Math 21A
Kouba
Rolle's Theorem, The Mean Value Theorem (MVT), and Other Important Theorems

(Some theorems are given without proof (You should try to prove them.) and proofs of some of the theorems are just loosely sketched.)

<u>*Definition*</u> : Function f takes on its *maximum* value at $x = c$ if $f(x) < f(c)$ for $x \neq c$. Function f takes on its *minimum* value at $x = c$ if $f(c) < f(x)$ for $x \neq c$.

<u>*Theorem A*</u> : If function f is differentiable at $x = c$, then f is continuous at $x = c$.

<u>*Theorem B*</u> : Assume that function f is differentiable and takes on its maximum value at $x = c$. Then $f'(c) = 0$.

<u>*Theorem C*</u> : Assume that function f is differentiable and takes on its minimum value at $x = c$. Then $f'(c) = 0$.

<u>*Extreme Value Theorem*</u> : If function f is continuous on a close interval $[a, b]$, then f attains both an maximum value M and a minimum value m in the interval $[a, b]$. (These maxima and minima can occur at an interior point where f' is zero, at an interior point where f' does not exist, or at an endpoint of the interval.)

<u>*Rolle's Theorem*</u> : Assume that function f is continuous on the closed interval $[a, b]$, differentiable on the open interval (a, b), and $f(a) = f(b)$. Then there is at least one number c, $a < c < b$, so that $f'(c) = 0$.

<u>*PROOF*</u> : Since f is continuous on a closed interval $[a, b]$ f has a maximum value M and a minimum value m. This follows from the Extreme Value Theorem.

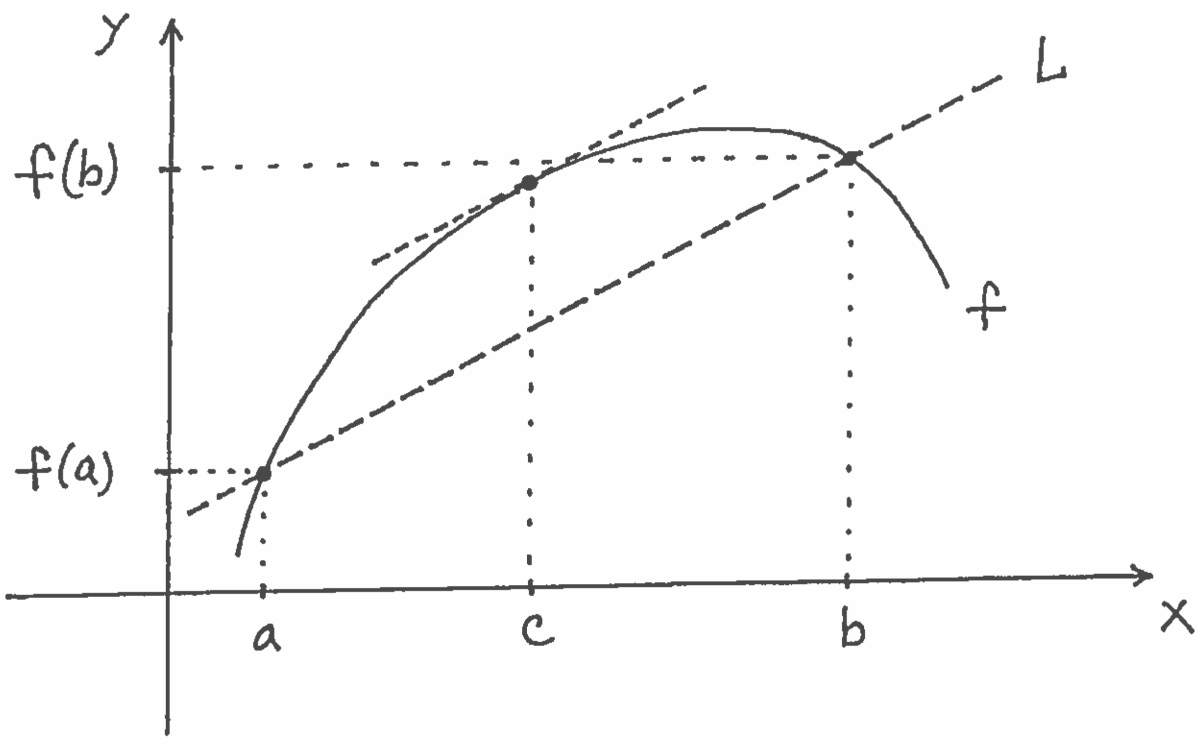
case 1. If $m = M$, then $f(x) = k$ for some constant k and all values x in $[a, b]$. Thus, $f'(x) = 0$ for all values of x in $[a, b]$. It follows that $f'(c) = 0$ for some value of c, $a < c < b$.

case 2. If $m < M$, then both m and M cannot occur at endpoints a and b since $f(a) = f(b)$. Thus, at least one occurs in the interior of the interval at $x = c$. It follows from Theorems B and C that $f'(c) = 0$. QED

<u>*Mean Value Theorem (MVT)*</u> : Assume that function f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Then there is at least one number c, $a < c < b$, so that

$$f'(c) = \frac{f(b) - f(a)}{b - a} .$$

PROOF : Consider line L in the diagram below.



The equation of line L in the diagram is

$$\frac{y - f(a)}{x - a} = \frac{f(b) - f(a)}{b - a} ,$$

so that

$$y = \frac{f(b) - f(a)}{b - a} \cdot (x - a) + f(a) .$$

Define a new function

$$s(x) = f(x) - y = f(x) - \left\{ \frac{f(b) - f(a)}{b - a} \cdot (x - a) + f(a) \right\} .$$

This function is differentiable on the open interval (a, b) since it is the difference of differentiable functions. This function is continuous on the closed interval $[a, b]$ since it is the difference of continuous functions. In addition, $s(a) = 0$ and $s(b) = 0$. It follows from Rolle's Theorem that there exists a number c, $a < c < b$, so that $s'(c) = 0$. Since

$$s'(x) = f'(x) - \left\{ \frac{f(b) - f(a)}{b - a} \cdot (1) + (0) \right\} = 0 ,$$

it follows that

$$s'(c) = f'(c) - \frac{f(b) - f(a)}{b - a} = 0 \quad \longrightarrow$$

$$f'(c) = \frac{f(b) - f(a)}{b - a} . \qquad \text{QED}$$

Theorem D : Assume that $f'(x) = 0$ for all values of x in the closed interval $[a, b]$. Then $f(x) = k$, a constant function on $[a, b]$.

PROOF : Consider any two arbitrary x-values w and z in $[a, b]$ with $w < z$. Consider the restriction of f to the new interval $[w, z]$. Since f is differentiable on the closed interval $[w, z]$ (and hence on the open interval (w, z), it follows from Theorem A that f is continuous on the open interval (w, z). By the MVT there is at least one number c, $w < c < z$, so that

$$f'(c) = \frac{f(z) - f(w)}{z - w} \longrightarrow$$

$$\frac{f(z) - f(w)}{z - w} = 0 \text{ (Since f'(c)=0)} \longrightarrow$$

$$f(z) - f(w) = 0 \longrightarrow$$

$$f(z) = f(w) \ .$$

Since w and z were chosen arbitrarily, it must be that $f(x) = k$ for some constant k and for all values of x in the closed interval $[a, b]$. QED

Theorem E : Assume that $f'(x) = g'(x)$ for all values of x in the closed interval $[a, b]$. Then $f(x) = g(x) + c$ for some constant c.

PROOF : Since $f'(x) = g'(x) \longrightarrow$

$$f'(x) - g'(x) = 0 \longrightarrow$$

$$D(f(x) - g(x)) = 0 \longrightarrow$$

$$f(x) - g(x) = c \text{ for some constant } c \text{ (by Theorem D)} \longrightarrow$$

$$f(x) = g(x) + c \text{ for some constant } c \ .$$ QED