Math 21D (Summer Session I 2020)
Kouba
Quiz 2

Printing and signing your name below is a verification that no other person assisted you in the completion of this Quiz.

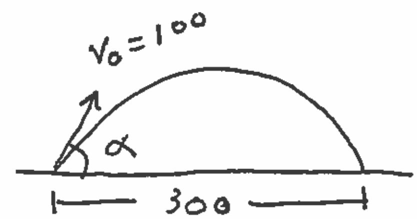
PRINT your name KEY SIGN your name ____________

Show clear, organized supporting work for your answers. Correct answers without supporting work may not receive full credit. Use of unapproved shortcuts may not receive full credit.

1.) (9 pts.) A softball is projected from ground level at an angle of α *degrees* and with an initial velocity of 100 $ft./sec.$ The softball strikes the ground 300 feet away from where it was projected. Start with the Projectile Motion equation and derive what you need from this equation. Do not use shortcuts.

a.) How long is the softball in the air before striking the ground ?

b.) What is the value of α ?



assume that

$$\vec{r}(t) = (v_0 \cos\alpha \cdot t)\,\vec{i} + (-16t^2 + v_0 \sin\alpha \cdot t + \cancel{y(0)}^{\,0})\,\vec{j}$$

$$= (100 \cos\alpha \cdot t)\,\vec{i} + (-16t^2 + 100 \sin\alpha \cdot t)\,\vec{j} \,;$$

Let $X = 300$ and $y = 0$:

$$\begin{cases} 100 \cos\alpha \cdot t = 300 \longrightarrow t = \dfrac{3}{\cos\alpha} \\ t(-16t + 100 \sin\alpha) = 0 \end{cases} \longrightarrow$$

$$-16\left(\frac{3}{\cos\alpha}\right) + 100 \sin\alpha = 0 \longrightarrow$$

$$-48 + 50 \cdot (2 \sin\alpha \cos\alpha) = 0 \rightarrow$$

$$50 \sin 2\alpha = 48 \rightarrow \sin 2\alpha = \frac{48}{50} = 0.96 \rightarrow$$

$$2\alpha \approx 73.74^\circ \text{ or } 106.26^\circ \longrightarrow$$

$$\begin{cases} \alpha \approx 36.87^\circ \rightarrow t = \dfrac{3}{\cos\alpha} \approx 3.75 \text{ sec.} \\ \alpha \approx 53.13^\circ \rightarrow t = \dfrac{3}{\cos\alpha} \approx 5.00 \text{ sec.} \end{cases}$$

2.) (12 pts.) (Work space continues on the next page.) Consider an object of mass 1/3 *grams* traveling along path C in 2D-Space determined by the vector function $\vec{r}(t) = (t)\vec{i} + (2\sqrt{t})\vec{j}$ for $t \geq 0$, where t is given in seconds and distance is given in centimeters.

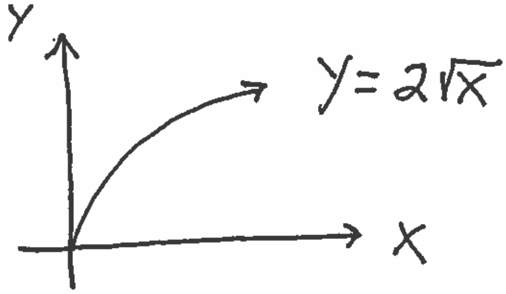
a.) Sketch path C on the given set of axes.

b.) Find the velocity vector $\vec{v}(t)$, the acceleration vector $\vec{a}(t)$, the unit tangent vector $\vec{T}(t)$, a formula for speed of motion at time t, and a formula for acceleration of motion at time t.

c.) Then (on the next page) plot and label $\vec{r}(1)$ on the given axes. Plot and label $\vec{v}(1), \vec{a}(1)$, and $\vec{T}(1)$, at the point determined by $\vec{r}(1)$ on the given axes. Find the speed and acceleration of motion when $t = 1$ second.

d.) Find and plot the force vector $\vec{F}(1)$ which is required to "turn" the object when $t = 1$.

$$a.)\ \begin{cases} x = t \\ y = 2\sqrt{t} \end{cases} \rightarrow y = 2\sqrt{x}$$



$$b.)\ \vec{r}(t) = (t)\vec{i} + (2\sqrt{t})\vec{j}$$

$$\xrightarrow{D} \vec{v}(t) = (1)\vec{i} + \left(\frac{1}{\sqrt{t}}\right)\vec{j}$$

$$\xrightarrow{D} \vec{a}(t) = (0)\vec{i} + \left(\frac{-1}{2t^{3/2}}\right)\vec{j} \quad ;$$

$$\vec{T}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|} = \frac{1}{\sqrt{(1)^2 + \left(\frac{1}{\sqrt{t}}\right)^2}} \left((1)\vec{i} + \left(\frac{1}{\sqrt{t}}\right)\vec{j}\right)$$

$$= \frac{1}{\sqrt{\frac{t}{t} + \frac{1}{t}}} \left((1)\vec{i} + \left(\frac{1}{\sqrt{t}}\right)\vec{j}\right) = \frac{\sqrt{t}}{\sqrt{t+1}} \left((1)\vec{i} + \left(\frac{1}{\sqrt{t}}\right)\vec{j}\right)$$

$$= \frac{\sqrt{t}}{\sqrt{t+1}}\vec{i} + \frac{1}{\sqrt{t+1}}\vec{j} \quad ;$$

speed $\frac{ds}{dt} = |\vec{v}(t)| = \frac{\sqrt{t+1}}{\sqrt{t}}$; acceleration

$$a(t) = \frac{\vec{v}(t) \cdot \vec{a}(t)}{|\vec{v}(t)|} = \frac{(1)(0) + \left(\frac{1}{\sqrt{t}}\right)\left(\frac{-1}{2t^{3/2}}\right)}{\sqrt{t+1}/\sqrt{t}}$$

$$= \frac{-1}{2t^{3/2}} \cdot \frac{1}{\sqrt{t+1}} ;$$

c.) $\vec{r}(1) = (1)\vec{i} + (2)\vec{j}$, $\vec{v}(1) = (1)\vec{i} + (1)\vec{j}$,

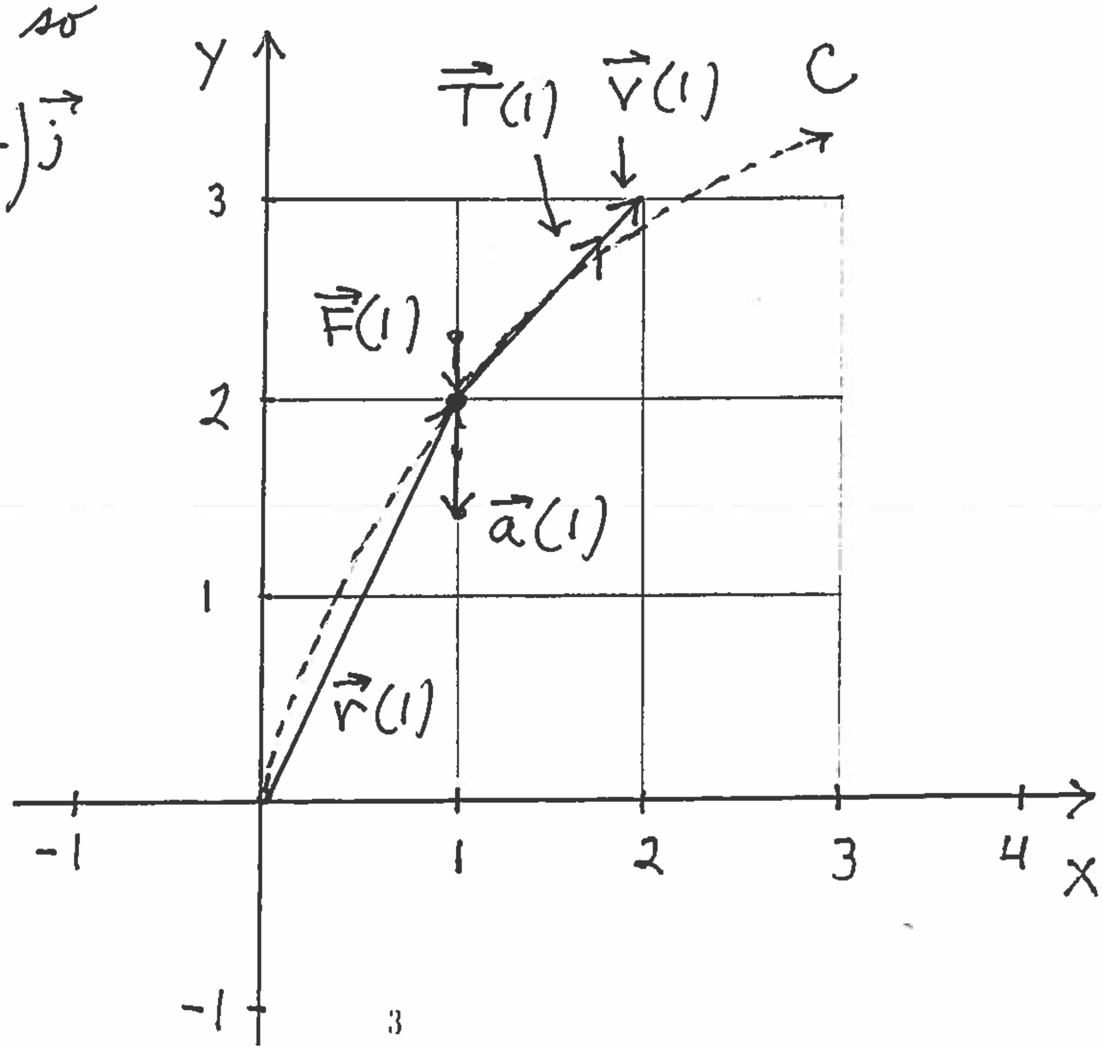
$\vec{a}(1) = (0)\vec{i} + \left(\frac{-1}{2}\right)\vec{j}$, $\frac{ds}{dt} = \sqrt{2}$ cm./sec. ,

$a(1) = \frac{-1}{2\sqrt{2}}$ cm./sec.^{2} , $\vec{T}(1) = \frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j}$.

d.) $\vec{F} = m\vec{a}$ so

$\vec{F}(1) = \left(\frac{1}{3}\right)\left(\frac{-1}{2}\right)\vec{j}$

$= \left(\frac{-1}{6}\right)\vec{j}$



3.) (9 pts.) Consider path C in 3D-Space given by $C: \begin{cases} x = (4/3)t^{3/2} \\ y = (1/2)t^2 \\ z = 2t \end{cases}$ for $t \geq 0$.

a.) Find Arc Length s for C from $t = 0$ to t as a function of t.

b.) Use your answer in part a.) to write t as a function of s.

a.) $$s = \int_0^t \sqrt{(x'(\tau))^2 + (y'(\tau))^2 + (z'(\tau))^2}\, d\tau$$

$$= \int_0^t \sqrt{(2\tau^{1/2})^2 + (\tau)^2 + (2)^2}\, d\tau$$

$$= \int_0^t \sqrt{4\tau + \tau^2 + 4}\, d\tau$$

$$= \int_0^t \sqrt{(\tau+2)^2}\, d\tau$$

$$= \int_0^t (\tau + 2)\, d\tau$$

$$= \left(\frac{1}{2}\tau^2 + 2\tau\right)\Big|_0^t = \frac{1}{2}t^2 + 2t, \text{ i.e.,}$$

arc Length $\boxed{s = \frac{1}{2}t^2 + 2t}$.

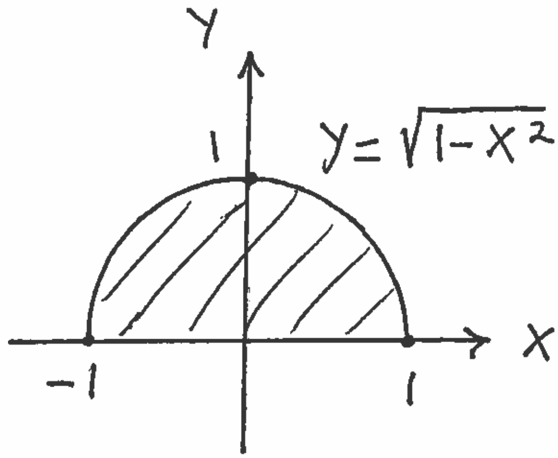
b.) $2s = t^2 + 4t \rightarrow 2s + 4 = t^2 + 4t + 4$

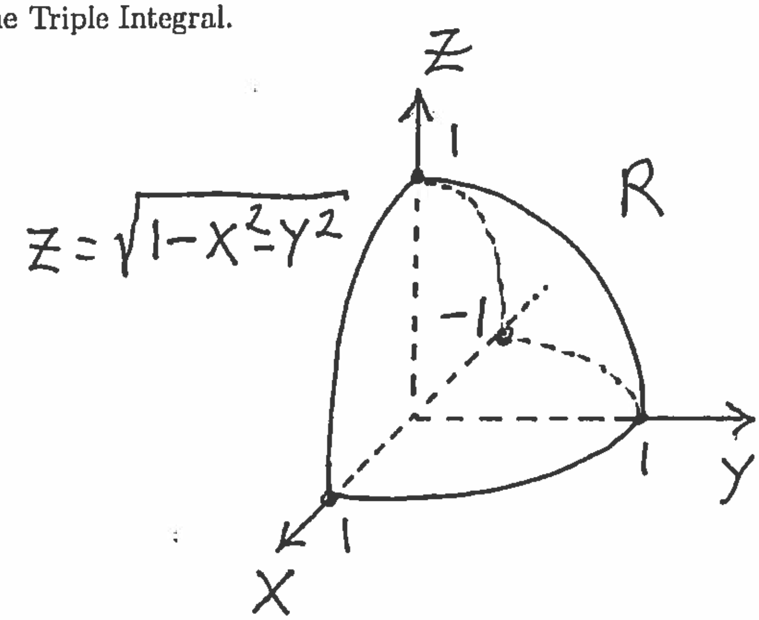
$\rightarrow 2s + 4 = (t+2)^2 \rightarrow t + 2 = \sqrt{2s+4}$

$\rightarrow \boxed{t = \sqrt{2s+4} - 2}$

4.) (9 pts.) Convert the following Triple Integral in Rectangular Coordinates to one in Spherical Coordinates. Then evaluate the Triple Integral.

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}}\int_{0}^{\sqrt{1-x^2-y^2}} z \; dz\, dy\, dx$$





$$R:\begin{cases} 0 \le \theta \le \pi \\ 0 \le \phi \le \frac{\pi}{2} \\ 0 \le \rho \le 1 \end{cases}, \text{ so}$$

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}}\int_{0}^{\sqrt{1-x^2-y^2}} z\; dz\,dy\,dx = \int_0^{\pi}\int_0^{\frac{\pi}{2}}\int_0^1 (\rho\cos\phi)(\rho^2\sin\phi)\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{\pi}\int_0^{\frac{\pi}{2}}\int_0^1 \rho^3\cos\phi\sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{\pi}\int_0^{\frac{\pi}{2}} \cos\phi\sin\phi\left(\frac{1}{4}\rho^4\Big|_{\rho=0}^{\rho=1}\right) d\phi\, d\theta$$

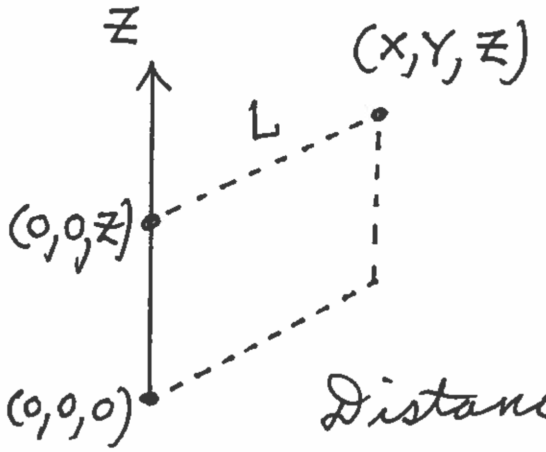
$$= \int_0^{\pi} \frac{1}{4}\cdot\left(\frac{1}{2}\sin^2\phi\Big|_{\phi=0}^{\phi=\frac{\pi}{2}}\right) d\theta$$

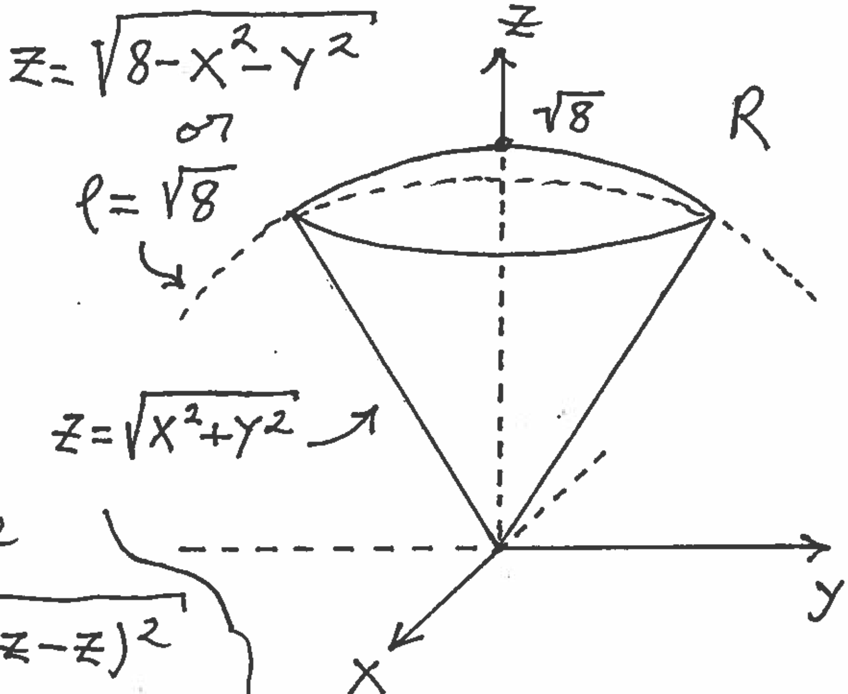
$$= \int_0^{\pi}\frac{1}{8}\left(\overset{1}{\cancel{\sin^2\frac{\pi}{2}}} - \overset{0}{\cancel{\sin^2 0}}\right) d\theta = \frac{1}{8}\theta\Big|_0^{\pi} = \frac{\pi}{8}$$

5.) (10 pts.) Consider a solid region R inside the cone $z = \sqrt{x^2+y^2}$ and below the hemi-sphere $z = \sqrt{8-x^2-y^2}$.

a.) Find a formula for the Distance L from point (x, y, z) in R to the z-axis.

b.) Use Spherical Coordinates to find the Average Value of the Distance formula in part a.). (SET UP ONLY.)





Distance

$$L = \sqrt{(X-0)^2 + (Y-0)^2 + (Z-Z)^2}$$

$$= \sqrt{X^2+Y^2} = \sqrt{r^2}$$

$$= r = \rho \sin\phi \ ;$$

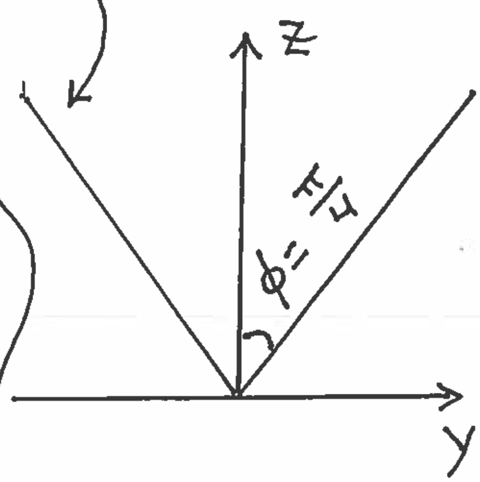
$Z = \sqrt{X^2+Y^2}$, $X = 0 \rightarrow$

$Z = \sqrt{Y^2} = |Y|$

Volume of R

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\sqrt{8}} 1 \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \ ;$$

then Average of L is

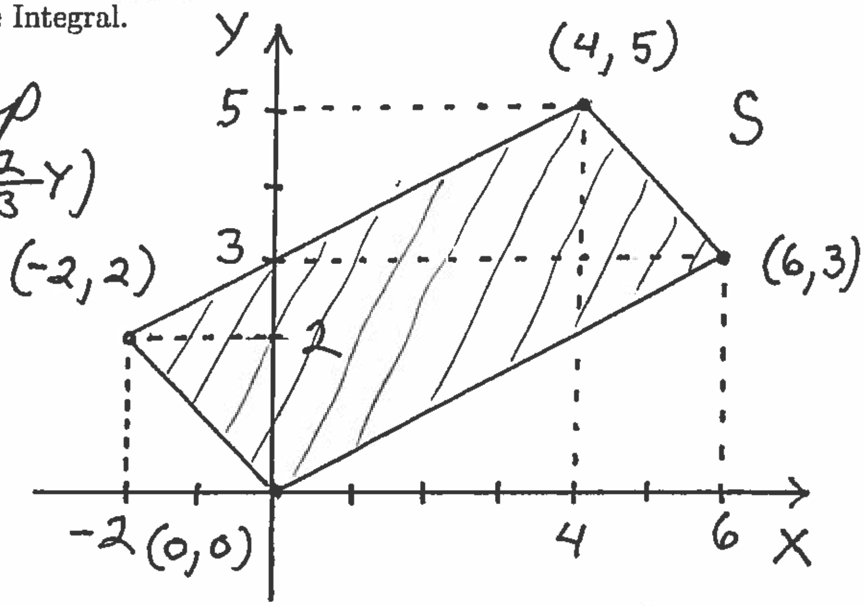


$$AVE = \frac{1}{Vol R} \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\sqrt{8}} (\rho \sin\phi)\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

6.) (10 pts.) (Work space continues on the next page.) Consider the rectangular region S given in the diagram below, and the Double Integral

$$\iint_S \sin(x+y)\, dy\, dx \,.$$ Make the following Change of Variable:

$u = (1/3)x + (1/3)y$ and $v = (-1/3)x + (2/3)y$, then apply the Change of Variable Theorem and evaluate the Double Integral.



Define Linear Map

$G(X,Y) = \left(\frac{1}{3}X + \frac{1}{3}Y,\ \frac{-1}{3}X + \frac{2}{3}Y\right)$

$= (u, v)$;

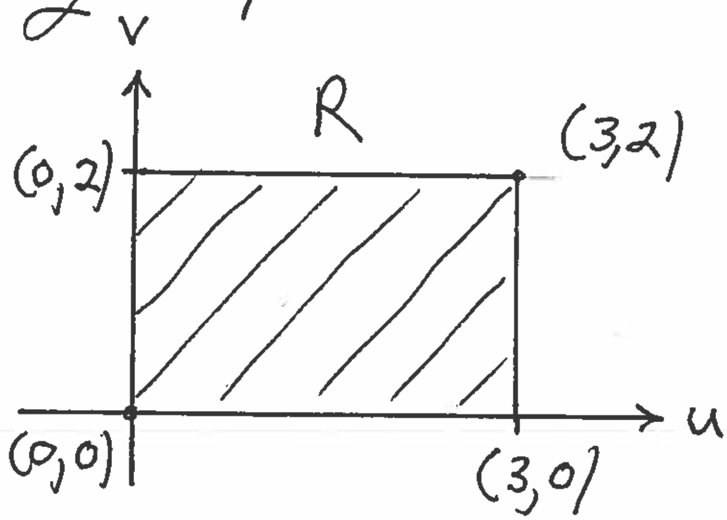
now map region S to the uv-plane. Call that region. Since G is a Linear Map, we need only map the corners of S :

$G(0,0) = (0,0)$,

$G(6,3) = (3,0)$,

$G(4,5) = (3,2)$,

$G(-2,2) = (0,2)$.



Now solve for X and y in terms of u and v :

$$\begin{cases} u = \frac{1}{3}X + \frac{1}{3}Y \\ v = \frac{-1}{3}X + \frac{2}{3}Y \end{cases} \rightarrow \boxed{u + v = Y} \text{ and } \boxed{2u - v = X} \,;$$

Now define $F(u,v) = (2u-v,\ u+v) = (X,Y)$;

the Jacobian is $$J(P) = \begin{vmatrix} X_u & X_v \\ Y_u & Y_v \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = 3.$$

Now apply the Change of Variable Theorem:

$$\iint_S \sin(X+Y)\,dy\,dX = \iint_R \sin(X+Y)\cdot|J(P)|\cdot dv\,du$$

$$= \int_0^3\int_0^2 \sin((2u-v)+(u+v))\cdot|3|\,dv\,du$$

$$= \int_0^3\int_0^2 3\sin 3u\,dv\,du$$

$$= \int_0^3 3\sin 3u\cdot\left(v\Big|_{v=0}^{v=2}\right)du$$

$$= \int_0^3 6\sin 3u\,du$$

$$= -2\cos 3u\Big|_0^3$$

$$= -2\cos 9 - (-2\cos 0)$$

$$= 2 - 2\cos 9$$

7.) (8 pts.) Consider path C determined by the vector function

$$\vec{r}(t) = (2t^2)\vec{i} - (t)\vec{j} + (1/3)(t^3)\vec{k}$$

Find parametric equations for the line L which is tangent to path C when $t = 3$.

D $\rightarrow$ $\vec{r}\,'(t) = (4t)\vec{i} + (-1)\vec{j} + (t^2)\vec{k}$

is tangent to path C ;

point : $t = 3 \rightarrow$

$\vec{r}(3) = (18)\vec{i} + (-3)\vec{j} + (9)\vec{k}$, so

pt. is $(18, -3, 9)$;

tangent (direction) vector :

$\vec{r}\,'(3) = (12)\vec{i} + (-1)\vec{j} + (9)\vec{k}$; so

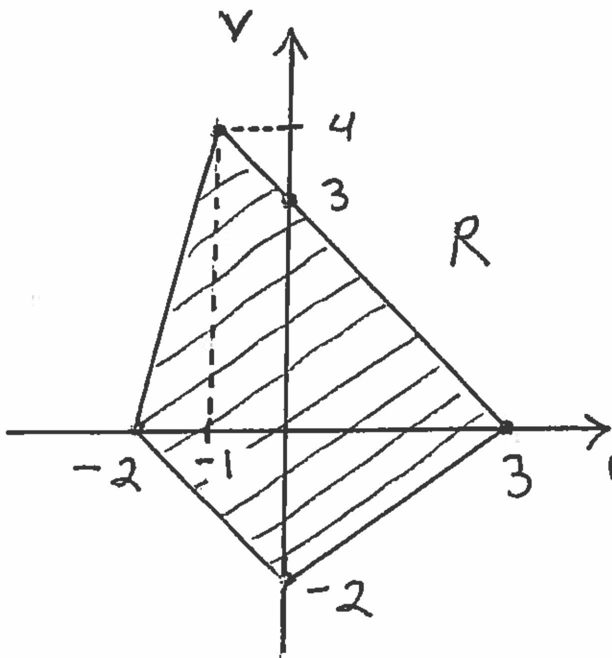
line L is

$$L : \begin{cases} x = 18 + (12)t \\ y = -3 + (-1)t \\ z = 9 + (9)t \end{cases}$$

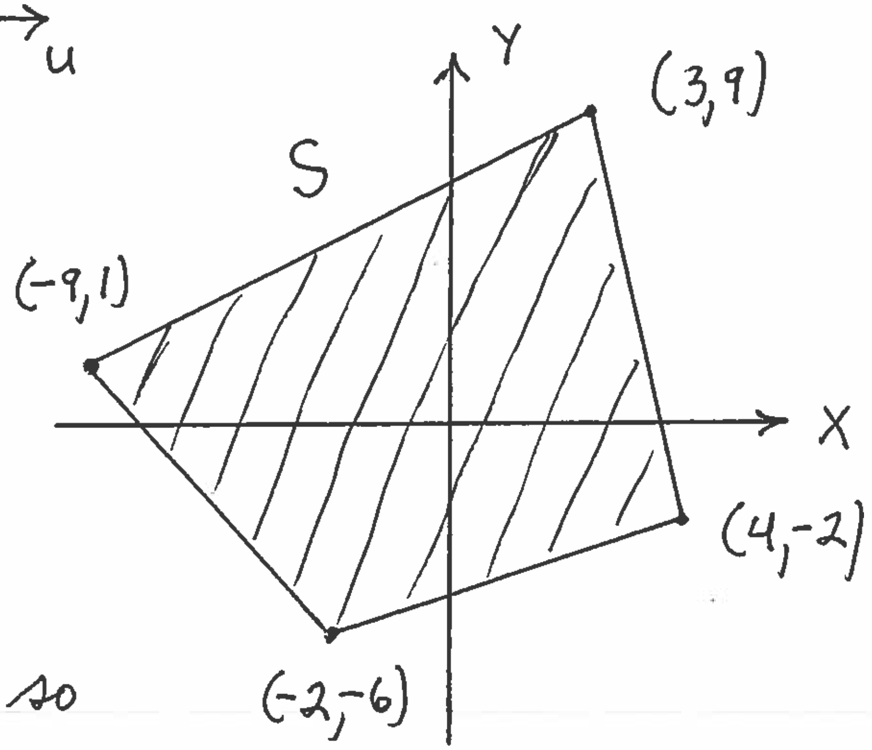
8.) (8 pts.) Consider the Linear Map $F(u,v) = (u - 2v, 3u + v)$ and the quadrilateral R in the uv-plane given in the diagram below. This Linear Map maps R to a region S in the xy-plane.

a.) Sketch region S.

b.) Use a Jacobian to find the area of S.



$F(-2,0) = (-2,-6)$,
$F(0,-2) = (4,-2)$,
$F(3,0) = (3,9)$,
$F(-1,4) = (-9,1)$



$$J(P) = \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix}$$

$= 1 - (-6) = 7$, so

Area $S = |J(P)| \cdot$ Area R

$= (7)\left\{\frac{1}{2}(5)(4) + \frac{1}{2}(5)(2)\right\}$

$= 7\{10 + 5\} = 105$

9.) (8 pts.) Consider the vector function $\vec{r}(t) = (2/3)t^{3/2}\vec{i} + (2\sqrt{t})\vec{j}$. Find and simplify it's Principal Unit Normal Vector, $\vec{N}(t)$.

$$\xrightarrow{D} \vec{v}(t) = (t^{1/2})\vec{i} + (t^{-1/2})\vec{j} \longrightarrow |\vec{v}(t)| = \sqrt{(\sqrt{t})^2 + \left(\frac{1}{\sqrt{t}}\right)^2}$$

$$= \sqrt{\frac{t}{1} + \frac{1}{t}} = \frac{\sqrt{t^2+1}}{\sqrt{t}} \text{, then}$$

$$\vec{T}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|} = \frac{\sqrt{t}\,\vec{i} + \frac{1}{\sqrt{t}}\vec{j}}{\frac{\sqrt{t^2+1}}{\sqrt{t}}} = \frac{t}{\sqrt{t^2+1}}\vec{i} + \frac{1}{\sqrt{t^2+1}}\vec{j}$$

$$\rightarrow \vec{T}'(t) = \frac{\sqrt{t^2+1}\,(1) - t \cdot \frac{1}{\cancel{2}}(t^2+1)^{-1/2} \cdot (\cancel{2}t)}{t^2+1}\vec{i} - \frac{1}{\cancel{2}}(t^2+1)^{-3/2}(\cancel{2}t)\vec{j}$$

$$= \frac{\frac{\sqrt{t^2+1}}{1} - \frac{t^2}{\sqrt{t^2+1}}}{t^2+1}\vec{i} - \frac{t}{(t^2+1)^{3/2}}\vec{j}$$

$$= \frac{(t^2+1) - t^2}{(t^2+1)^{3/2}}\vec{i} - \frac{t}{(t^2+1)^{3/2}}\vec{j} \longrightarrow$$

$$\vec{T}'(t) = \frac{1}{(t^2+1)^{3/2}}\vec{i} + \frac{-t}{(t^2+1)^{3/2}}\vec{j} \longrightarrow$$

$$|\vec{T}'(t)| = \sqrt{\left(\frac{1}{(t^2+1)^{3/2}}\right)^2 + \left(\frac{-t}{(t^2+1)^{3/2}}\right)^2}$$

$$= \sqrt{\frac{1}{(t^2+1)^3} + \frac{t^2}{(t^2+1)^3}} = \sqrt{\frac{1+t^2}{(t^2+1)^3}}$$

$$= \sqrt{\frac{1}{(t^2+1)^2}} = \frac{1}{t^2+1} \to |\vec{T}'(t)| = \frac{1}{t^2+1} ;$$

then

$$\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}$$

$$= \frac{\frac{1}{(t^2+1)^{3/2}}\vec{i} + \frac{-t}{(t^2+1)^{3/2}}\vec{j}}{\frac{1}{t^2+1}} \longrightarrow$$

$$\vec{N}(t) = \frac{1}{\sqrt{t^2+1}}\vec{i} - \frac{t}{\sqrt{t^2+1}}\vec{j}$$

10.) (8 pts.) Consider the force vector field $\vec{F}(x, y, z) = (xy)\vec{i} + (z)\vec{j} + (y)\vec{k}$ (units: pounds). Find the Work done by $\vec{F}$ along path C given by

$$C : \begin{cases} x = t^2 \\ y = t \\ z = t^3 \end{cases} \quad \text{for } 0 \le t \le 1 \quad \text{(units: feet)}$$

$$\text{Work} = \int_C \vec{F} \cdot \vec{T} \, ds$$

$$= \int_C \left(M \frac{dx}{dt} + N \frac{dy}{dt} + P \frac{dz}{dt} \right) dt$$

$$= \int_C \left[(xy) \cdot (2t) + (z)(1) + (y)(3t^2) \right] dt$$

$$= \int_0^1 \left[(t^2)(t)(2t) + (t^3) + (t)(3t^2) \right] dt$$

$$= \int_0^1 \left[2t^4 + t^3 + 3t^3 \right] dt$$

$$= \int_0^1 (2t^4 + 4t^3) \, dt$$

$$= \left(\frac{2}{5} t^5 + t^4 \right) \Big|_0^1$$

$$= \frac{2}{5} + 1 = \frac{7}{5} \text{ (ft.)(lbs.)}$$

11.) (9 pts.) Derive the Curvature formula for path C given by $C: \begin{cases} x = 2t - 1 \\ y = t^2 \end{cases}$.

$$\vec{v}(t) = (2)\vec{i} + (2t)\vec{j} \to |\vec{v}(t)| = \sqrt{4+4t^2} = 2\sqrt{1+t^2},$$

$$\vec{T}(t) = \frac{(2)\vec{i} + (2t)\vec{j}}{2\sqrt{1+t^2}} = (1+t^2)^{-1/2}\vec{i} + \frac{t}{\sqrt{1+t^2}}\vec{j} \xrightarrow{D}$$

$$\vec{T}'(t) = \frac{-1}{\cancel{2}}(1+t^2)^{-3/2}(\cancel{2}t)\vec{i} + \frac{\sqrt{1+t^2}(1) - t\cdot\frac{1}{\cancel{2}}(1+t^2)^{-1/2}(\cancel{2}t)}{(1+t^2)}\vec{j}$$

$$= \frac{-t}{(1+t^2)^{3/2}}\vec{i} + \left(\frac{\sqrt{1+t^2}}{1} - \frac{t^2}{\sqrt{1+t^2}}\right)\frac{1}{1+t^2}\vec{j}$$

$$= \frac{-t}{(1+t^2)^{3/2}}\vec{i} + \frac{(1+t^2) - t^2}{\sqrt{1+t^2}}\cdot\frac{1}{1+t^2}\vec{j}$$

$$= \frac{-t}{(1+t^2)^{3/2}}\vec{i} + \frac{1}{(1+t^2)^{3/2}}\vec{j} \longrightarrow$$

$$|\vec{T}'(t)| = \sqrt{\left(\frac{-t}{(1+t^2)^{3/2}}\right)^2 + \left(\frac{1}{(1+t^2)^{3/2}}\right)^2}$$

$$= \sqrt{\frac{t^2}{(1+t^2)^3} + \frac{1}{(1+t^2)^3}} = \sqrt{\frac{1+t^2}{(1+t^2)^3}}$$

$$= \sqrt{\frac{1}{(1+t^2)^2}} = \frac{1}{1+t^2},$$ so curvature

$$K = \frac{1}{|\vec{v}(t)|}|\vec{T}'(t)| = \frac{1}{2\sqrt{1+t^2}}\cdot\frac{1}{1+t^2} \longrightarrow$$

$$K = \frac{1}{2(1+t^2)^{3/2}}$$