}(s_{\alpha_i})$, $\widetilde{\rho}(s_{\beta_i})$ on S^1. (Note that $p^{-1}(U_\alpha)$ (resp. $p^{-1}(U_\beta)$) is *not* an expanding subset for the action $\widetilde{\rho}(s_{\alpha_i})$ (resp. $\widetilde{\rho}(s_{\beta_i})$).) The reader will verify that this action is S-hyperbolic.

The same construction works for surface group actions; see Example 7.3.

A trivial example where $U_\alpha = \emptyset$ for an index $\alpha \in \mathcal{I}$ is the action of a cyclic group $\Gamma = \langle \gamma \rangle \cong \mathbb{Z}$ generated by a loxodromic transformation $\gamma(z) = mz$, $|m| > 1$, on $M = \mathbb{C}$ (with the standard Euclidean metric) and $\Lambda = \{0\}$. Any open subset of M containing 0 is an expanding subset for γ, while the expanding subset for γ^{-1} is empty.

In Sections 6, 7.2 and 7.3, we give more complicated examples of S-hyperbolic actions.

4 Proof of Sullivan's theorem

We work out the details of Sullivan's proof of Theorem 3.27. The assertion (1) will be proved in Sections 4.1-4.6, and the assertions (2)-(4) in Sections 4.7-4.9, respectively.

Let $\rho : \Gamma \to \mathrm{Homeo}(M; \Lambda)$ be an S-hyperbolic action. Then we have the S-expansion data $(\mathcal{I}, \mathcal{U}, \Sigma, \Delta, L, \lambda)$ from Definition 3.3 (S-expansion), along with a constant $\delta \in (0, \Delta)$ and an integer $N \geq 1$ from Definition 3.21 (S-hyperbolicity).

4.1 Specifying small perturbations ρ'

We first need to determine a compact $K \supset \Lambda$ and a constant $\epsilon = \epsilon(\delta, N) > 0$ in order to specify the open set $U(\rho; K, \epsilon)$ of all (K, ϵ)-perturbations ρ' of ρ. We set them as follows:

$$K := \overline{N}_\Delta(\Lambda),$$
$$\epsilon := \frac{\lambda - 1}{2} \cdot \min\left\{\frac{\delta}{(N+1)L^N},\ 1\right\}.$$

Note that K is compact since (M, d) is assumed to be proper. Since $\Delta \leq \Delta_{\mathcal{U}}$ is a Lebesgue number of $\mathcal{U}$, it follows that $\mathcal{U}$ is a cover of $N_\Delta(\Lambda)$.

Now, we suppose

$$\rho' \in U(\rho; K, \epsilon).$$

Then, by the definitions (3.24) and (3.26), we have

$$d_{\mathrm{Lip},K}(\rho(s_\alpha), \rho'(s_\alpha)) < \epsilon \tag{4.1}$$

for all $s_\alpha \in \Sigma$.

We first observe that

$$\rho'(s_\alpha^{-1}) \text{ is } \lambda'\text{-expanding on } U_\alpha \cap K \tag{4.2}$$

for each $\alpha \in \mathcal{I}$, where

$$\lambda' := \lambda - \epsilon \geq \lambda - \frac{\lambda - 1}{2} = \frac{\lambda + 1}{2} > 1.$$

Indeed, since $\rho(s_\alpha^{-1})$ is λ-expanding on U_α, we see from (4.1) that

$$\frac{d(\rho'(s_\alpha^{-1})(x), \rho'(s_\alpha^{-1})(y))}{d(x,y)} > \frac{d(\rho(s_\alpha^{-1})(x), \rho(s_\alpha^{-1})(y))}{d(x,y)} - \epsilon \geq \lambda - \epsilon = \lambda'$$

for all distinct $x, y \in U_\alpha \cap K$.

Moreover, we also note that

$$\rho'(s_\alpha) \text{ is } (L + \epsilon)\text{-Lipschitz on } N_\Delta(\Lambda) \tag{4.3}$$

for every $\alpha \in \mathcal{I}$. To see this recall that the maps $\rho(s_\alpha)$ are L-Lipschitz on $N_\Delta(\Lambda)$ by Definition 3.3(i), thus by (4.1) again

$$\frac{d(\rho'(s_\alpha)(x), \rho'(s_\alpha)(y))}{d(x,y)} < \frac{d(\rho(s_\alpha)(x), \rho(s_\alpha)(y))}{d(x,y)} + \epsilon \leq L + \epsilon$$

for all distinct $x, y \in N_\Delta(\Lambda) \subset K$. Note that $L + \epsilon \geq \lambda + \epsilon > \lambda - \epsilon = \lambda'$.

4.2 Definition of ϕ

We first construct a map $\phi : \Lambda \to M$.

Let $x \in \Lambda$. In order to define $\phi(x)$, choose a δ-code (α, p) for x as in Section 3.2. Then from Lemma 3.15 we know that x has an exponentially shrinking nested sequence of neighborhoods $\rho(c_i^\alpha)[B_\delta(p_{i+1})]$ ($i \in \mathbb{N}_0$), so that

$$\{x\} = \bigcap_{i=0}^{\infty} \rho(c_i^\alpha)[B_\delta(p_{i+1})].$$

Now, consider a perturbation $\rho' \in U(\rho; K, \epsilon)$ of ρ as specified in Section 4.1. Then, under ρ', the sequence $\rho(c_i^\alpha)[B_\delta(p_{i+1})]$ ($i \in \mathbb{N}_0$) of neighborhoods of x will be perturbed slightly to a sequence of subsets

$$\rho'(c_i^\alpha)[B_\delta(p_{i+1})] \quad (i \in \mathbb{N}_0).$$

Nonetheless, we claim that this new sequence of subsets is still nested and exponentially shrinking. Since M is complete, the intersection of this collection of subsets is a singleton in M and we can *define* $\phi_\alpha(x)$ by the formula

$$\{\phi_\alpha(x)\} = \bigcap_{i=0}^{\infty} \rho'(c_i^\alpha)[B_\delta(p_{i+1})].$$

It remains to prove the above claim. In fact, we shall prove a bit more general statement, which will often be used later, for example, when we show that ϕ is well-defined (Section 4.3) and is continuous (Section 4.5).

4.4 Lemma. *Let $\rho : \Gamma \to \mathrm{Homeo}(M; \Lambda)$ be an S-hyperbolic action with data (δ, N). Consider a number $0 < \varepsilon \leq \min\{\frac{\delta}{(N+1)L^N}, 1\}$ (so that $\frac{\lambda-1}{2}\varepsilon \leq \epsilon$) and let*

$$\rho' \in U(\rho; K, \tfrac{\lambda-1}{2}\varepsilon) \subset U(\rho; K, \epsilon).$$

If (α, p) is a δ-code for $x \in \Lambda$ and $\Delta^\alpha := \max\{\eta \in (0, \Delta] \mid \alpha \in \mathrm{Code}^{x,\eta}\} \geq \delta$, then for every $\eta \in [\varepsilon, \Delta^\alpha]$ the sequence of subsets

$$\rho'(c_i^\alpha)[B_\eta(p_{i+1})] \quad (i \in \mathbb{N}_0)$$

is nested and exponentially shrinking. Consequently, we have

$$\{\phi_\alpha(x)\} = \bigcap_{i=0}^{\infty} \rho'(c_i^\alpha)[B_\eta(p_{i+1})] = \bigcap_{i=0}^{\infty} \rho'(c_i^\alpha)[B_\delta(p_{i+1})]$$

and thus, for every $i \in \mathbb{N}_0$,

$$\rho'(c_i^\alpha)^{-1}(\phi_\alpha(x)) \in B_\eta(p_{i+1}).$$

If the δ-code (α, p) is special, then

$$d(x, \phi_\alpha(x)) < \varepsilon.$$

4.5 Remark. Throughout Section 4, except in Section 4.7, we have to set

$$\varepsilon = \min\left\{\frac{\delta}{(N+1)L^N},\ 1\right\}$$

(so that $\frac{\lambda-1}{2}\varepsilon = \epsilon$) in this lemma, because we are considering (K,ϵ)-perturbations ρ' of ρ.

Proof. Since $\varepsilon \le \eta$, we see from the assumption $\rho' \in U(\rho; K, \frac{\lambda-1}{2}\varepsilon)$ that

$$\sup_{x\in K} d(\rho(s_\alpha^{-1})(x), \rho'(s_\alpha^{-1})(x)) \le d_{\mathrm{Lip},K}(\rho(s_\alpha^{-1}), \rho'(s_\alpha^{-1})) < \frac{\lambda-1}{2}\varepsilon \le \frac{1}{2}(\lambda\eta - \eta) \tag{4.6}$$

for all $\alpha \in \mathcal{I}$. Since $\eta \le \Delta^\alpha \le \Delta$, we have by Definition 3.3(ii) the inclusion (3.8) for ρ

$$B_\eta(p_{i+1}) \subset B_{\lambda\eta}(p_{i+1}) \subset \rho(s_{\alpha(i)}^{-1})[B_\eta(p_i)] \subset \rho(s_{\alpha(i)}^{-1})[U_{\alpha(i)}] \tag{4.7}$$

for all $i \in \mathbb{N}$, while $B_\eta(p_i) \subset B_{\Delta^\alpha}(p_i) \subset U_{\alpha(i)} \cap K$ as α is a Δ^α-code. Therefore, as for ρ', we conclude from (4.6) and (4.7) that

$$B_\eta(p_{i+1}) \subset \rho'(s_{\alpha(i)}^{-1})[B_\eta(p_i)] \subset \rho'(s_{\alpha(i)}^{-1})[U_{\alpha(i)} \cap K]. \tag{4.8}$$

Thus

$$\rho'(s_{\alpha(i)})[B_\eta(p_{i+1})] \subset B_\eta(p_i) \tag{4.9}$$

for all $i \in \mathbb{N}$, and we check as in (3.14) the nesting property

$$\rho'(c_i^\alpha)[B_\eta(p_{i+1})] \subset \rho'(c_{i-1}^\alpha)[B_\eta(p_i)]$$

for all $i \in \mathbb{N}$.

Furthermore, the diameter of

$$\rho'(c_i^\alpha)[B_\eta(p_{i+1})] = \rho'(s_{\alpha(0)}s_{\alpha(1)}\cdots s_{\alpha(i)})[B_\eta(p_{i+1})]$$

is at most $2\eta(L+\epsilon)/(\lambda')^i$, because we have (4.8) and each $\rho'(s_{\alpha(j)})$ $(1 \le j \le i)$ is $(1/\lambda')$-contracting on $\rho'(s_{\alpha(j)}^{-1})[U_{\alpha(j)} \cap K]$ by (4.2), and the last map $\rho'(s_{\alpha(0)})$ is $(L+\epsilon)$-Lipschitz on $B_\eta(p_1) \subset N_\Delta(\Lambda)$ by (4.3). Hence the exponentially shrinking property also holds.

If the δ-code (α, p) is special, then the inclusion (4.8) as well as (4.9) hold for $i = 0$. Thus

$$\phi_\alpha(x) \in \rho'(c_0^\alpha)[B_\eta(p_1)] = \rho'(s_{\alpha(0)})[B_\eta(p_1)] \subset B_\eta(x)$$

for all $\eta \in [\varepsilon, \Delta^\alpha]$. It follows that $d(x, \phi_\alpha(x)) < \varepsilon$. $\square$

4.3 ϕ is well-defined

We would like to show that $\phi_\alpha(x) = \phi_\beta(x)$ for $(\alpha, p), (\beta, q) \in \mathrm{Code}_\rho^{x,\delta}$. Since ρ is S-hyperbolic, the corresponding rays $c^\alpha, c^\beta \in \mathrm{Ray}_\rho^{x,\delta}(\Gamma)$ are N-equivalent, that is, $c^\alpha \sim_N c^\beta$. By definition, the relation $\approx_N$ generates the equivalence relation $\sim_N$ (see Definition 3.20(a)). Thus, it suffices to show the equality $\phi_\alpha(x) = \phi_\beta(x)$ when $c^\alpha \approx_N c^\beta$, that is, there exist infinite subsets P, $Q \subset \mathbb{N}_0$ such that the subsets $c^\alpha(P), c^\beta(Q)$ are within Hausdorff distance N from each other in (Γ, d_Σ).

Suppose to the contrary that

$$\bigcap_{i\in P} \rho'(c_i^\alpha)[B_\delta(p_{i+1})] = \{\phi_\alpha(x)\} \neq \{\phi_\beta(x)\} = \bigcap_{i\in Q} \rho'(c_i^\beta)[B_\delta(q_{i+1})].$$

Since the open sets $\rho'(c_i^\beta)[B_\delta(q_{i+1})]$ shrink to $\phi_\beta(x)$, there exists an integer $n \in Q$ such that $\phi_\alpha(x) \notin \rho'(c_n^\beta)[B_\delta(q_{n+1})]$, that is,

$$\rho'(c_n^\beta)^{-1}(\phi_\alpha(x)) \notin B_\delta(q_{n+1}). \tag{4.10}$$

Since the Hausdorff distance between $\{c_i^\alpha\}_{i\in P}$ and $\{c_i^\beta\}_{i\in Q}$ is at most N, there is an integer $m \in P$ such that $d_\Sigma(c_n^\beta, c_m^\alpha) \leq N$. Set

$$r = (c_n^\beta)^{-1} c_m^\alpha,$$

so that $|r|_\Sigma \leq N$. Note from (3.13) that $\rho(r)$ maps p_{m+1} to q_{n+1}:

$$\rho(r)(p_{m+1}) = \rho(r)[\rho(c_m^\alpha)^{-1}(x)] = \rho(c_n^\beta)^{-1}(x) = q_{n+1}.$$

See Figure 3.

In view of Lemma 4.4 (with $\varepsilon = \eta = \min\{\frac{\delta}{(N+1)L^N}, 1\}$), we may assume that we used η-balls in the definition of $\phi_\alpha(x)$, so that

$$\rho'(c_m^\alpha)^{-1}(\phi_\alpha(x)) \in B_\eta(p_{m+1}).$$

Now, if we show that $\rho'(r)$ maps $B_\eta(p_{m+1})$ into $B_\delta(q_{n+1})$ and hence

$$\begin{aligned}\rho'(c_n^\beta)^{-1}(\phi_\alpha(x)) &= \rho'(r)[\rho'(c_m^\alpha)^{-1}(\phi_\alpha(x))] \\ &\in \rho'(r)[B_\eta(p_{m+1})] \subset B_\delta(q_{n+1}),\end{aligned}$$

then we are done, since we are in contradiction with (4.10).

To show that $\rho'(r)[B_\eta(p_{m+1})] \subset B_\delta(q_{n+1})$, we first need the following:

Claim. *For every $w \in \Gamma$ such that $|w|_\Sigma = k \leq N$, we have*

$$d(\rho(w)(y), \rho'(w)(y)) < kL^{k-1}\epsilon$$

for all $y \in B_\eta(p_{m+1})$.

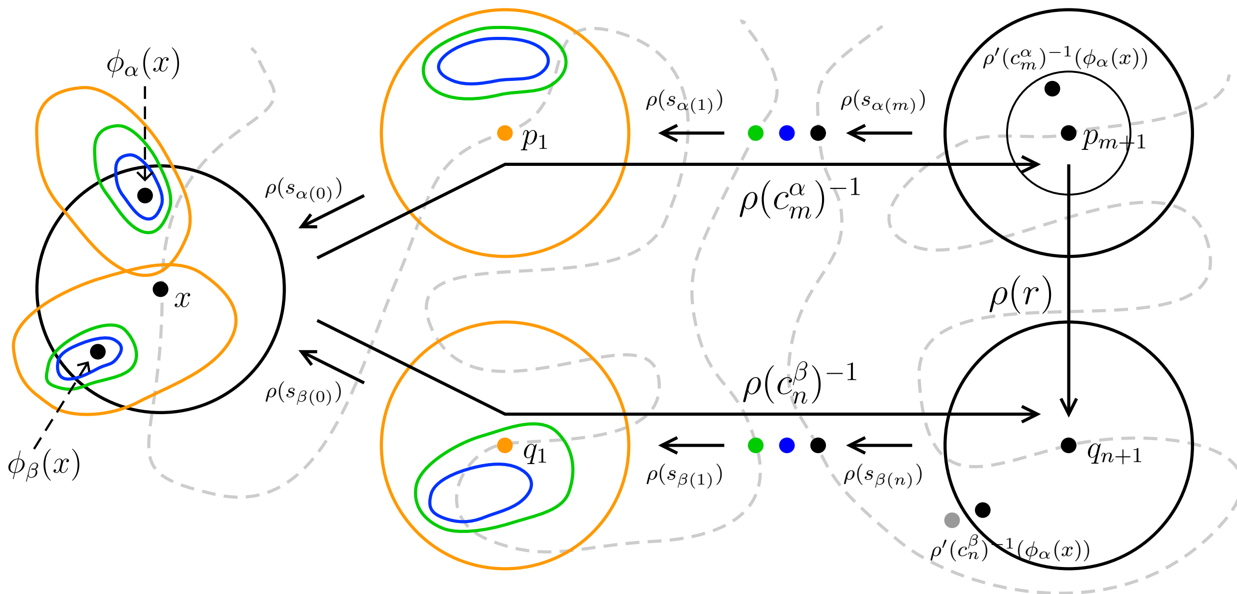

Figure 3: The points $\phi_\alpha(x)$ and $\phi_\beta(x)$.

Proof. We prove this by induction on k. If $|w|_\Sigma = 1$, the claim is true by (4.1) since $B_\eta(p_{m+1}) \subset N_\Delta(\Lambda)$. Suppose $w = st$ with $s \in \Sigma$, $|t|_\Sigma = k-1$ and $|w|_\Sigma = k$. Since $\rho(t)$ is L^{k-1}-Lipschitz on $N_{\Delta/L^{k-2}}(\Lambda) \supset B_\eta(p_{m+1})$ by (3.4), we have $\rho(t)(y) \in N_{L^{k-1}\eta}(\Lambda) \subset N_\Delta(\Lambda)$. By the induction hypothesis $d(\rho(t)(y), \rho'(t)(y)) < (k-1)L^{k-2}\epsilon$, we then have $\rho'(t)(y) \in N_\Delta(\Lambda)$ as well, since $\epsilon = \frac{\lambda-1}{2}\eta < L\eta$ and thus

$$L^{k-1}\eta + (k-1)L^{k-2}\epsilon < L^{k-1}\eta + (k-1)L^{k-2}L\eta = kL^{k-1}\eta < \delta < \Delta.$$

As $\rho(s)$ is L-Lipschitz on $N_\Delta(\Lambda)$, we obtain

$$d\big(\rho(s)[\rho(t)(y)], \rho(s)[\rho'(t)(y)]\big) < L \cdot (k-1)L^{k-2}\epsilon.$$

Furthermore, we have from (4.1)

$$d\big(\rho(s)[\rho'(t)(y)], \rho'(s)[\rho'(t)(y)]\big) < \epsilon$$

since $\rho'(t)(y) \in N_\Delta(\Lambda)$. Thus

$$\begin{aligned} d(\rho(w)(y), \rho'(w)(y)) &\le d\big(\rho(s)[\rho(t)(y)], \rho(s)[\rho'(t)(y)]\big) + d\big(\rho(s)[\rho'(t)(y)], \rho'(s)[\rho'(t)(y)]\big) \\ &< (k-1)L^{k-1}\epsilon + \epsilon \\ &< (k-1)L^{k-1}\epsilon + L^{k-1}\epsilon = kL^{k-1}\epsilon \end{aligned}$$

and the claim is proved. $\square$

Now, we let $k = |r|_\Sigma \leq N$ and verify that $\rho'(r)$ maps $B_\eta(p_{m+1})$ into $B_\delta(q_{n+1})$. If $y \in B_\eta(p_{m+1})$ then

$$\begin{aligned} d(\rho'(r)(y), q_{n+1}) &= d(\rho'(r)(y), \rho(r)(p_{m+1})) \\ &\leq d(\rho'(r)(y), \rho(r)(y)) + d(\rho(r)(y), \rho(r)(p_{m+1})) \\ &< kL^{k-1}\epsilon + L^k\eta \\ &< NL^{N-1}L\eta + L^N\eta \leq \delta, \end{aligned}$$

where the second inequality holds by the above claim and since $\rho(r)$ is L^k-Lipschitz on $N_{\Delta/L^{k-1}}(\Lambda) \supset B_\eta(p_{m+1})$ by (3.4), and the third inequality holds since $\epsilon = \frac{\lambda-1}{2}\eta < L\eta$.

This completes the proof of the equality $\phi_\alpha(x) = \phi_\beta(x)$.

From now on we may write $\phi(x)$ for $x \in \Lambda$ without ambiguity. An immediate consequence of this is that we are henceforth free to choose a *special* δ-code for x. Then from Lemma 4.4 (with $\varepsilon = \min\{\frac{\delta}{(N+1)L^N}, 1\}$) we conclude that

$$d(x, \phi(x)) < \varepsilon \leq \frac{\delta}{(N+1)L^N} < \frac{\delta}{2} \tag{4.11}$$

for all $x \in \Lambda$.

4.4 ϕ is equivariant

To show the equivariance of ϕ, it suffices to check it on the generating set Σ of Γ.

Given $x \in \Lambda$ and $s \in \Sigma$, set $y = \rho(s^{-1})(x)$. Let (β, q) be a *special* δ-code for y, so that $B_\delta(y) \subset U_{\beta(0)}$ (see Definition 3.6). Then we consider a δ-code (α, p) for x defined by $s_{\alpha(0)} = s$ and

$$\begin{aligned} \alpha(i) &= \beta(i-1), \\ p_i &= q_{i-1} \end{aligned}$$

for $i \in \mathbb{N}$: indeed, we verify the requirement that

$$B_\delta(p_i) = B_\delta(q_{i-1}) \subset U_{\beta(i-1)} = U_{\alpha(i)}$$

for $i \in \mathbb{N}$. The associated rays $\{c_i^\alpha\}_{i\in\mathbb{N}_0}$ and $\{c_i^\beta\}_{i\in\mathbb{N}_0}$ (see Definition 3.11) are related by

$$c_i^\alpha = s_{\alpha(0)}(s_{\alpha(1)} \cdots s_{\alpha(i)}) = s(s_{\beta(0)} \cdots s_{\beta(i-1)}) = s\, c_{i-1}^\beta$$

for $i \in \mathbb{N}$. Therefore, we have

$$\begin{aligned}\{\phi[\rho(s)(y)]\} = \{\phi(x)\} &= \bigcap_{i=0}^{\infty} \rho'(c_i^\alpha)[B_\delta(p_{i+1})] \\ &= \bigcap_{i=1}^{\infty} \rho'(c_i^\alpha)[B_\delta(p_{i+1})] \\ &= \bigcap_{i=1}^{\infty} \rho'(s)\rho'(c_{i-1}^\beta)[B_\delta(q_i)] \\ &= \rho'(s)\left[\bigcap_{i=0}^{\infty} \rho'(c_i^\beta)[B_\delta(q_{i+1})]\right] = \rho'(s)\{\phi(y)\},\end{aligned}$$

which implies the equivariance of ϕ.

4.5 ϕ is continuous

Let $\varepsilon_1 > 0$ be given. In order to show that ϕ is continuous at $x \in \Lambda$, assign a δ-code (α, p) for x which comes from a Δ-code for x. (See Remark 3.7(b) and we note that Lemma 4.4 applies to the δ-code (α, p) with $\eta = \Delta^\alpha = \Delta$). Choose an integer $k \in \mathbb{N}_0$ such that $2\Delta(L+\epsilon)/(\lambda')^k < \varepsilon_1$. Since $\rho(c_k^\alpha)^{-1}$ maps x to p_{k+1} and is continuous, there exists $\delta_1 > 0$ such that $\rho(c_k^\alpha)^{-1}[B_{\delta_1}(x)] \subset B_{\Delta-\delta}(p_{k+1})$. Below we will show that if $y \in \Lambda$ satisfies $d(x, y) < \delta_1$ then $d(\phi(x), \phi(y)) < \varepsilon_1$ thereby proving that ϕ is continuous at x.

Let $y \in \Lambda$ be such that $d(x, y) < \delta_1$. Then $\rho(c_k^\alpha)^{-1}(y) \in B_{\Delta-\delta}(p_{k+1})$, hence

$$y \in \rho(c_k^\alpha)[B_{\Delta-\delta}(p_{k+1})] \subset \rho(c_{k-1}^\alpha)[B_{\Delta-\delta}(p_k)] \subset \cdots \subset \rho(c_0^\alpha)[B_{\Delta-\delta}(p_1)]$$

by Lemma 3.15(i). In other words, for $0 \leq i \leq k$, we have

$$\begin{aligned}&\rho(c_i^\alpha)^{-1}(y) \in B_{\Delta-\delta}(p_{i+1}),\\ \text{hence}\quad &B_\delta(\rho(c_i^\alpha)^{-1}(y)) \subset B_\Delta(p_{i+1}) \subset U_{\alpha(i+1)}.\end{aligned}$$

This means that there is a δ-code (β, q) for $y \in \Lambda$ with a property that

$$\beta(i) = \alpha(i) \quad \text{for } 0 \leq i \leq k.$$

In particular, $c_k^\beta = c_k^\alpha$ and hence $q_{k+1} = \rho(c_k^\beta)^{-1}(y) = \rho(c_k^\alpha)^{-1}(y) \in B_{\Delta-\delta}(p_{k+1})$. Consequently, we have $B_\delta(q_{k+1}) \subset B_\Delta(p_{k+1})$ and

$$\begin{aligned}\{\phi(y)\} &= \bigcap_{i=0}^{\infty} \rho'(c_i^\beta)[B_\delta(q_{i+1})] \\ &\subset \rho'(c_k^\beta)[B_\delta(q_{k+1})] \\ &= \rho'(c_k^\alpha)[B_\delta(q_{k+1})] \\ &\subset \rho'(c_k^\alpha)[B_\Delta(p_{k+1})],\end{aligned}$$

By (the proof of) Lemma 4.4, the diameter of the last set $\rho'(c_k^\alpha)[B_\Delta(p_{k+1})]$ is at most $2\Delta(L+\epsilon)/(\lambda')^k < \varepsilon_1$. Since $\phi(x) \in \rho'(c_k^\alpha)[B_\Delta(p_{k+1})]$ as well, we showed

$$d(\phi(x), \phi(y)) \leq 2\Delta(L+\epsilon)/(\lambda')^k < \varepsilon_1$$

as desired.

4.6 ϕ is injective

Suppose, to the contrary, that $\phi(x) = \phi(y)$ but $x \neq y$. Since ϕ is equivariant, we then have $\phi[\rho(g)(x)] = \phi[\rho(g)(y)]$ for any $g \in \Gamma$, hence, by (4.11),

$$d(\rho(g)(x), \rho(g)(y)) < d(\rho(g)(x), \phi[\rho(g)(x)]) + d(\phi[\rho(g)(y)], \rho(g)(y)) < \delta/2 + \delta/2 = \delta$$

for all $g \in \Gamma$. But this contradicts Corollary 3.16, since ρ has an expansivity constant $\Delta > \delta$. Therefore, ϕ is injective.

So far, we have proved the claim that ϕ is an equivariant homeomorphism. This completes the proof of Theorem 3.27(1).

4.7 ϕ depends continuously on ρ'

We will show that if ρ' is close to ρ then ϕ is close to the identity map, and hence $\phi(\Lambda) = \Lambda'$ is close to Λ. More precisely, we claim that the map

$$\begin{aligned} \mathsf{Hom}(\Gamma, \mathrm{Homeo}(M)) \supset U(\rho; K, \epsilon) &\to C^0(\Lambda, (M, d)) \\ \rho' &\mapsto \phi \end{aligned}$$

is continuous at ρ, where we equip $\mathsf{Hom}(\Gamma, \mathrm{Homeo}(M))$ with the topology of "algebraic convergence" as in Section 3.4 and $C^0(\Lambda, (M, d))$ with the uniform topology. As for Λ and Λ', this will imply that the map $\rho' \mapsto d_{\mathrm{Haus}}(\Lambda, \Lambda')$ is continuous at ρ, where d_{Haus} stands for the Hausdorff distance.

To prove the claim, suppose a sufficiently small constant $\varepsilon > 0$ is given. Of course, we may assume $\varepsilon \leq \min\{\frac{\delta}{(N+1)L^N}, 1\}$. Then we have to find a neighborhood $U(\rho; K', \epsilon')$ of ρ such that $\sup\{d(x, \phi(x)) \mid x \in \Lambda\} < \varepsilon$ for every $\rho' \in U(\rho; K', \epsilon')$. So, we let

$$K' = K \quad \text{and} \quad \epsilon' = \frac{\lambda - 1}{2}\varepsilon.$$

Suppose $\rho' \in U(\rho; K', \epsilon')$. Let $x \in \Lambda$ and choose a *special* δ-code (α, p) for x. Then by Lemma 4.4 we have $d(x, \phi(x)) = d(x, \phi_\alpha(x)) < \varepsilon$. Since $x \in \Lambda$ is arbitrary, the proof is complete.

4.8 ρ' is S-expanding

We will now check that $\rho' : \Gamma \to \mathrm{Homeo}(M; \Lambda')$ is S-expanding. In order to verify Definition 3.3 for ρ', we use the same data $\mathcal{I}' = \mathcal{I}$ and $\Sigma' = \Sigma$, but use

$$\mathcal{U}' = \{U'_\alpha := \operatorname{int} U_\alpha^{\delta/2} \cap N_\Delta(\Lambda) \mid \alpha \in \mathcal{I}\}$$

as well as $L + \epsilon$ and λ' (instead of L and λ, respectively). If $x' \in \Lambda'$, then $d(\phi^{-1}(x'), x') < \delta/2$ by (4.11), so $\overline{B}_{\delta/2}(x') \subset B_\delta(\phi^{-1}(x')) \subset U_\alpha \cap N_\Delta(\Lambda)$ for some $\alpha \in \mathcal{I}$, and hence $x' \in U'_\alpha$. Therefore, $\mathcal{U}'$ covers Λ'. Let $\Delta' = \min\{\Delta_{\mathcal{U}'}, \Delta - \delta/2\}$. Then $N_{\Delta'}(\Lambda') \subset N_\Delta(\Lambda)$.

Now, by (4.2), the map $\rho'(s_\alpha^{-1})$ is (λ', U'_α)-expanding for every $\alpha \in \mathcal{I}$. Moreover, by (4.3), the maps $\rho'(s_\alpha^{-1})$ are $(L + \epsilon)$-Lipschitz on $N_{\Delta'}(\Lambda')$. Lastly, in view of Remark 3.5(e), we can further modify $\mathcal{U}'$ and Δ' (retaining the names $\mathcal{U}'$ and Δ' for the sake of simplicity of notation), so that the properties (i) and (ii) of Definition 3.3 are satisfied for ρ' with data $(\mathcal{I}, \mathcal{U}', \Sigma, \Delta', L + \epsilon, \lambda')$. Therefore, ρ' is S-expanding.

For later use we note that, for the modified $\mathcal{U}' = \{U'_\alpha \mid \alpha \in \mathcal{I}\}$ as above, it holds that $U'_\alpha \subset U_\alpha^{\delta/2} \subset U_\alpha$ for $\alpha \in \mathcal{I}$.

4.9 ρ' is again uniformly S-hyperbolic

Assume that the action $\rho : \Gamma \to \mathrm{Homeo}(M; \Lambda)$ is uniformly S-hyperbolic (Definition 3.21). Then there exists a constant $\delta \in (0, \Delta)$ such that for all $\eta \in (0, \delta]$ there is an integer $N = N(\eta) \geq 1$ such that, for every $x \in \Lambda$, all rays in $\mathrm{Ray}_\rho^{x,\eta}(\Gamma)$ are $N(\eta)$-equivalent. Set $K \supset \Lambda$ and $\epsilon > 0$ as in Section 4.1. We will show that every $\rho' \in U(\rho; K, \epsilon)$ is uniformly S-hyperbolic.

We know from Section 4.8 that ρ' is S-expanding. We continue to use the same data found there, for example, the cover $\mathcal{U}'$ and a Lebesgue number Δ'.

In order to check that ρ' is uniformly S-hyperbolic, set $\delta' = \min\{\Delta', \delta\}$ and let $\eta \in (0, \delta']$. We claim that if (α, p') is an η-code for $\phi(x) \in \Lambda'$ then $(\alpha, \phi^{-1} \circ p')$ is an η-code for $x \in \Lambda$. Since $\eta \leq \delta' \leq \delta$, uniform S-hyperbolicity of ρ' will then follow immediately from that of ρ.

It remains to prove the claim. Recall Definition 3.6. Since (α, p') is an η-code for $\phi(x) \in \Lambda'$, we have

$$\begin{aligned} p'_0 &= \phi(x), \\ p'_{i+1} &= \rho'(s_{\alpha(i)}^{-1})(p'_i), \\ B_\eta(p'_i) &\subset U'_{\alpha(i)} \end{aligned}$$

for all $i \in \mathbb{N}$. To show that $(\alpha, \phi^{-1} \circ p')$ is an η-code for $x \in \Lambda$, we first check

$$(\phi^{-1} \circ p')(0) = \phi^{-1}(\phi(x)) = x$$

and then, for $i \in \mathbb{N}$, check

$$(\phi^{-1} \circ p')(i+1) = \phi^{-1}[\rho'(s_{\alpha(i)}^{-1})(p'_i)] = \rho(s_{\alpha(i)}^{-1})[\phi^{-1}(p'_i)] = \rho(s_{\alpha(i)}^{-1})[(\phi^{-1} \circ p')(i)],$$

where the second equality is due to the equivariance of ϕ. Lastly, since $B_\eta(p'_i) \subset U'_{\alpha(i)} \subset U^{\delta/2}_{\alpha(i)}$ for all $i \in \mathbb{N}$, we have $B_{\eta+(\delta/2)}(p'_i) \subset U_{\alpha(i)}$. Since $d(\phi^{-1}(p'_i), p'_i) < \delta/2$ by (4.11), we obtain

$$B_\eta((\phi^{-1} \circ p')(i)) = B_\eta(\phi^{-1}(p'_i)) \subset B_{\eta+(\delta/2)}(p'_i) \subset U_{\alpha(i)}$$

as desired. The claim is proved.

5 Actions of hyperbolic groups

In this section we first prove that S-expansion implies uniform S-fellow-traveling for certain nice actions of hyperbolic groups, and then explore to which extent S-hyperbolic actions of hyperbolic groups arise from their actions on Gromov boundaries.

5.1 S-expansion implies S-fellow-traveling

5.1 Theorem. *Let Γ be a non-elementary hyperbolic group. Suppose that an action $\rho : \Gamma \to \mathrm{Homeo}(M; \Lambda)$ is S-expanding and there exists an equivariant continuous nowhere constant map $f : \Lambda \to \partial_\infty\Gamma$. Then ρ is uniformly S-fellow-traveling and f is the coding map of ρ.*

Proof. Let (Δ, Σ) be the S-expansion data of ρ (see Definition 3.3). Recall that d_Σ denotes the word metric on Γ with respect to Σ.

We claim that if $x \in \Lambda$ and $\eta \in (0, \Delta]$ then, for every η-code α for x, the α-ray $c^\alpha \in \mathrm{Ray}^{x,\eta}_\rho(\Gamma)$ is a uniform quasi-geodesic ray in (Γ, d_Σ) asymptotic to $f(x)$. To see this, we first note that by Corollary 3.17 the ray c^α is indeed an (A, C)-quasi-geodesic ray with A and C independent of x and η.

By Theorem 3.18(1), there is a subsequence (g_j) of (c^α_i) such that $(\rho(g_j))$ converges to x on some ball $B_{\eta/2}(q) \subset \Lambda$. Since f is nowhere constant, the image $S := f(B_{\eta/2}(q)) \subset \partial_\infty\Gamma$ is not a singleton. By the equivariance of f, the subsequence (g_j) converges to $\xi := f(x)$ pointwise on S. Moreover, the initial point $c^\alpha_0 = s_{\alpha(0)} \in \Sigma$ is a generator of Γ. Therefore, Lemma 2.7 applies to the ray c^α and we conclude that the image $c^\alpha(\mathbb{N}_0)$ is D-Hausdorff close to a geodesic ray $e\xi$ in (Γ, d_Σ), where the constant D depends only on the hyperbolicity constant of (Γ, d_Σ) and the quasi-isometry constants (A, C).

Now, suppose that $c^\alpha, c^\beta \in \mathrm{Ray}^{x,\eta}_\rho(\Gamma)$ are rays associated to η-codes α, β for $x \in \Lambda$, respectively. Then the images of c^α, c^β are within Hausdorff distance D from a geodesic ray $e\xi$, hence, these images are $2D$-Hausdorff close. Therefore, the rays $2D$-fellow travel each other. Since $x \in \Lambda$ is arbitrary, we conclude that the action is uniformly S-fellow-traveling with data $(\delta, N(\eta)) = (\Delta, 2D)$; recall Definitions 3.20 and 3.21. □

5.2 Corollary. *Let Γ be a non-elementary hyperbolic group. Suppose that the action of Γ on its Gromov boundary $\partial_\infty\Gamma$ is S-expanding with respect to a metric d_∞ compatible with the topology. Then this action is uniformly S-fellow-traveling.*

Proof. We set $M = \Lambda = \partial_\infty\Gamma$ and $f = \mathrm{id} : \partial_\infty\Gamma \to \partial_\infty\Gamma$ in the above theorem. □

Let Γ be a non-elementary hyperbolic group and d_a a visual metric on $\partial_\infty\Gamma$. Coornaert [Coo93, Proposition 3.1, Lemma 6.2] showed that the Γ-action on $(\partial_\infty\Gamma, d_a)$ is S-expanding. Thus, we have:

5.3 Corollary. *Let Γ be a non-elementary hyperbolic group with the Gromov boundary $\partial_\infty\Gamma$ equipped with a visual metric d_a. Then the action of Γ on $(\partial_\infty\Gamma, d_a)$ is uniformly S-fellow-traveling.*

5.2 S-hyperbolic actions of hyperbolic groups

It is natural to ask to what extent the converse of Corollary 5.3 is true:

5.4 Question. Does every S-hyperbolic action $\Gamma \to \mathrm{Homeo}(M;\Lambda)$ come from an action of a hyperbolic group on its Gromov boundary?

Assume first that $\mathrm{card}(\Lambda) \geq 3$ and the action of Γ on Λ in Question 5.4 is a convergence action (see Section 2.1 for definition). Then, in view of the S-expansion condition, it is also a uniform convergence action; see [KLP17, Lemma 3.13] or [KL18, Theorem 8.8] for a different argument. If we assume, in addition, that Λ is perfect (or that Λ is the limit set of the action of Γ on Λ, see Theorem 2.2), then Γ is hyperbolic and Λ is equivariantly homeomorphic to the Gromov boundary $\partial_\infty\Gamma$ (Theorem 2.8). To summarize:

5.5 Proposition. *Suppose an S-expanding action $\Gamma \to \mathrm{Homeo}(M;\Lambda)$ is a convergence action with limit set Λ satisfying $\mathrm{card}(\Lambda) \geq 3$. Then Γ is hyperbolic and Λ is equivariantly homeomorphic to $\partial_\infty\Gamma$.*

Note that we do not even need to assume faithfulness of the action of Γ on Λ since, by the convergence action assumption, such an action necessarily has finite kernel.

As another application of the formalism of convergence group actions we obtain:

5.6 Proposition. *Suppose that Γ is hyperbolic, d_∞ is a compatible metric on the Gromov boundary $\partial_\infty\Gamma$, and the Γ-action on $(\partial_\infty\Gamma, d_\infty)$ is S-expanding. Define the subset $\Sigma_0 \subset \Sigma$ of the finite generating set Σ, consisting of elements s_α with non-empty expansion subsets $U_\alpha \subset \Lambda$. Then Σ_0 generates a finite index subgroup $\Gamma_0 < \Gamma$.*

Proof. The action of Γ_0 on $\partial_\infty\Gamma$ is still S-expanding and convergence. Therefore, as noted above, the action of Γ_0 on $T(\partial_\infty\Gamma)$ is also cocompact. Since the action of Γ on $T(\partial_\infty\Gamma)$ is properly discontinuous, it follows that Γ_0 has finite index in Γ. □

Next, assuming hyperbolicity of Γ in Question 5.4, we can relate Λ and $\partial_\infty\Gamma$. Recall the coding map $\pi : \Lambda \to \partial_\infty\Gamma$ in Definition 3.23.

5.7 Theorem. *Let Γ be a non-elementary hyperbolic group. If $\Gamma \to \mathrm{Homeo}(M;\Lambda)$ is an S-hyperbolic action, then the following hold.*

(1) *The coding map $\pi : \Lambda \to \partial_\infty\Gamma$ is an equivariant continuous surjective map.*

(2) *For each minimal non-empty closed* Γ*-invariant subset* $\Lambda_\mu \subset \Lambda$*, the restriction* $\pi_\mu : \Lambda_\mu \to \partial_\infty\Gamma$ *of* π *to* Λ_μ *is a surjective quasi-open map.*
(3) *Every* Λ_μ *as above is perfect.*

Proof. (1) Continuity of $\pi : \Lambda \to \partial_\infty\Gamma$ can be seen as in Section 4.5, where we showed that if $x, y \in \Lambda$ are close then there exist δ-codes α and β for x and y, respectively, such that $\alpha(i) = \beta(i)$ for all $0 \le i \le n$, where $n \in \mathbb{N}$ is sufficiently large. Then

$$c_i^\alpha = s_{\alpha(0)} s_{\alpha(1)} \cdots s_{\alpha(i)} = s_{\beta(0)} s_{\beta(1)} \cdots s_{\beta(i)} = c_i^\beta$$

for all $0 \le i \le n$. This means $\pi(y) \in V_n(\pi(x))$ for a sufficiently large n, hence $\pi(x)$ and $\pi(y)$ are close.

Since Λ is compact, the image $\pi(\Lambda)$ is closed and Γ-invariant. By the minimality of the action of Γ on $\partial_\infty\Gamma$, we have $\pi(\Lambda) = \partial_\infty\Gamma$.

(2) Surjectivity of π_μ follows from the minimality of the action of Γ on $\partial_\infty\Gamma$ as in part (1). We now prove that each π_μ is quasi-open. Since Λ_μ is compact, it is locally compact, hence it suffices to prove that for every compact subset $K \subset \Lambda_\mu$ with non-empty interior, the image $\pi(K) \subset \partial_\infty\Gamma$ also has non-empty interior.

In view of the minimality of the Γ-action on Λ_μ and compactness of Λ_μ, there exists a finite subset $\{g_1, \dots, g_n\} \subset \Gamma$ such that

$$\rho(g_1)(\operatorname{int} K) \cup \cdots \cup \rho(g_n)(\operatorname{int} K) = \Lambda_\mu.$$

By the equivariance of π and surjectivity of $\pi_\mu : \Lambda_\mu \to \partial_\infty\Gamma$, we also have

$$g_1(\pi(K)) \cup \cdots \cup g_n(\pi(K)) = \partial_\infty\Gamma.$$

Since a finite collection (even a countable collection) of nowhere dense subsets cannot cover $\partial_\infty\Gamma$, it follows that $\pi(K)$ has non-empty interior.

(3) Suppose that Λ_μ has an isolated point z. Since the action of Γ on Λ_μ is minimal, the compact subset $\Lambda_\mu \subset \Lambda$ consists entirely of isolated points, i.e. is finite. Therefore, $\pi(\Lambda_\mu) \subset \partial_\infty\Gamma$ is a finite non-empty Γ-invariant subset. This contradicts the minimality of the action of Γ on $\partial_\infty\Gamma$. $\square$

5.8 Remark. For some minimal S-hyperbolic actions of hyperbolic groups $\Gamma \to \operatorname{Homeo}(\Lambda)$ the map π is not open; see Example 7.6.

5.9 Corollary. *Let* Γ *be a non-elementary hyperbolic group. If* $\Gamma \to \operatorname{Homeo}(M; \Lambda)$ *is an S-hyperbolic action, then:*
(1) Γ *acts on* Λ *with finite kernel* K*.*
(2) *If* (g_i) *is a sequence in* Γ *converging to the identity on* Λ *then the projection of this sequence to* Γ/K *is eventually equal to* $e \in \Gamma/K$*.*

Proof. Both statements are immediate consequences of Theorem 5.7(1) and the convergence property for the action of a Γ on $\partial_\infty\Gamma$; see Section 2.2. $\square$

As another immediate corollary of the theorem and Theorem 5.1 we obtain:

5.10 Corollary. *Let* Γ *be a non-elementary hyperbolic group. Then every S-hyperbolic action of* Γ *is in fact uniformly S-fellow-traveling.*

These are positive results regarding Question 5.4. In Section 7, however, we present several examples which show that in general the question has negative answer.

6 Anosov subgroups

Our goal in this section is to characterize Anosov subgroups in terms of the S-expansion condition on suitable subsets of partial flag manifolds (Theorem 6.3) and show that the corresponding actions are S-hyperbolic. As an application, we give an alternative proof of the stability of Anosov subgroups (Corollary 6.5). We end the section with a brief historical remark on stability of group actions.

6.1 Definition

Let us first recall several necessary definitions regarding the geometry of symmetric spaces. For more details, see [BGS85], [Ebe96] or [KLP17, Section 2].

Let X be a symmetric space of non-compact type. Then the identity component, denoted by G, of the group of isometries of X is a connected semisimple Lie group with trivial center and no compact factors. For a point $x \in X$, its stabilizer K in G is a maximal compact subgroup and the quotient space G/K is G-equivariantly diffeomorphic to X.

The visual boundary $\partial_\infty X$ of X has a topological spherical building structure, the Tits building associated to X. Let a_{mod} denote the model apartment of this building and W the Weyl group acting isometrically on a_{mod}. We will fix a chamber $\sigma_{\text{mod}} \subset a_{\text{mod}}$ and call it the model chamber; it is a fundamental domain for the W-action on a_{mod}. Let $w_0 \in W$ denote the unique element sending σ_{mod} to the opposite chamber $-\sigma_{\text{mod}}$. Then the *opposition involution* ι of the model chamber σ_{mod} is defined as $\iota = -w_0$.

Consider the induced action of G on $\partial_\infty X$. Every orbit intersects every chamber exactly once, so there is a natural identification $\partial_\infty X/G \cong \sigma_{\text{mod}}$. The projection $\theta : \partial_\infty X \to \partial_\infty X/G$ is called the *type* map. Let τ_{mod} be a face of σ_{mod}. The τ_{mod}-*flag manifold* $\text{Flag}(\tau_{\text{mod}})$ is the space of simplices of type τ_{mod} in $\partial_\infty X$. It has a structure of a compact smooth manifold and can be identified with the quotient space G/P_τ, where $P_\tau < G$ is the stabilizer subgroup of a simplex $\tau \subset \partial_\infty X$ of type $\theta(\tau) = \tau_{\text{mod}}$. Two simplices τ_1 and τ_2 in $\partial_\infty X$ are said to be *antipodal* if they are opposite in an apartment containing both of them; their types are related by $\theta(\tau_2) = \iota\theta(\tau_1)$. A subset E of $\text{Flag}(\tau_{\text{mod}})$ is said to be *antipodal* if any two distinct elements of E are antipodal. A map into $\text{Flag}(\tau_{\text{mod}})$ is *antipodal* if it is injective and has antipodal image. We shall always assume that $\text{Flag}(\tau_{\text{mod}})$ is equipped with an auxiliary Riemannian metric.

We are now ready to define Anosov subgroups of G. The following definition of *Anosov* and *non-uniformly Anosov* subgroups is given by Kapovich, Leeb and Porti in [KLP17, Definitions 5.43 and 5.62]. Many other equivalent characterizations of the Anosov subgroups are established in [KLP17, Theorem 5.47] and their equivalence with the original definitions given by Labourie [Lab06] and Guichard and Wienhard [GW12] is proven in [KLP17, Section 5.11].

6.1 Definition. Let $\tau_{\mathrm{mod}} \subset \sigma_{\mathrm{mod}}$ be an ι-invariant face. A subgroup $\Gamma < G$ is τ_{mod}-*boundary embedded* if

(a) it is a hyperbolic group;

(b) there is an antipodal Γ-equivariant continuous map (called a *boundary embedding*)

$$\beta : \partial_\infty \Gamma \to \mathrm{Flag}(\tau_{\mathrm{mod}});$$

A τ_{mod}-boundary embedded subgroup $\Gamma < G$ with a boundary embedding α is τ_{mod}-*Anosov* if

(c) for every $\xi \in \partial_\infty \Gamma$ and for every geodesic ray $r : \mathbb{N}_0 \to \Gamma$ starting at $e \in \Gamma$ and asymptotic to ξ, the expansion factor (see (2.1)) satisfies

$$\epsilon(r(n)^{-1}, \alpha(\xi)) \geq A\, e^{Cn}$$

for $n \in \mathbb{N}_0$ with constants $A, C > 0$ independent of the point ξ and the ray r.

A τ_{mod}-boundary embedded subgroup $\Gamma < G$ with a boundary embedding α is *non-uniformly* τ_{mod}-*Anosov* if

(d) for every $\xi \in \partial_\infty \Gamma$ and for every geodesic ray $r : \mathbb{N}_0 \to \Gamma$ starting at $e \in \Gamma$ and asymptotic to ξ, the expansion factor satisfies

$$\sup_{n \in \mathbb{N}_0} \epsilon(r(n)^{-1}, \alpha(\xi)) = +\infty.$$

A τ_{mod}-boundary embedded subgroup $\Gamma < G$ is said to be *non-elementary* if it is a non-elementary hyperbolic group.

6.2 Remark. (a) In fact, for τ_{mod}-boundary embedded subgroups, the conditions (c) and (d) are equivalent; see [KLP17, Theorem 5.47]. For the purpose of this paper, the definition of non-uniformly τ_{mod}-Anosov subgroups will suffice.

(b) A τ_{mod}-Anosov subgroup $\Gamma < G$ may have other boundary embeddings $\beta : \partial_\infty \Gamma \to \mathrm{Flag}(\tau_{\mathrm{mod}})$ besides the map α which appears in the definition; see [KLP14, Example 6.20]. However, the boundary embedding α as in the conditions (c) and (d) is unique; its image is the τ_{mod}-*limit set* of Γ in $\mathrm{Flag}(\tau_{\mathrm{mod}})$,

$$\alpha(\partial_\infty \Gamma) = \Lambda_\Gamma(\tau_{\mathrm{mod}}) \subset \mathrm{Flag}(\tau_{\mathrm{mod}}).$$

We thus will refer to the map α as the *asymptotic embedding* for $\Gamma < G$.

6.2 S-hyperbolicity and stability for Anosov subgroups

For the sake of simplicity, we shall restrict our attention to the case of non-elementary hyperbolic groups and make use of Corollary 3.17 and Theorems 5.1 and 5.7. Then Lemma 6.4 below says that the condition (d) in Definition 6.1 is also equivalent to the S-expansion condition (Definition 3.3) at the image of the boundary embedding. Consequently, we obtain the following characterization of Anosov subgroups:

6.3 Theorem. *For a non-elementary hyperbolic subgroup $\Gamma < G$ the following are equivalent.*

(1) Γ *is non-uniformly $\tau_{\rm mod}$-Anosov with asymptotic embedding $\alpha : \partial_\infty\Gamma \to \Lambda_\Gamma(\tau_{\rm mod})$.*

(2) Γ *is $\tau_{\rm mod}$-boundary embedded with a boundary embedding $\beta : \partial_\infty\Gamma \to \mathrm{Flag}(\tau_{\rm mod})$ and the Γ-action on $\mathrm{Flag}(\tau_{\rm mod})$ is S-expanding at $\beta(\partial_\infty\Gamma)$; in this case, β equals the asymptotic embedding α.*

(3) *There exists a closed Γ-invariant antipodal subset $\Lambda \subset \mathrm{Flag}(\tau_{\rm mod})$ such that the action $\Gamma \to \mathrm{Homeo}(\mathrm{Flag}(\tau_{\rm mod}); \Lambda)$ is S-hyperbolic with injective coding map $\pi : \Lambda \to \partial_\infty\Gamma$; in this case, π equals the inverse of the asymptotic embedding α.*

Proof. $(2 \Rightarrow 3)$ Let $\Lambda = \beta(\partial_\infty\Gamma) \subset \mathrm{Flag}(\tau_{\rm mod})$. It is a closed Γ-invariant antipodal subset such that the action $\Gamma \to \mathrm{Homeo}(\mathrm{Flag}(\tau_{\rm mod}); \Lambda)$ is S-expanding and the equivariant homeomorphism $\beta^{-1} : \Lambda \to \partial_\infty\Gamma$ is nowhere constant. By Theorem 5.1, this action is S-hyperbolic with coding map β^{-1}.

$(3 \Rightarrow 2)$ By Theorem 5.7(1) the coding map π is equivariant, continuous and surjective. Since π is assumed to be injective and Λ is compact, π is in fact a homeomorphism. Since Λ is antipodal, the inverse $\pi^{-1} : \partial_\infty\Gamma \to \Lambda \subset \mathrm{Flag}(\tau_{\rm mod})$ is a boundary embedding. The Γ-action on Λ is S-hyperbolic, in particular, S-expanding.

$(1 \Leftrightarrow 2)$ This equivalence reduces to the lemma below. □

6.4 Lemma. *Suppose $\Gamma < G$ is $\tau_{\rm mod}$-boundary embedded with a boundary embedding $\beta : \partial_\infty\Gamma \to \mathrm{Flag}(\tau_{\rm mod})$.*

(1) *If Γ is non-uniformly $\tau_{\rm mod}$-Anosov with the asymptotic embedding β, then the Γ-action on $\mathrm{Flag}(\tau_{\rm mod})$ is S-expanding at $\beta(\partial_\infty\Gamma)$.*

(2) *If Γ is non-elementary and the Γ-action on $\mathrm{Flag}(\tau_{\rm mod})$ is S-expanding at $\beta(\partial_\infty\Gamma)$, then it is non-uniformly $\tau_{\rm mod}$-Anosov and β is the asymptotic embedding for Γ.*

Proof. (1) This is a special case of [KLP17, Equivalence Theorem 1.1]: every $\tau_{\rm mod}$-Anosov subgroup $\Gamma < G$ is expanding at $\beta(\partial_\infty\Gamma)$. It is also an immediate consequence of the condition (d) in Definition 6.1 combined with Remark 3.5(c).

(2) Suppose the Γ-action on $\mathrm{Flag}(\tau_{\rm mod})$ is S-expanding at $\beta(\partial_\infty\Gamma)$ with data (Δ). Since $\partial_\infty\Gamma$ is perfect, Corollary 3.17 applies. Thus, for any $\eta \in (0, \Delta]$, the rays c^α associated to η-codes α for $f(\xi) \in f(\partial_\infty\Gamma)$ are uniform quasi-geodesic rays in Γ.

Let $\xi \in \partial_\infty\Gamma$ and let $r : \mathbb{N}_0 \to \Gamma$ be a geodesic ray starting at $e \in \Gamma$ and asymptotic to ξ. If α is an η-code for $\beta(\xi)$ then, as in the proof of Lemma 2.7, the Hausdorff distance between $\{r(i)\}_{i\in\mathbb{N}_0}$ and $\{c^\alpha_j\}_{j\in\mathbb{N}_0}$ is bounded above by a uniform constant $C > 0$. This means that for each $i \in \mathbb{N}_0$, there exist $n_i \in \mathbb{N}_0$ and an element $g_i \in \Gamma$ with $|g_i|_\Sigma \le C$ such that $r(i) = c^\alpha_{n_i} g_i$.

Then we have

$$\epsilon(r(i)^{-1}, f(\xi)) = \epsilon(g_i^{-1}(c_{n_i}^{\alpha})^{-1}, \beta(\xi)) \geq A \cdot \epsilon((c_{n_i}^{\alpha})^{-1}, \beta(\xi)),$$

where $A = \inf\{\epsilon(g, \beta(\zeta)) \mid \zeta \in \partial_\infty \Gamma,\ g \in \Gamma \text{ and } |g|_\Sigma \leq C\}$. Since $\epsilon((c_j^{\alpha})^{-1}, \beta(\xi))$ tends to infinity as j tends to infinity (by the last statement of Lemma 3.15), it follows that

$$\sup_{i \in \mathbb{N}} \epsilon(r(i)^{-1}, \beta(\xi)) = +\infty.$$

Therefore, the Γ-action satisfies the condition (d) of Definition 6.1. □

A corollary of this theorem is the stability of Anosov subgroups; see, for example, [GW12, Theorem 1.2] and [KLP14, Theorem 1.10]. Let us denote by

$$\mathrm{Hom}^{\tau}(\Gamma, G)$$

the space of faithful representations $\Gamma \to G$ with (non-uniformly) τ_{mod}-Anosov images.

6.5 Corollary. *Suppose that Γ is a non-elementary hyperbolic group. Then*
(1) $\mathrm{Hom}^{\tau}(\Gamma, G)$ *is open in* $\mathrm{Hom}(\Gamma, G)$.
(2) *For any sequence of representations $\rho_i \in \mathrm{Hom}^{\tau}(\Gamma, G)$ converging to $\rho \in \mathrm{Hom}^{\tau}(\Gamma, G)$, the asymptotic embeddings $\alpha_i : \partial_\infty \Gamma \to \Lambda_{\rho_i(\Gamma)}(\tau_{\mathrm{mod}})$ converge uniformly to the asymptotic embedding $\alpha : \partial_\infty \Gamma \to \Lambda_{\rho(\Gamma)}(\tau_{\mathrm{mod}})$.*

Proof. We start with an embedding $\rho \in \mathrm{Hom}^{\tau}(\Gamma, G)$; let $\Lambda := \Lambda_{\rho(\Gamma)}(\tau_{\mathrm{mod}})$ and

$$\alpha : \partial_\infty \Gamma \to \Lambda \subset \mathrm{Flag}(\tau_{\mathrm{mod}})$$

denote the asymptotic embedding of ρ. By Theorem 6.3, the Γ-action on Λ is S-hyperbolic. By Theorem 3.27 there exists a small neighborhood U' of ρ in $\mathrm{Hom}(\Gamma, G)$ such that, for each $\rho' \in U'$, there exists a ρ'-invariant compact $\Lambda' \subset \mathrm{Flag}(\tau_{\mathrm{mod}})$ at which the ρ'-action is S-expanding and there is an equivariant homeomorphism $\phi : \Lambda \to \Lambda'$.

By Corollary 3.28, for every $\rho' \in U'$, the kernel of the action of $\Gamma' = \rho'(\Gamma)$ on Λ' equals the kernel of the action of Γ on Λ. Since Γ is assumed to be non-elementary, it acts on Λ with finite kernel Φ (Corollary 5.9). Therefore, the kernel of ρ' is contained in the finite subgroup $\Phi < \Gamma$. As explained in [KLP14, proof of Corollary 7.34], rigidity of finite subgroups of Lie groups implies that U' contains a smaller neighborhood U of ρ such that every $\rho' \in U$ is injective on Φ. Therefore, every $\rho' \in U$ is faithful.

Since Λ' depends continuously on ρ' (see Section 4.7), the antipodality of Λ leads to the antipodality of Λ'. (In order to guarantee this, one may further reduce the size of U if necessary.) Thus $\phi \circ \alpha : \partial_\infty \Gamma \to \Lambda'$ is a boundary embedding of Γ'. From Lemma 6.4(2) we conclude that $\Gamma' < G$ is again (non-uniformly) τ_{mod}-Anosov and the boundary embedding $\phi \circ \alpha$ of Γ' is uniformly close to α. □

6.3 Historical remarks on stability

The history of stability for convex-cocompact (and, more generally, geometrically finite) Kleinian groups goes back to the pioneering work of Marden [Mar74] (in the case of subgroups of $\mathrm{PSL}(2,\mathbb{C})$). It appears that the first proof of stability of geometrically finite subgroups of $\mathrm{Isom}(\mathbb{H}^n)$ (the isometry group of the hyperbolic n-space) was given by Bowditch in [Bow98a], although many arguments are already contained in [CEG87]. Bowditch in his paper also credits this result to Tukia.

For convex-cocompact subgroups of rank one Lie groups the first proof of stability was given by Corlette [Cor90]. Corlette's proof also goes through in the setting of C^1-stability as it was observed by Yue [Yue96]. Unlike the proofs of Sullivan and Bowditch, Corlette's proof is based on an application of Anosov flows; the same tool is used in the subsequent proofs of stability of Anosov representations by Labourie [Lab06] and by Guichard and Wienhard [GW12].

An alternative proof of stability of Anosov subgroups is given by Kapovich, Leeb and Porti in [KLP14] using coarse-geometric ideas and is quite different from the arguments of Sullivan, Bowditch and Corlette–Labourie–Guichard–Wienhard. A generalization of the Sullivan's stability theorem for subgroups of $\mathrm{Isom}(\mathbb{H}^n)$, proving the existence of a quasiconformal conjugation on the entire sphere at infinity, was given by Izeki [Ize00].

7 Examples

We present a number of examples and non-examples of S-hyperbolic actions. We emphasize that all examples of S-hyperbolic actions presented here are in fact S-fellow-traveling. Examples of non-S-fellow-traveling S-hyperbolic actions will be discussed elsewhere.

7.1 S-expansion does not imply S-hyperbolicity

In general, even for hyperbolic groups, the S-expansion condition alone does not imply the S-hyperbolicity condition.

7.1 Example (Action with infinite kernel)**.** Suppose that Γ' and Γ are non-elementary hyperbolic groups and $\phi : \Gamma' \to \Gamma$ is an epimorphism with infinite kernel. We equip the Gromov boundary $\Lambda = \partial_\infty \Gamma$ with a visual metric. Then the action $\rho : \Gamma \to \mathrm{Homeo}(\Lambda)$ is S-expanding (compare Corollary 5.3). Thus, the associated Γ'-action $\rho \circ \phi : \Gamma' \to \mathrm{Homeo}(\Lambda)$ is S-expanding as well. But this action cannot be S-hyperbolic: see Corollary 5.9.

7.2 Example (Non-discrete action)**.** Suppose that Γ is a non-elementary hyperbolic group and $\rho : \Gamma \to G = \mathrm{Isom}(\mathbb{H}^n)$ is a representation with dense image. Then the associated action $\rho : \Gamma \to \mathrm{Homeo}(\Lambda)$ on the visual boundary $\Lambda = \partial_\infty \mathbb{H}^n$ is S-expanding due to the density of $\rho(\Gamma)$ in G. But the action $\rho : \Gamma \to \mathrm{Homeo}(\Lambda)$ cannot be S-hyperbolic. Indeed, by the density of $\rho(\Gamma)$ in G, there is a sequence of distinct elements $g_i \in \Gamma$ such that $\rho(g_i)$ converges to the identity element of G. If ρ were S-hyperbolic, then Corollary 5.9 would give a contradiction that the set $\{g_i\}$ is finite.

More generally, the same argument works for representations $\rho : \Gamma \to G$ of non-elementary hyperbolic groups to a semisimple Lie group G with dense images $\rho(\Gamma)$. The associated actions ρ of Γ on the partial flag manifolds G/P are S-expanding but not S-hyperbolic.

7.2 S-hyperbolic actions of hyperbolic groups

In addition to the toy examples in Section 3.5 we now give more interesting examples of S-hyperbolic actions of hyperbolic groups. As we proceed, the associated coding maps $\pi : \Lambda \to \partial_\infty \Gamma$ will be increasingly more complicated.

Recall that a discrete subgroup $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ is *convex-cocompact* if its limit set $\Lambda = \Lambda_\Gamma \subset S^{n-1} = \partial_\infty \mathbb{H}^n$ is not a singleton and Γ acts cocompactly on the closed convex hull of Λ in $\mathbb{H}^n$. We refer to [Bow93] for details on convex-cocompact and, more generally, geometrically finite isometry groups of hyperbolic spaces.

Every convex-cocompact (discrete) subgroup Γ of isometries of $\mathbb{H}^n$ (and, more generally, a rank one symmetric space) is S-hyperbolic. More precisely, for $\Lambda = \Lambda_\Gamma$, the action $\Gamma \to \mathrm{Homeo}(S^{n-1}; \Lambda)$ is S-hyperbolic. This can be either proven directly using a *Ford fundamental domain* (as in [Sul85, Theorem I] by considering the conformal ball model of $\mathbb{H}^n$ inside $\mathbb{R}^n$) or regarded as a special case of S-hyperbolicity of Anosov subgroups (Theorem 6.3).

Unlike the convex-cocompact or Anosov examples, the invariant compact set Λ is not equivariantly homeomorphic to the Gromov boundary in the examples below.

7.3 Example (k-fold non-trivial covering)**.** Let S_g be a closed oriented hyperbolic surface of genus $g \geq 2$; it is isometric to the quotient $\mathbb{H}^2/\Gamma$, where $\Gamma \cong \pi_1(S_g)$ is a discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$. Take any $k \geq 2$ dividing $2g - 2$. Since the Euler number of the unit circle bundle of S_g is $2 - 2g$, same as the Euler number of the action of Γ on $S^1 = \partial_\infty \mathbb{H}^2$, it follows that the action of Γ lifts to a smooth action

$$\widetilde{\rho} : \Gamma \to \mathrm{Homeo}(S^1)$$

with respect to the degree k covering $p : \Lambda = S^1 \to S^1$. We pull-back the Riemannian metric from the range to the domain Λ via the map p. Since $\Gamma < \mathrm{PSL}(2, \mathbb{R})$ is convex-cocompact, its action on $\partial_\infty \mathbb{H}^2$ is S-hyperbolic. Let $\{U_\alpha \mid \alpha \in \mathcal{I}\}$ be a collection of expanding subsets (arcs) in $S^1 = \partial_\infty \mathbb{H}^2$ corresponding to a generating set $\Sigma = \{s_\alpha \mid \alpha \in \mathcal{I}\}$ of Γ. As in Example 3.29, we lift these arcs to connected components

$$\{\widetilde{U}_{\alpha_i} \subset p^{-1}(U_\alpha) \mid \alpha \in \mathcal{I},\ i = 1, \ldots, k\}.$$

These will be expanding subsets for the generators $s_{\alpha_i} = s_\alpha$ ($\alpha \in \mathcal{I}$, $i = 1, \ldots, k$) of Γ. The action $\widetilde{\rho}$ will be minimal and S-hyperbolic (because the original action of Γ is).

Thus, we obtain an example of a minimal S-hyperbolic action of a hyperbolic group on a set Λ which is not equivariantly homeomorphic to $\partial_\infty \Gamma$.

Below is a variation of the above construction.

7.4 Example (Trivial covering)**.** Let Γ_0 be a non-elementary hyperbolic group with the Gromov boundary $\Lambda_0 = \partial_\infty \Gamma_0$ equipped with a visual metric. Let $\Lambda = \Lambda_0 \times \{0, 1\}$ and $\Gamma_0 \to \mathrm{Homeo}(\Lambda)$ be the product action where Γ_0 acts trivially on $\{0, 1\}$. This action is S-hyperbolic but, obviously, non-minimal.

We extend this action of Γ_0 to an action of $\Gamma = \Gamma_0 \times \mathbb{Z}_2$, where the generator of $\mathbb{Z}_2$ acts by the map

$$(\xi, i) \mapsto (\xi, 1 - i) \quad (\text{for } \xi \in \Lambda_0 \text{ and } i = 0, 1)$$

with an empty expansion subset. The action $\Gamma \to \mathrm{Homeo}(\Lambda)$ is easily seen to be faithful, S-hyperbolic and minimal. It is clear, however, that Λ is not equivariantly homeomorphic to $\partial_\infty \Gamma \cong \partial_\infty \Gamma_0$.

In the above examples we had an equivariant finite covering map $\pi : \Lambda \to \partial_\infty \Gamma$. In the next example the coding map $\pi : \Lambda \to \partial_\infty \Gamma$ is a finite-to-one open map but not a local homeomorphism; one can regard the map π as a *generalized branched covering* (in the sense that it is an open finite-to-one map which is a covering map away from a codimension 2 subset).

7.5 Example (Generalized branched covering)**.** Let $\Gamma < \mathrm{PSL}(2, \mathbb{R})$ be a *Schottky subgroup*, that is, a convex-cocompact non-elementary free subgroup. Its limit set $\Lambda_\Gamma \subset S^1$ is homeomorphic to the Cantor set; it is also equivariantly homeomorphic to the Gromov boundary $\partial_\infty \Gamma$.

We regard Γ as a subgroup of $\mathrm{PSL}(2, \mathbb{C})$ via the standard embedding $\mathrm{PSL}(2, \mathbb{R}) \to \mathrm{PSL}(2, \mathbb{C})$. The domain of discontinuity of the action of Γ on S^2 is $\Omega_\Gamma = S^2 - \Lambda_\Gamma$; the quotient surface $S = \Omega_\Gamma / \Gamma$ is compact and its genus equals to the rank r of Γ. We let $\chi : \pi_1(S) \to F$ be a homomorphism to a finite group F which is non-trivial on the image of $\pi_1(\Omega_\Gamma)$ in $\pi_1(S)$. For concreteness, we take the following homomorphism $\chi : \pi_1(S) \to F = \mathbb{Z}_2$. We let $\{a_1, b_1, \ldots, a_r, b_r\}$ denote a generating set of $\pi_1(S)$ such that $a_1, \ldots, a_r$ lie in the kernel of the natural homomorphism $\phi : \pi_1(S) \to \Gamma$, while ϕ sends $b_1, \ldots, b_r$ to (free) generators of Γ. Then take χ such that $\chi(a_1) = 1 \in \mathbb{Z}_2$, while χ sends the rest of the generators to $0 \in \mathbb{Z}_2$. This homomorphism to F, therefore, lifts to an epimorphism $\widetilde{\chi} : \pi_1(\Omega_\Gamma) \to F$ with Γ-invariant kernel $K < \pi_1(\Omega_\Gamma)$. Hence, there exists a non-trivial 2-fold covering

$$p : \widetilde{\Omega} \to \Omega_\Gamma$$

associated to K and the action of the group Γ on Ω_Γ lifts to an action of Γ on $\widetilde{\Omega}$. One verifies that p is a proper map which induces a surjective but not injective map $p_\infty : \mathrm{End}(\widetilde{\Omega}) \to \mathrm{End}(\Omega_\Gamma)$ between the spaces of ends of the surfaces $\widetilde{\Omega}$ and Ω_Γ. Since p is a 2-fold covering map, the induced map p_∞ is at most 2-to-1 (that is, the fibers of p_∞ have cardinality ≤ 2).

We let ds^2 denote the restriction of the standard Riemannian metric on S^2 to the domain Ω_Γ and let $\widetilde{ds^2}$ denote the pull-back of ds^2 to $\widetilde{\Omega}$. The Riemannian metric ds^2 is, of course, incomplete; the Cauchy completion of the associated Riemannian distance function d_{Ω_Γ} on Ω_Γ is naturally homeomorphic to S^2 (which is also the end-compactification of Ω_Γ), as a sequence in Ω_Γ is Cauchy with respect to the metric d_{Ω_Γ} if and only if it converges in S^2. (Here we are using the assumption that Λ_Γ is contained in the circle S^1.)

We therefore let (M, d) denote the Cauchy completion of the Riemannian distance function of $(\widetilde{\Omega}, \widetilde{ds^2})$. One verifies that M is compact and is naturally homeomorphic to the end-compactification of $\widetilde{\Omega}$. In particular, the covering map $p : \widetilde{\Omega} \to \Omega_\Gamma$ extends to a continuous open finite-to-one map

$$p : M \to S^2$$

sending $\Lambda := M - \widetilde{\Omega}$ to Λ_Γ. The map $p : M \to S^2$ is locally one-to-one on $\widetilde{\Omega}$ but fails to be a local homeomorphism at Λ. Since every element of Γ acts as a Lipschitz map to $(\widetilde{\Omega}, \widetilde{ds^2})$, the action of Γ on $(\widetilde{\Omega}, \widetilde{ds^2})$ extends to an action of Γ on M so that every element of Γ is a Lipschitz map and Λ is a Γ-invariant compact subset of M. The map $p : M \to S^2$ is equivariant with respect to the actions of Γ on M and on S^2. Similarly to our covering maps examples, the action $\Gamma \to \mathrm{Homeo}(M; \Lambda)$ is S-hyperbolic at Λ. The coding map $\pi : \Lambda \to \Lambda_\Gamma = \partial_\infty \Gamma$ equals the restriction of p to Λ and, hence, is not a local homeomorphism.

In the next example the coding map $\pi : \Lambda \to \partial_\infty \Gamma$ is not even an open map.

7.6 Example (Denjoy blow-up)**.** We let Γ be the fundamental group of a closed hyperbolic surface M^2. Let $c \subset M^2$ be a simple closed geodesic representing the conjugacy class $[\gamma]$ in Γ. The Gromov boundary of Γ is the circle S^1. We perform a *blow-up* of S^1 at the set $\Phi \subset S^1$ of fixed points of the elements in the conjugacy class $[\gamma]$, replacing every fixed point by a pair of points. See Figure 4. The resulting topological space Λ is homeomorphic to the Cantor set; the quotient map

$$q : \Lambda \to S^1$$

is 1-to-1 over $S^1 - \Phi$ and is 2-to-1 over Φ. The map q is quasi-open (with $O_q = \Lambda - q^{-1}(\Phi)$) but not open. (This map is an analogue of the Cantor function $f : C \to [0, 1]$ mentioned in the beginning of Section 2.) The action of Γ on S^1 lifts to a continuous action of Γ on Λ with every $g \in [\gamma]$ fixing all the points of the preimage of the fixed-point set of g in S^1. In particular, the action of Γ on Λ is minimal. One can metrize Λ so that the action $\Gamma \to \mathrm{Homeo}(\Lambda)$ is S-hyperbolic with the coding map $\pi : \Lambda \to \partial_\infty \Gamma$ being equal to the quotient map $q : \Lambda \to S^1$.

More generally, one can define a Denjoy blow-up for actions of fundamental groups of higher-dimensional compact hyperbolic manifolds. Let $M^n = \mathbb{H}^n/\Gamma$ be a compact hyperbolic n-manifold containing a compact totally geodesic hypersurface C. Let $A \subset \mathbb{H}^n$ denote the preimage of C in $\mathbb{H}^n$. The visual boundary of each component A_i of A is an $(n-2)$-dimensional sphere $S_i \subset S^{n-1} = \partial_\infty \mathbb{H}^n$. The blow-up Λ of S^{n-1} is then performed by replacing each sphere S_i with two copies of this sphere. The result is a compact topological space Λ equipped with a quotient map $q : \Lambda \to S^{n-1}$ such that q is 1-to-1 over every point not in $S = \cup_i S_i$ and is 2-to-1 over every point in S. Each connected component of Λ is either a singleton or is homeomorphic to the $(n-2)$-dimensional Sierpinsky carpet. The action of Γ on S^{n-1} lifts to a continuous action of Γ on Λ which is S-hyperbolic for a suitable choice of a metric on Λ.

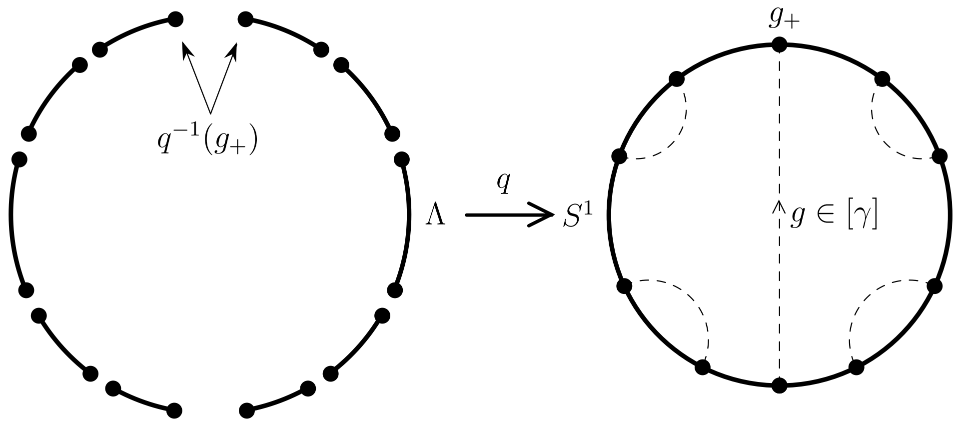

Figure 4: Denjoy blow-up.

7.3 S-hyperbolic actions of non-hyperbolic groups

Here we consider non-hyperbolic groups and examples of their S-hyperbolic actions.

The following example shows that faithfulness of an S-hyperbolic action on M does not imply faithfulness of perturbed actions.

7.7 Example (Non-discrete representation)**.** Suppose that M is the standard compactification of the hyperbolic space $\mathbb{H}^4$, $P = \mathbb{H}^2 \subset \mathbb{H}^4$ is a hyperbolic plane, and $\Lambda = S^1 \subset S^3 = \partial_\infty \mathbb{H}^4$ is the ideal boundary of P. We let

$$\Gamma = \Gamma_1 \times \Gamma_2,$$

where Γ_1 is a hyperbolic surface group and $\Gamma_2 \cong \mathbb{Z}$. We consider a faithful isometric action ρ of Γ on $\mathbb{H}^4$ where Γ_1 preserves P and acts on it properly discontinuously and cocompactly, while Γ_2 acts as a group of elliptic isometries fixing P pointwise. This action admits a conformal extension to M. Since the subgroup $\Gamma_1 < \mathrm{Isom}(\mathbb{H}^4)$ is convex-cocompact, it is S-expanding at its limit set, which is equal to Λ. We take $\Sigma = \Sigma_1 \times \{e\} \cup \{e\} \times \{r, r^{-1}\}$ as a symmetric generating set of Γ, where Σ_1 is a finite generating set of Γ_1 (given by its S-expanding action) and r is a single generator of Γ_2. The expansion subsets U_r, $U_{r^{-1}}$ of r, r^{-1} (as in the definition of an S-expanding action) are defined to be the empty set. The action ρ on M is uniformly S-hyperbolic: the S-expansion property is clear; for uniform S-hyperbolicity, we observe that all the rays in $\mathrm{Ray}_\rho^{x,\delta}(\Gamma)$ for the action of Γ have the form

$$(s\, c_i^\alpha)_{i \in \mathbb{N}}$$

where $s \in \Sigma$ and c^α is an α-ray in $\Gamma_1 \times \{e\}$ for a special code α to $x \in \Lambda$.

The image $\rho(r)$ is an infinite order elliptic rotation. Therefore, we can approximate ρ by isometric actions ρ_i of Γ on $\mathbb{H}^4$ such that $\rho_i|_{\Gamma_1} = \rho|_{\Gamma_1}$, while $\rho_i(r)$ is an elliptic transformation of order i fixing P pointwise. In particular, the representations ρ_i are not faithful.

In the next two examples, a non-hyperbolic Γ acts faithfully and S-hyperbolically on Λ. Moreover, the action even satisfies the S-fellow-traveling property, see Definition 3.21(a).

7.8 Example (Actions of product groups)**.** For $i = 1, 2$, we consider S-hyperbolic actions $\rho_i : \Gamma_i \to \mathrm{Homeo}(\Lambda_i)$. Let $(\mathcal{I}_i, \mathcal{U}_i, \Sigma_i)$ be the respective S-expanding data. Since Λ_i's are non-empty, the groups Γ_i are infinite. Consider the group

$$\Gamma = \Gamma_1 \times \Gamma_2.$$

This group is non-hyperbolic since its Cayley graph contains a quasi-flat (the product of complete geodesics in the Cayley graphs of Γ_1 and Γ_2). Let $\mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2$ and we equip Γ with a symmetric generating set

$$\Sigma = \Sigma_1 \times \{e\} \sqcup \{e\} \times \Sigma_2.$$

Define the space $\Lambda = \Lambda_1 \sqcup \Lambda_2$. The group Γ acts on Λ as follows:

$$(\gamma_1, \gamma_2)(x) = \gamma_i(x) \;\text{ if } x \in \Lambda_i,$$

where $(\gamma_1, \gamma_2) \in \Gamma_1 \times \Gamma_2$. If the actions of Γ_i on Λ_i are both faithful, so is the action of Γ on Λ. As a cover $\mathcal{U}$ of Λ we take the union $\mathcal{U}_1 \cup \mathcal{U}_2$ of respective covers. We leave it to the reader to verify that the action of Γ on Λ is S-hyperbolic.

This example can be modified to a minimal action. Namely, take identical actions $\rho_1 = \rho_2$ of the same group $\Gamma_1 = \Gamma_2$ and then extend the action of $\Gamma = \Gamma_1 \times \Gamma_1$ on Λ to a minimal action of $\Gamma \rtimes \mathbb{Z}_2$ as in Example 7.4. Here the generator of $\mathbb{Z}_2$ swaps the direct factors of Γ.

7.9 Example ($\mathbb{Z}^n$ acting on $\mathsf{P}^n(\mathbb{R})$)**.** There is an S-hyperbolic action of $\mathbb{Z}^n$ on $M = \mathsf{P}^n(\mathbb{R})$ by projective transformations.

Let $\{E_i \mid i = 0, 1, \dots, n\}$ be the standard basis of $\mathbb{R}^{n+1}$. Let $\mathbb{Z}^n < \mathrm{GL}(n+1, \mathbb{R})$ be the free abelian group of rank n generated by *bi-proximal* diagonal matrices G_j $(1 \le j \le n)$ for which E_0 (resp. E_j) is the eigenvector of the biggest (resp. smallest) modulus eigenvalue. Denote by $e_i \in \mathsf{P}^n(\mathbb{R})$ and $g_j \in \mathrm{PGL}(n+1, \mathbb{R})$ the projectivizations of E_i and G_j, respectively. Let $\Lambda = \{e_i \mid i = 0, 1, \dots, n\} \subset \mathsf{P}^n(\mathbb{R}) = M$ and $\Sigma = \{g_j, g_j^{-1} \mid 1 \le j \le n\}$. We claim that the action $\mathbb{Z}^n \to \mathrm{Homeo}(M; \Lambda)$ is S-hyperbolic.

Let $U_{g_j} \subset M$ denote an expanding subset of g_j and similarly $U_{g_j^{-1}}$ for g_j^{-1}. We *assume* that $U_{g_j^{-1}} = \emptyset$ for $j = 2, 3, \dots, n$ but that $U_{g_1^{-1}}$ as well as U_{g_j} $(1 \le j \le n)$ are non-empty. Then $\mathcal{U} = \{U_{g_j}, U_{g_j^{-1}} \mid 1 \le j \le n\}$ covers Λ, since $e_0 \in U_{g_1^{-1}}$ and $e_j \in U_{g_j}$ for $1 \le j \le n$. Thus the action is S-expanding.

A ray associated to $e_j \in \Lambda$ is of the form $\big(g(g_j^{-1})^k\big)_{k \in \mathbb{N}_0}$ with $g \in \Sigma$ and a ray associated to $e_0 \in \Lambda$ is of the form $\big(g(g_1)^k\big)_{k \in \mathbb{N}_0}$ with $g \in \Sigma$. In any case, each point in Λ has only a finite number of rays associated to it. Hence the S-hyperbolicity follows.

7.4 Embedding into Lie group actions on homogeneous manifolds

Examples 7.3, 7.4 and 7.8 can be embedded in smooth Lie group actions on homogeneous manifolds.

For instance, consider the action of $\mathrm{GL}(3,\mathbb{R})$ on the space of oriented lines in $\mathbb{R}^3$, which we identify with the 2-sphere S^2 equipped with its standard metric. We have the equivariant 2-fold covering $p : \mathsf{S}^2 \to \mathsf{P}^2(\mathbb{R})$ with the covering group generated by the antipodal map $-I \in \mathrm{GL}(3,\mathbb{R})$. Let $\mathbb{H}^2 \subset \mathsf{P}^2(\mathbb{R})$ be the Klein model of the hyperbolic plane invariant under a subgroup $\mathrm{PSO}(2,1) < \mathrm{PSL}(3,\mathbb{R})$. Then $p^{-1}(\mathbb{H}^2)$ consists of two disjoint copies of the hyperbolic plane bounded by two circles Λ_1 and Λ_2. Taking a discrete closed surface subgroup $\Gamma_1 < \mathrm{SO}(2,1) < \mathrm{SL}(3,\mathbb{R})$, $\Gamma = \Gamma_1 \times \langle -I \rangle \cong \Gamma_1 \times \mathbb{Z}_2$ and $\Lambda = \Lambda_1 \cup \Lambda_2$, we obtain an S-hyperbolic action

$$\Gamma \to \mathrm{Homeo}(\mathsf{S}^2; \Lambda),$$

which restricts on Λ to Example 7.4.

Below is a more general version of this construction. Recall that the set $\mathrm{Hom}^{\tau}(\Gamma, G)$ of τ_{mod}-Anosov representations $\Gamma \to G$ forms an open subset of the representation variety $\mathrm{Hom}(\Gamma, G)$ (see Corollary 6.5). Let $\Gamma = \pi_1(S_g)$ ($g \geq 2$) and $G = \mathrm{PSL}(n,\mathbb{R})$ ($n \geq 3$). We will consider two types of simplices τ_{mod} for the Lie group G:

- σ_{mod}; the corresponding flag manifold $\mathrm{Flag}(\sigma_{\mathrm{mod}})$ consists of full flags in $\mathbb{R}^n$.
- τ_{mod} of the type "pointed hyperplanes"; the corresponding flag manifold $\mathrm{Flag}(\tau_{\mathrm{mod}})$ consists of pairs $V_1 \subset V_{n-1} \subset \mathbb{R}^n$ of lines contained in hyperplanes in $\mathbb{R}^n$.

In both cases we have a natural fibration $q : \mathrm{Flag}(\tau_{\mathrm{mod}}) \to \mathsf{P}(\mathbb{R}^n)$ sending each flag to the line in the flag.

7.10 Example (Hitchin and Barbot representations)**.** The Hitchin representations are σ_{mod}-Anosov representations belonging to a connected component $\mathrm{Hom}^{\mathrm{Hit}}(\Gamma, G)$ of $\mathrm{Hom}^{\sigma}(\Gamma, G)$ containing a representation

$$\Gamma \hookrightarrow \mathrm{PSL}(2,\mathbb{R}) \hookrightarrow \mathrm{PSL}(n,\mathbb{R}),$$

where the first map is a Fuchsian representation of Γ and the second an irreducible embedding of $\mathrm{PSL}(2,\mathbb{R})$.

We may also consider the standard reducible embedding $\iota : \mathrm{SL}(2,\mathbb{R}) \hookrightarrow \mathrm{SL}(n,\mathbb{R})$ given by $A \mapsto \left(\begin{smallmatrix} A & 0 \\ 0 & I \end{smallmatrix}\right)$. Let $\varphi : \Gamma \hookrightarrow \mathrm{PSL}(2,\mathbb{R})$ be a Fuchsian representation and $\tilde{\varphi} : \Gamma \hookrightarrow \mathrm{SL}(2,\mathbb{R})$ one of the $2g$ lifts of φ. Let $p : \mathrm{SL}(n,\mathbb{R}) \to \mathrm{PSL}(n,\mathbb{R})$ denote the covering map, which is of degree 2 if and only if n is even. Representations of the form

$$p \circ \iota \circ \tilde{\varphi} : \Gamma \hookrightarrow \mathrm{SL}(2,\mathbb{R}) \hookrightarrow \mathrm{SL}(n,\mathbb{R}) \to \mathrm{PSL}(n,\mathbb{R})$$

are τ_{mod}-Anosov, where τ_{mod} has the type of pointed hyperplanes. Let $\mathrm{Hom}^{\mathrm{Bar}}(\Gamma, G)$ be the union of connected components of $\mathrm{Hom}^{\tau}(\Gamma, G)$ containing such representations. We say representations in $\mathrm{Hom}^{\mathrm{Bar}}(\Gamma, G)$ are of *Barbot type*; see [Bar10] for the case $n = 3$.

Let $\rho : \Gamma \to \mathrm{PSL}(n,\mathbb{R})$ be a Hitchin representation. It is known [Lab06] that the projection $q \circ \alpha : \partial_\infty \Gamma \to \mathrm{Flag}(\sigma_{\mathrm{mod}}) \to \mathsf{P}^{n-1}(\mathbb{R})$ of the asymptotic embedding α is a hyper-convex curve. This curve is homotopically trivial if and only if n is odd. On the other hand, if ρ is of Barbot type, the curve $q \circ \alpha : \partial_\infty \Gamma \to \mathrm{Flag}(\tau_{\mathrm{mod}}) \to \mathsf{P}^{n-1}(\mathbb{R})$ is always homotopically non-trivial.

We now lift to the space of oriented full flags (resp. oriented line-hyperplane flags). Accordingly, we lift the action of Γ on the sphere S^{n-1}. The preimage Λ of $(q \circ \alpha)(\partial_\infty \Gamma)$ in S^{n-1} is either a Jordan curve or a disjoint union of two Jordan curves, with the 2-fold equivariant covering map

$$\Lambda \to (q \circ \alpha)(\partial_\infty \Gamma) \cong \mathsf{P}^1(\mathbb{R}) \cong S^1.$$

The result is an S-hyperbolic action $\tilde{\rho} : \Gamma \to \mathrm{Diff}(\mathsf{S}^n; \Lambda)$ where $\tilde{\rho}(\Gamma)$ is contained in the image of the group $\mathrm{SL}(n, \mathbb{R})$ in $\mathrm{Diff}(\mathsf{S}^n)$. The restrictions of the Γ-actions to Λ are as in Example 7.3 (with $k = 2$) and Example 7.4.

7.11 Example (Embedded product examples)**.** We embed Example 7.8 in the action of $\mathrm{SL}(4, \mathbb{R})$ on $\mathsf{P}^3(\mathbb{R})$. Consider $G_1 \times G_2 = \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R}) < G = \mathrm{SL}(4, \mathbb{R})$. The action of $G_1 \times G_2$ on $\mathbb{R}^4$ is reducible, preserving a direct sum decomposition

$$\mathbb{R}^4 = V_1 \oplus V_2,$$

where V_1 and V_2 are 2-dimensional subspaces and G_i acts trivially on V_{3-i} for $i = 1, 2$. Let τ be an involution of $\mathbb{R}^4$ swapping V_1 and V_2. For $i = 1, 2$, take $\Gamma_i < G_i$ to be an infinite (possibly elementary) convex-cocompact subgroup with the limit set $\Lambda_i \subset \mathsf{P}(V_i)$. Then the subgroup $\Gamma = \Gamma_1 \times \Gamma_2 < G$ acts on $\mathsf{P}^3(\mathbb{R})$ preserving the union $\Lambda = \Lambda_1 \sqcup \Lambda_2 \subset \mathsf{P}(V_1) \sqcup \mathsf{P}(V_2)$. We equip $\mathsf{P}^3(\mathbb{R})$ with its standard Riemannian metric. The action

$$\Gamma \to \mathrm{Homeo}(\mathsf{P}^3(\mathbb{R}); \Lambda)$$

is S-hyperbolic and restricts on Λ to Example 7.8. As in Example 7.8, taking $\Gamma_1 = \Gamma_2$ and an index two extension Γ of $\Gamma_1 \times \Gamma_2$ we can extend this action to an S-hyperbolic action minimal on Λ using the involution τ.

7.5 Algebraically stable but not convex-cocompact

We provide an example to the claim made in the introduction that for groups with torsion the implication $(4 \Rightarrow 1)$ is false.

7.12 Example (Quasiconformally stable non-convex-cocompact subgroups of $\mathrm{PSL}(2, \mathbb{C})$)**.** We recall also that a *von Dyck group* $D(p, q, r)$ is given by the presentation

$$\langle a, b, c \mid a^p = b^p = c^r = 1,\ abc = 1 \rangle.$$

Such groups are called *hyperbolic* (resp. *parabolic*, *elliptic*) if the number

$$\chi = \chi(p, q, r) = \frac{1}{p} + \frac{1}{q} + \frac{1}{r}$$

is < 1 (resp. $= 1$, > 1). Depending on the type $D(p, q, r)$ can be embedded (uniquely up to conjugation) as a discrete cocompact subgroup of isometries of hyperbolic plane (if $\chi < 1$), a discrete cocompact subgroup of the group $\mathrm{Aff}(\mathbb{C})$ of complex affine transformations of $\mathbb{C}$ (if $\chi = 1$), or is finite and embeds in the group of isometries of the 2-sphere (if $\chi > 1$).

Let Γ_i ($i = 1, 2$) be two discrete elementary subgroups of $\mathrm{Aff}(\mathbb{C}) < \mathrm{PSL}(2, \mathbb{C})$ isomorphic to parabolic Von Dyck groups

$$D(p_i, q_i, r_i)\ (i = 1, 2).$$

These groups consist of elliptic and parabolic elements and are virtually free abelian of rank 2; hence they cannot be contained in a convex-cocompact group. Subgroups of $\mathrm{PSL}(2, \mathbb{C})$ isomorphic to Von Dyck groups are (locally) rigid. One can choose embeddings of the groups Γ_1 and Γ_2 into $\mathrm{PSL}(2, \mathbb{C})$ such that they generate a free product

$$\Gamma = \Gamma_1 \star \Gamma_2 < \mathrm{PSL}(2, \mathbb{C}),$$

which is geometrically finite and every parabolic element of Γ is conjugate into one of the free factors. (The group Γ is obtained via the Klein combination of Γ_1 and Γ_2, see [Kap01, §4.18] for example). The discontinuity domain Ω of Γ in $\mathsf{P}^1(\mathbb{C})$ is connected and the quotient orbifold $\mathcal{O} = \Omega/\Gamma$ is a sphere with six cone points of the orders p_i, q_i and r_i ($i = 1, 2$).

Being geometrically finite, the group Γ is relatively stable (relative its parabolic elements): let

$$\mathrm{Hom}_{\mathrm{par}}(\Gamma, \mathrm{PSL}(2, \mathbb{C}))$$

denote the *relative representation variety*, which is the subvariety in the representation variety defined by the condition that images of parabolic elements of Γ are again parabolic. Let $\iota_\Gamma : \Gamma \to \mathrm{PSL}(2, \mathbb{C})$ denote the identity embedding. Then there is a small neighborhood U of ι_Γ in $\mathrm{Hom}_{\mathrm{par}}(\Gamma, \mathrm{PSL}(2, \mathbb{C}))$ which consists entirely of faithful geometrically finite representations which are, moreover, given by quasiconformal conjugations of Γ. Since the subgroups Γ_1 and Γ_2 are rigid, there is a neighborhood V of ι_Γ such that

$$V \cap \mathrm{Hom}_{\mathrm{par}}(\Gamma, \mathrm{PSL}(2, \mathbb{C})) = V \cap \mathrm{Hom}(\Gamma, \mathrm{PSL}(2, \mathbb{C})).$$

It follows that the action of Γ on its limit set is structurally stable in $\mathrm{PSL}(2, \mathbb{C})$, in particular, algebraically stable. However, Γ is not convex-cocompact. Lastly, the group Γ is not rigid, the (complex) dimension of the character variety

$$X(\Gamma, \mathrm{PSL}(2, \mathbb{C})) /\!\!/ \mathrm{PSL}(2, \mathbb{C})$$

near $[\iota_\Gamma]$ equals the (complex) dimension of the Teichmüller space of $\mathcal{O}$, which is 3.

On the other hand, one can show that if $\Gamma < \mathrm{PSL}(2, \mathbb{C})$ is a finitely generated discrete subgroup which is not a lattice and contains no parabolic von Dyck subgroups, then algebraic stability of Γ implies quasiconvexity of Γ.

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