

Math 16A

Kouba

Derive the Product Rule Using the Limit definition of the Derivative

Let  $F(x) = f(x)g(x)$  . It's derivative is

$$\begin{aligned} F'(x) &= \lim_{h \rightarrow 0} \frac{F(x+h) - F(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\ &= \lim_{h \rightarrow 0} \left[ \frac{f(x+h)g(x+h) - f(x+h)g(x)}{h} + \frac{f(x+h)g(x) - f(x)g(x)}{h} \right] \\ &= \lim_{h \rightarrow 0} \left[ f(x+h) \cdot \frac{g(x+h) - g(x)}{h} + \frac{f(x+h) - f(x)}{h} \cdot g(x) \right] \\ &= f(x) \cdot g'(x) + f'(x) \cdot g(x) \\ &= f(x) \cdot g'(x) + f'(x) \cdot g(x) \end{aligned}$$

i.e.,

$$D\{f(x)g(x)\} = f(x) \cdot g'(x) + f'(x) \cdot g(x)$$