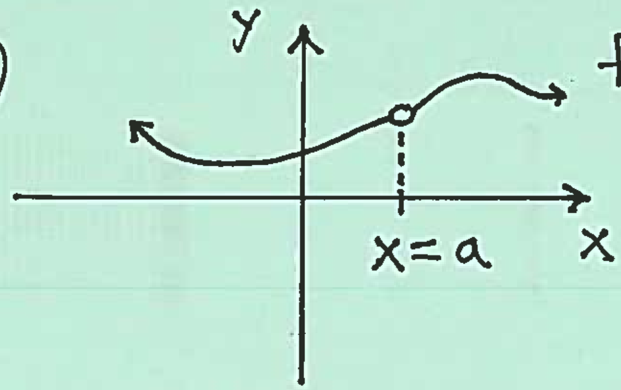
Math 16A
Section 1.6

Continuity of a Function, $y=f(x)$

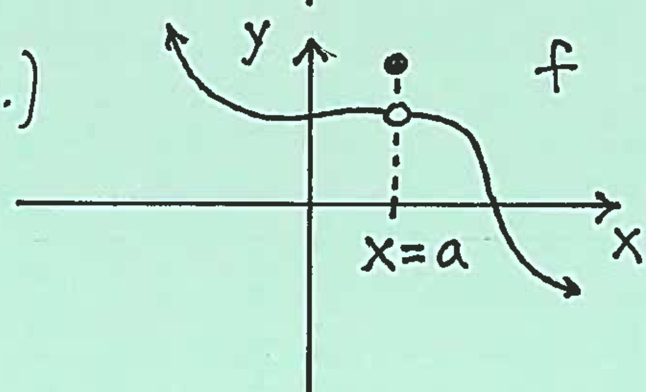
Before we formally define continuity, let's look at graphs of functions which are <u>NOT</u> continuous at $x=a$.

1.)



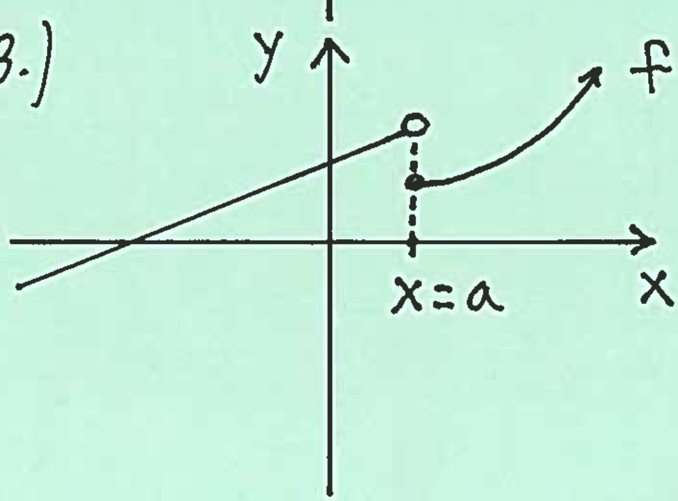
f is NOT continuous at $x=a$.

2.)



f is NOT continuous at $x=a$.

3.)



f is NOT continuous at $x=a$.

Definition: (Three-Step Definition)
Function $y = f(x)$ is <u>continuous</u> at $x = a$ if

i.) $f(a)$ exists (is finite)

ii.) $\lim_{x \to a} f(x) = L$ (a finite #)

iii.) $\lim_{x \to a} f(x) = L = f(a)$

Example: Use the Three-Step Definition of Continuity to determine if each function is continuous at the given x-value.

1.) $$f(x) = \begin{cases} x^2 - 3x + 2, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$$

@ $x = 0$

i.) $f(0) = 1$ ✓

ii.) $\lim_{x \to 0} f(x) = \lim_{x \to 0} (x^2 - 3x + 2)$

$= (0)^2 - 3(0) + 2 = 2$ ✓

iii.) $\lim_{x\to 0} f(x) = 2 \neq 1 = f(0)$, so

f is NOT continuous at $x=0$.

2.) $f(x) = \begin{cases} x+2 & , \text{ if } x>1 \\ x^2+x+1 & , \text{ if } x \leq 1 \end{cases}$

@ $x=1$

i.) $f(1) = (1)^2 + (1) + 1 = 3$ ✓

ii.) $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x+2)$

$= (1) + 2 = 3$;

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x^2+x+1)$

$= (1)^2 + (1) + 1 = 3$; then

$\lim_{x\to 1} f(x) = 3$ ✓

iii.) $\lim_{x\to 1} f(x) = 3 = f(1)$, so

f IS continuous at $x=1$

3.) $f(x) = \begin{cases} \dfrac{x^2-3}{2-x} & , \text{ if } x \neq 2 \\ 4 & , \text{ if } x = 2 \end{cases}$

@ $x = 2$

i.) $f(2) = 4$ ✓

ii.) $\displaystyle\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^2-3}{2-x}$

$= \dfrac{(2)^2-3}{2-(2)} = \text{"}\dfrac{1}{0}\text{"} = \pm\infty$

(NOT finite), so

f is NOT continuous at $x=2$

4.) $f(x) = \begin{cases} 3-x & , \text{ if } x \geq 1 \\ 2 & , \text{ if } -2 \leq x < 1 \\ x+3 & , \text{ if } x < -2 \end{cases}$

a.) @ $x = 1$ b.) @ $x = -2$

a.) i.) $f(1) = 3-(1) = 2$ ✓

ii.) $\displaystyle\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (3-x) = 3-1 = 2$,

$\displaystyle\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 2 = 2$, so

$$\lim_{x \to 1} f(x) = 2 \quad \checkmark$$

iii.) $\lim_{x \to 1} f(x) = 2 = f(1)$ ✓ , then

f IS continuous at $x = 1$.

b.) i.) $f(-2) = 2$ ✓

ii.) $\lim_{x \to -2^+} f(x) = \lim_{x \to -2^+} 2 = 2$,

$\lim_{x \to -2^-} f(x) = \lim_{x \to -2^-} (x+3) = -2+3 = 1$,

so that $\lim_{x \to -2} f(x)$ D.N.E. ,

so that f is NOT continuous at $x = -2$.

Continuity and Shortcuts

Following is a list of Shortcuts and well-known continuous functions, which are given without proof. They will be used to determine the x-values for which a function is continuous.

SHORTCUT 1: Every polynomial is continuous for all values of x.

SHORTCUT 2: Sums, differences, products, quotients (denominator $\neq 0$), and compositions of continuous functions are continuous.

HERE is a short list of WELL-KNOWN CONTINUOUS FUNCTIONS:

1. $\sin x$ (for all x-values)
2. $\cos x$ (for all x-values)
3. $\sqrt{x}$ (for all $x \geq 0$)
4. $x^{1/3}$ (for all x-values)
5. e^x (for all x-values)- formally introduced in Math 16B
6. $\ln x$ (for all $x > 0$)- formally introduced in Math 16B

<u>Example</u>: Determine the x-values for which each function is continuous. Briefly explain your answers using shortcuts.

1.) $y = 3x^4 - 2x^3 + 4x^2 - 5x + 1$:
This function is continuous for all x-values (polynomial).

2.) $y = \sin x + e^x$: $y = \sin x$ is continuous for all x-values (well-known); $y = e^x$ is continuous for all x-values (well-known); so $y = \sin x + e^x$ is continuous for all x-values (sum).

3.) $f(x) = \dfrac{x^3 + x}{x^2 - 9}$: $y = x^3 + x$ and $y = x^2 - 9$ are both continuous for all values of x (polynomials). So $f(x) = \dfrac{x^3 + x}{x^2 - 9}$ (quotient) is continuous for all x-values except when $x^2 - 9 = (x-3)(x+3) = 0$, i.e., except when $x = 3$, $x = -3$.

4.) $y = \cos x \cdot \ln x$: $y = \cos x$ is continuous for all x-values (well-known) ; $y = \ln x$ is continuous for $x > 0$ (well-known); then $y = \cos x \cdot \ln x$ (product) is continuous for $x > 0$.

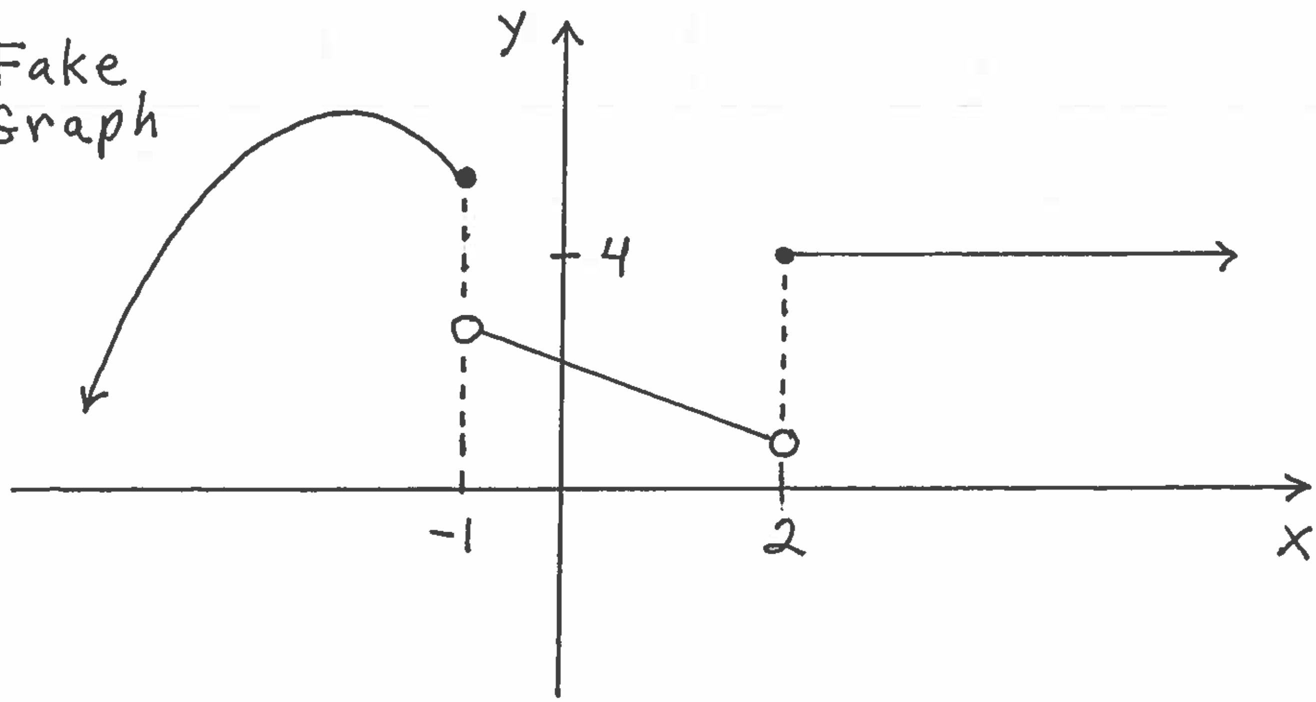
5.) $y = \sin(x^2 - 3x + 5)$:
$f(x) = \sin x$ (well-known) and $g(x) = x^2 - 3x + 5$ (polynomial) are continuous for all x-values. Thus, $y = \sin(x^2 - 3x + 5) = f(g(x))$ (functional composition) is continuous for all x-values.

Continuity and "Fake Graphs"

Example: Use limits to determine constants A and B so that each function is continuous for all x-values.

1.)
$$f(x)=\begin{cases} Ax^2+Bx, & \text{if } x\le -1 \\ Ax-2B, & \text{if } -1<x<2 \\ 4, & \text{if } x\ge 2 \end{cases}$$

Fake Graph



"Glue" the graphs together at $x=-1$:

$$\lim_{x\to -1^-}(Ax^2+Bx)=\lim_{x\to -1^+}(Ax-2B) \longrightarrow$$

$$A(-1)^2 + B(-1) = A(-1) - 2B \longrightarrow$$

$$A - B = -A - 2B \longrightarrow \quad 2A = -B \longrightarrow$$

$$\boxed{B = -2A} \quad ;$$

"glue" the graphs together at $x = 2$:

$$\lim_{x \to 2^-} (Ax - 2B) = \lim_{x \to 2^+} (4) \longrightarrow$$

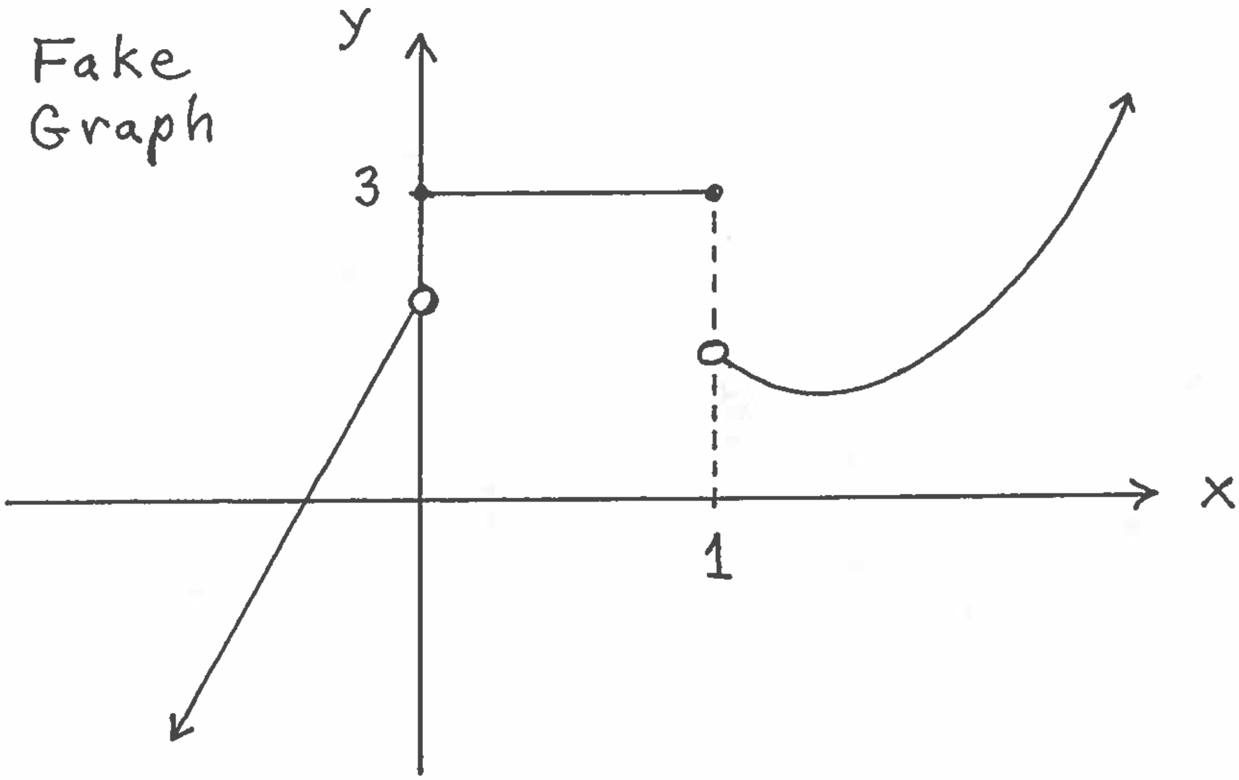
$$A(2) - 2B = 4 \rightarrow \quad 2A - 2B = 4 \longrightarrow$$

$$A - B = 2 \longrightarrow \quad \boxed{A = B + 2} \quad ; \text{ then}$$

$$A = B + 2 = (-2A) + 2 \rightarrow \quad 3A = 2 \rightarrow$$

$$\boxed{A = \frac{2}{3}} \text{ and } \boxed{B = \frac{-4}{3}}$$

2.) $f(x) = \begin{cases} Ax^2 + Bx - 2, & \text{if } x > 1 \\ 3, & \text{if } 0 \le x \le 1 \\ 3A - B + 3x, & \text{if } x < 0 \end{cases}$



"Glue" the graphs together at $x=0$:

$$\lim_{x\to 0^-} (3A-B+3x) = \lim_{x\to 0^+} (3) \longrightarrow$$

$$3A-B+3(0) = 3 \rightarrow \boxed{B = 3A-3} \;;$$

"Glue" the graphs together at $x=1$:

$$\lim_{x\to 1^-} (3) = \lim_{x\to 1^+} (Ax^2+Bx-2) \rightarrow$$

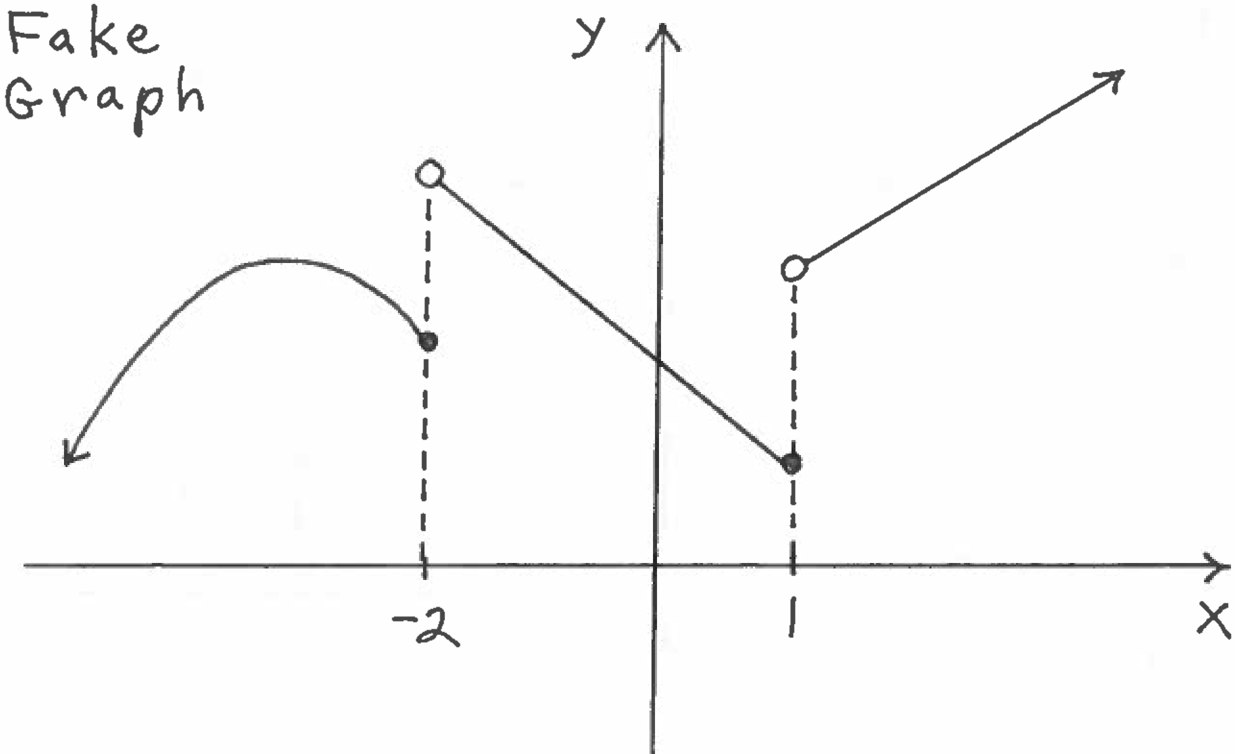
$$3 = A(1)^2 + B(1) - 2 \rightarrow \boxed{A+B=5} \;;$$

then $A+B=5 \rightarrow A+(3A-3)=5 \rightarrow$

$4A = 8 \longrightarrow \boxed{A=2}$ and $\boxed{B=3}$.

3.) $f(x) = \begin{cases} 2Bx - 3 + 2A \ , & \text{if } x > 1 \\ Ax + B \ , & \text{if } -2 < x \leq 1 \\ Ax^2 + Bx \ , & \text{if } x \leq -2 \end{cases}$

Fake Graph



"Glue" the graphs together at $x = -2$:

$\lim\limits_{x \to -2^-} (Ax^2 + Bx) = \lim\limits_{x \to -2^+} (Ax + B) \rightarrow$

$A(-2)^2 + B(-2) = A(-2) + B \rightarrow$

$4A - 2B = -2A + B \rightarrow 6A = 3B$

$\rightarrow \boxed{B = 2A}$;

"Glue" the graphs together at $x = 1$:

$$\lim_{x\to 1^-} (Ax+B) = \lim_{x\to 1^+} (2Bx-3+2A) \to$$

$$A(1)+B = 2B(1)-3+2A \to \boxed{A=3-B};$$

then $A=3-B=3-(2A) \to$

$3A=3 \to \boxed{A=1}$ and $\boxed{B=2}$.