Math 16A
Section 2.1

The Derivative of a Function

<u>Definition</u>: Let $y = f(x)$ be a function. The <u>derivative</u> <u>of</u> <u>f</u> <u>at</u> <u>x</u> is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

<u>Example</u>: Find the derivative of each function.

1.) $f(x) = x^2 + 1 \xrightarrow{D}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 + 1 - (x^2+1)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 + \cancel{1} - \cancel{x^2} - \cancel{1}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}}$$

$= 2x + (0) = 2x$, i.e., $f'(x) = 2x$.

2.) $f(x) = 3x - 4 \xrightarrow{D}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{3(x+h) - 4 - (3x-4)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{3x} + 3h - \cancel{4} - \cancel{3x} + \cancel{4}}{h}$$

$$= \lim_{h \to 0} \frac{3\cancel{h}}{\cancel{h}} = 3,$$ i.e.,

$f'(x) = 3$.

3.) $f(x) = \sqrt{x+4} \xrightarrow{D}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{(x+h)+4} - \sqrt{x+4}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h+4} - \sqrt{x+4}}{h} \cdot \frac{\sqrt{x+h+4} + \sqrt{x+4}}{\sqrt{x+h+4} + \sqrt{x+4}}$$

$$= \lim_{h \to 0} \frac{(\sqrt{x+h+4})^2 - (\sqrt{x+4})^2}{h(\sqrt{x+h+4} + \sqrt{x+4})}$$

$$= \lim_{h \to 0} \frac{\cancel{x}+h+\cancel{4} - (\cancel{x}+\cancel{4})}{h(\sqrt{x+h+4} + \sqrt{x+4})}$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h+4} + \sqrt{x+4})}$$

$$= \frac{1}{\sqrt{x+(0)+4} + \sqrt{x+4}} = \frac{1}{2\sqrt{x+4}} \text{, i.e.,}$$

$$f'(x) = \frac{1}{2\sqrt{x+4}} .$$

4.) $f(x) = \dfrac{x+2}{3-x} \xrightarrow{D}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\dfrac{(x+h)+2}{3-(x+h)} - \dfrac{x+2}{3-x}}{\dfrac{h}{1}}$$

$$= \lim_{h\to 0} \frac{(x+h+2)(3-x)-(3-x-h)(x+2)}{(3-x-h)\ (3-x)} \cdot \frac{1}{h}$$

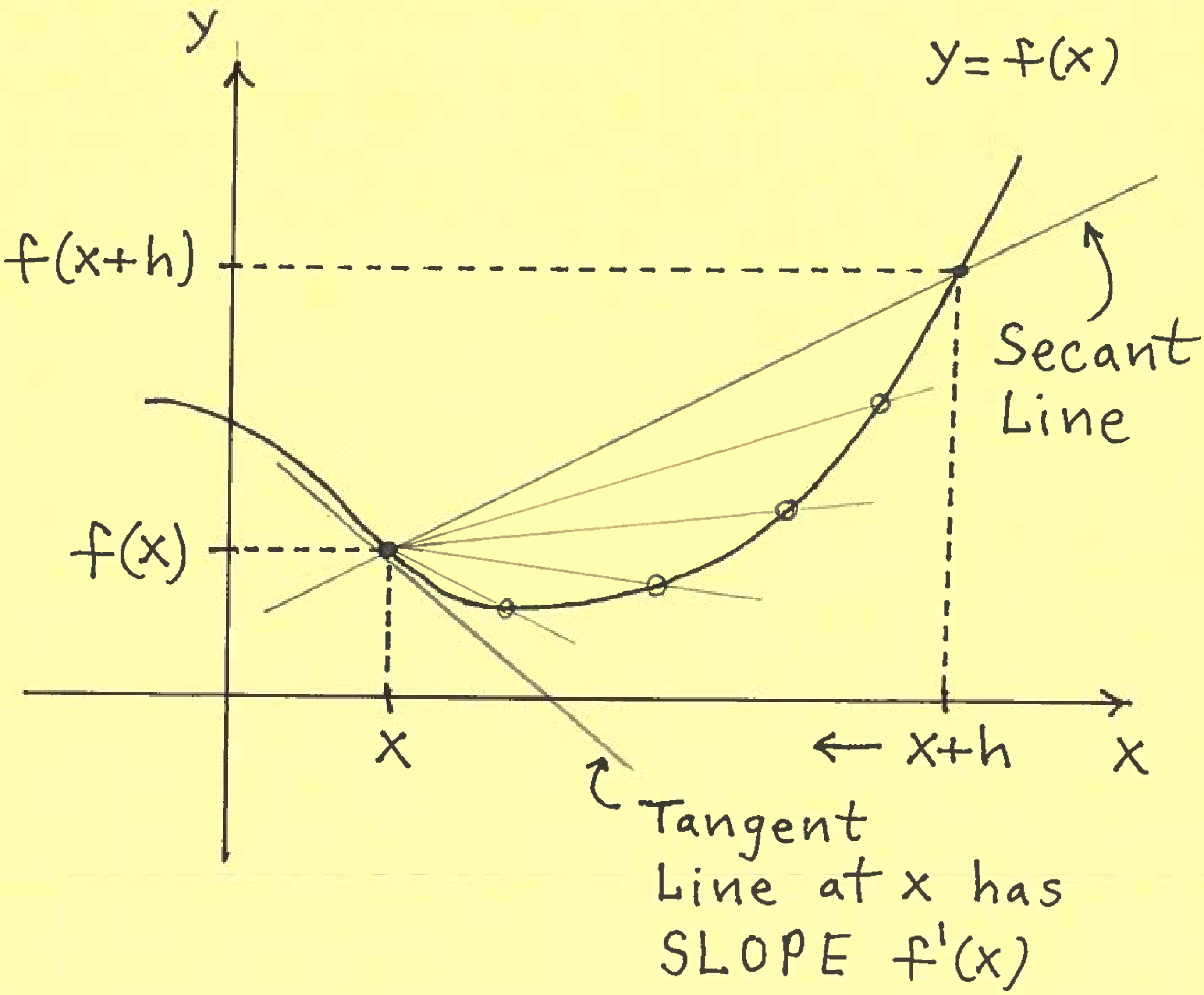
$$= \lim_{h\to 0} \frac{\cancel{3x}+3h+\cancel{6}-\cancel{x^2}-\cancel{hx}-\cancel{2x}-(\cancel{3x}-\cancel{x^2}-\cancel{hx}+\cancel{6}-\cancel{2x}-2h)}{(3-x-h)(3-x)\,h}$$

$$= \lim_{h\to 0} \frac{5\cancel{h}}{(3-x-h)(3-x)\cancel{h}}$$

$$= \frac{5}{(3-x-(0))(3-x)} = \frac{5}{(3-x)^2}, \text{ i.e.,}$$

$$f'(x) = \frac{5}{(3-x)^2}.$$

Question: What is the graphical meaning of the derivative of function $y=f(x)$?



As $h \to 0$ the Secant Line between points $(x, f(x))$ and $(x+h, f(x+h))$ becomes the Tangent Line at $(x, f(x))$. Thus, the SLOPE of the Secant Line becomes the

SLOPE of the Tangent Line,
i.e., SLOPE (Secant Line)

$$m = \frac{f(x+h) - f(x)}{(x+h) - x}$$

$$= \frac{f(x+h) - f(x)}{h}$$

becomes SLOPE (Tangent Line)

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x) .$$

<u>Example</u> : Consider the graph of the parabola

$$f(x) = 3x^2 - 12x + 4 .$$

Find the SLOPES of the Tangent Lines at $x = -1$, $x = 2$, and $x = 5$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{3(x+h)^2 - 12(x+h) + \cancel{4} - (3x^2 - 12x + \cancel{4})}{h}$$

$$= \lim_{h \to 0} \frac{3(x^2 + 2hx + h^2) - \cancel{12x} - 12h - 3x^2 + \cancel{12x}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{3x^2} + 6hx + 3h^2 - 12h - \cancel{3x^2}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(6x + 3h - 12)}{\cancel{h}}$$

$= 6x + 3(0) - 12 = 6x - 12$, i.e.,

$f'(x) = 6x - 12$; SLOPES of Tangent Lines are

a.) $x = -1$: $m = f'(-1) = 6(-1) - 12 = -18$

b.) $x = 2$: $m = f'(2) = 6(2) - 12 = 0$

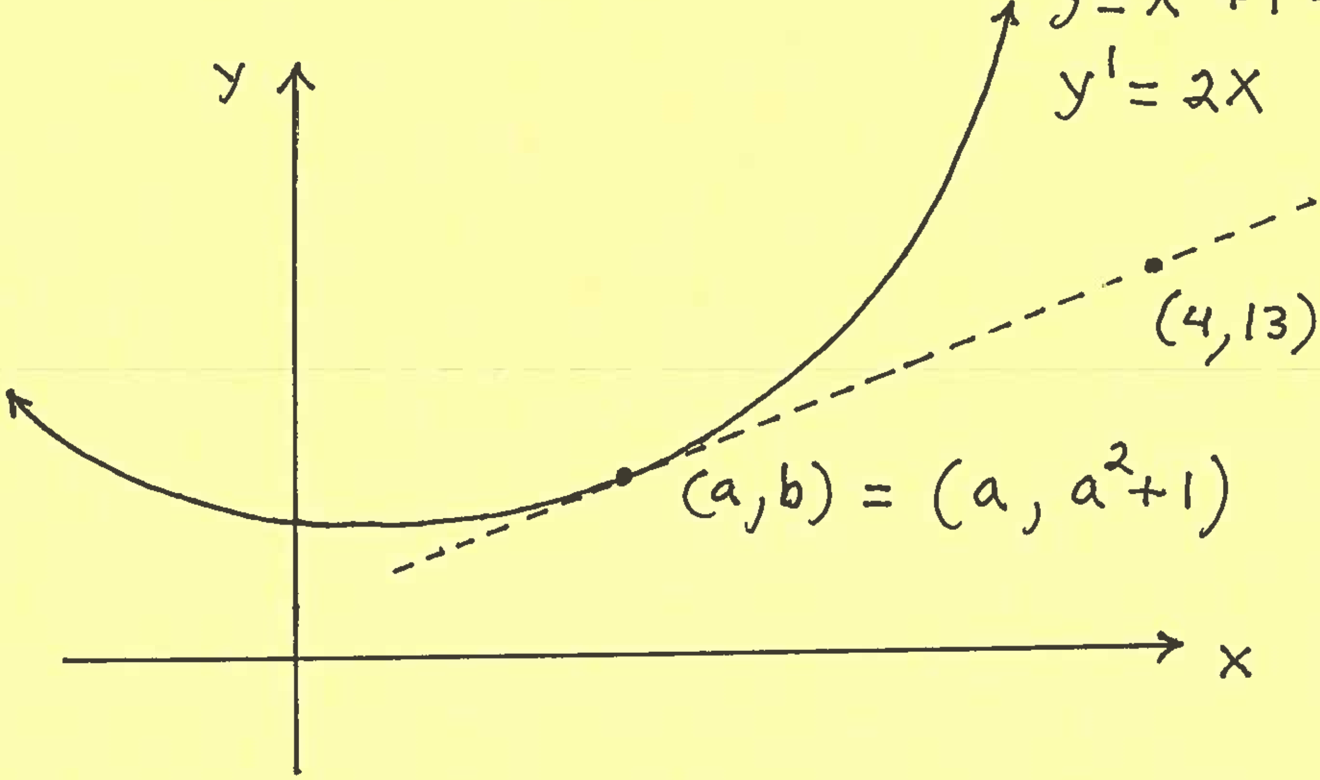
c.) $x = 5$: $m = f'(5) = 6(5) - 12 = 18$

<u>Example</u> : a spaceship travels through space along the curve $y = x^2 + 1$ (from left to right). a

payload is to be released from the spaceship so that it will travel on a linear path tangent to the curve and to be intercepted at the point (4, 13). At what point (a, b) should the payload be released?

$y = x^2 + 1 \xrightarrow{D}$

$y' = 2x$



The SLOPE of the tangent line at $x = a$ is:

1.) $y' = 2a$

2.) $m = \dfrac{(a^2+1) - 13}{a - 4} = \dfrac{a^2 - 12}{a - 4}$

SET SLOPES EQUAL:

$$2a = \frac{a^2-12}{a-4} \longrightarrow$$

$$2a(a-4) = a^2-12 \longrightarrow$$

$$2a^2-8a = a^2-12 \longrightarrow$$

$$a^2-8a+12=0 \longrightarrow (a-2)(a-6)=0 \longrightarrow$$

$a=6$ (NO, why?) or $\boxed{a=2,\ b=5}$

<u>Derivative Notation</u>:

$$y' = f'(x) = Df(x) = \frac{dy}{dx}$$