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Can We Use the First Derivative to Determine Inflection Points?

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Points on the graph of a function f where the concavity changes are called *points of inflection*, and because concavity is determined by the sign of the second derivative, finding the points of inflection is a typical application of the second derivative in introductory calculus courses. But more than one attentive student has suggested a plausible shortcut that uses only the first derivative to find certain inflection points.

Example. Let $f(x) = 3x^4 - 8x^3$. The first derivative is $f'(x) = 12x^3 - 24x^2 = 12x^2(x-2)$ and its sign is

We conclude that f achieves its absolute minimum value of $y = -16$ at $x = 2$. Proceed to compute f'' in order to determine the inflection points for f.

But wait! At this point an enthusiastic student interjects that the *first* derivative f' tells us that f must have an inflection point at $x = 0$, since the graph of f is flat at $x = 0$ and decreases on either side of $x = 0$. The student argues that the sketch in Figure 1 must look somewhat like the graph of f near $x = 0$. The student

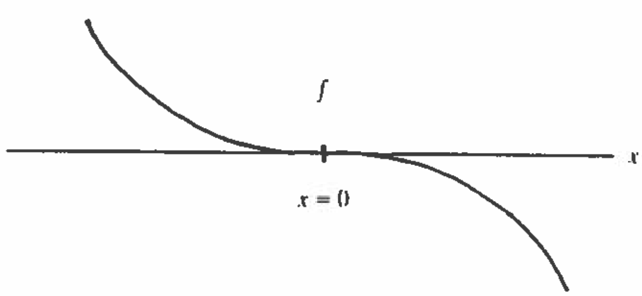


Figure 1

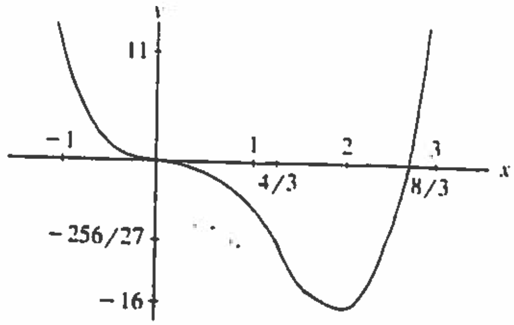
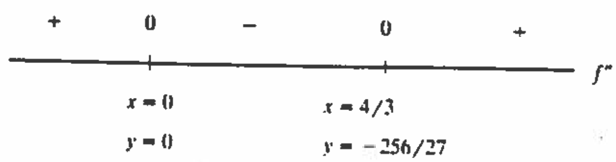


Figure 2

certainly *appears* to be correct. It looks as if $x = 0$ does in fact determine an inflection point for f.

Let's check by computing the second derivative, which is $f''(x) = 36x^2 - 48x = 12x(3x - 4)$, and its sign:



Indeed, there is an inflection point at $x = 0$! There is an additional one at $x = 4/3$. We can now sketch a detailed graph of f: see Figure 2.

Question. Assume that f is twice differentiable on (a, b) and suppose the first derivative $f'(x)$ is strictly positive at all points x in (a, b) (or strictly negative at all such points), except that $f'(c) = 0$ at one point $c, a < c < b$. Does it then follow that f has an inflection point at $x = c$?

A counterexample. Unfortunately, unless we make additional assumptions about the function f, the answer is no! Consider the function

$$f(x) = \begin{cases} x^3 + x^4 \sin(1/x) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0. \end{cases}$$

For $x \neq 0$ the derivative is $f'(x) = 3x^2 - x^2\cos(1/x) + 4x^3\sin(1/x)$, and using the limit definition of the derivative at $x = 0$, we get

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{h^3 + h^4 \sin(1/h)}{h}$$
$$= \lim_{h \to 0} \left[h^2 + h^3 \sin(1/h)\right] = 0.$$

The second derivative is $f''(x) = 6x - \sin(1/x) - 6x\cos(1/x) + 12x^2\sin(1/x)$ for

$x \neq 0$; at $x = 0$

$$\begin{aligned} f''(0) &= \lim_{h \to 0} \frac{f'(0+h) - f'(0)}{h} \\ &= \lim_{h \to 0} \frac{3h^2 - h^2\cos(1/h) + 4h^3\sin(1/h)}{h} \\ &= \lim_{h \to 0} \left[3h - h\cos(1/h) + 4h^2\sin(1/h)\right] = 0. \end{aligned}$$

Thus this function is twice differentiable. (Note that f'' is not continuous at $x = 0$ since the limit of $f''(x)$ does not exist as x approaches zero.)

We next show that the sign of f' is positive on some two-sided neighborhood of zero. Since the first derivative is $f'(x) = 3x^2 - x^2\cos(1/x) + 4x^3\sin(1/x)$, we can insure that $f'(x) > 0$ for $x > 0$ if an appropriate inequality is solved for x. Since $-1 \le \cos(1/x), \sin(1/x) \le 1$, then for positive x,

$$\begin{aligned} f'(x) &= 3x^2 - x^2\cos(1/x) + 4x^3\sin(1/x) \\ &> 3x^2 - x^2 - 4x^3 = 2x^2(1 - 2x). \end{aligned}$$

Thus, $f'(x) > 0$ if $0 < x < 1/2$. Similarly, for negative x,

$$\begin{aligned} f'(x) &= 3x^2 - x^2\cos(1/x) + 4x^3\sin(1/x) \\ &> 3x^2 - x^2 + 4x^3 = 2x^2(1 + 2x), \end{aligned}$$

so that $f'(x) > 0$ for $-1/2 < x < 0$. Hence, f' is positive throughout $(-1/2, 1/2)$ except that $f'(0) = 0$.

The last point to be verified is that this function does *not* have an inflection point at $x = 0$. We are motivated by the fact that the graph of the derivative f' appears to have infinitely many horizontal tangent lines in every neighborhood of zero. The graph of f' is shown in Figure 3.

Recall that the second derivative is $f''(x) = 6x - \sin(1/x) - 6x\cos(1/x) + 12x^2\sin(1/x)$ for $x \neq 0$. Consider the sequence of x-values given by $x_n = 1/2n\pi$ and satisfying $-1/2 < x_n < 1/2$ for $n = 1, 2, 3, \ldots$. Then

$$\begin{aligned} f''(x_n) &= 6(x_n) - \sin\left(\frac{1}{x_n}\right) - 6(x_n)\cos\left(\frac{1}{x_n}\right) + 12(x_n)^2\sin\left(\frac{1}{x_n}\right) \\ &= \frac{6}{2n\pi} - \sin(2n\pi) - \frac{6}{2n\pi}\cos(2n\pi) + 12\left(\frac{1}{2n\pi}\right)^2\sin(2n\pi) \\ &= \frac{6}{2n\pi} - 0 - \frac{6}{2n\pi}(1) + 12\left(\frac{1}{2n\pi}\right)^2(0) \\ &= 0. \end{aligned}$$

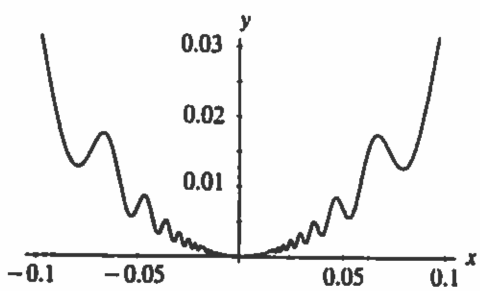


Figure 3

Since the sequence x_n converges to zero, it follows that there is no number t such that f is strictly concave up or concave down on $(0, t)$. Since f'' is an odd function, the same argument applied to the sequence $-x_n$ shows that f is not strictly concave up or down on any interval $(-t, 0)$. We conclude that f does not have an inflection point at $x = 0$.

Remark. Even if we require that the second derivative be continuous, this shortcut is still not valid. Consider the function

$$f(x) = \begin{cases} x^3 + x^{13/3} \sin(1/x) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0. \end{cases}$$

It can be shown (with some difficulty) that f is twice continuously differentiable, $f'(0) = 0$, and $f'(x) > 0$ for x-values near zero. However, $x = 0$ does not determine an inflection point for f.

The key property of both our counterexamples is that the second derivative has zeros arbitrarily close to the origin. It is reassuring to note that such functions are rarely encountered in a standard calculus textbook, and the following theorem shows that, except for such anomalies, the students were right all along about their shortcut for identifying inflection points where the tangent line to the graph is horizontal.

Theorem. *Let function f be twice continuously differentiable on the interval $[a, b]$, and suppose the second derivative f'' has only a finite number of zeros on this interval. Suppose $f'(x)$ is strictly positive at all points x in (a, b) (or strictly negative at all such points), except that $f'(c) = 0, a < c < b$. Then f has an inflection point at $x = c$.*

Proof. It suffices to show that there exist numbers $s > 0$ and $t > 0$ so that

$$f''(x) > 0 \quad \text{for all } x \text{ in } (c, c + t), \tag{1}$$

$$f''(x) < 0 \quad \text{for all } x \text{ in } (c - s, c). \tag{2}$$

We will prove (1) by contradiction, assuming $f'(x)$ is positive at all points $x \neq c$ in (a, b). The other cases are entirely similar.

Since f'' has only a finite number of zeros in $[a, b]$, there exists a number $t > 0$ such that $c + t < b$ and f'' is never zero in $(c, c + t)$. The continuity of f'' together with the intermediate value theorem guarantees that f'' has constant sign on the interval $(c, c + t)$, and we wish to show that it is in fact positive. (If f'' changed sign, the intermediate value theorem would imply that it had a zero somewhere in this interval, contradicting our choice of t.)

That f'' must be positive on $(c, c + t)$ now follows by applying the mean value theorem to the first derivative: $[f'(c + t) - f'(c)]/t = f''(z)$ for some z in $(c, c + t)$. Since by our assumptions $f'(c + t)$ and t are positive, and $f'(c) = 0$, it follows that $f''(z) > 0$, which completes the proof. ■