Math 16A
Section 2.5

<u>Chain Rule</u> (for functional composition)

$$D f(g(x)) = f'(g(x)) \cdot g'(x)$$

<u>Example</u>: Find $D\,(4x^3+7x)^5$;

Let $f(x) = x^5$ and $g(x) = 4x^3+7x \xrightarrow{D}$
$f'(x) = 5x^4$ and $g'(x) = 12x^2+7$;

then $D\,(4x^3+7x)^5 = D f(g(x))$

$$= f'(g(x)) \cdot g'(x)$$
$$= f'(4x^3+7x) \cdot (12x^2+7)$$
$$= 5(4x^3+7x)^4 \cdot (12x^2+7).$$

<u>Note</u>: That's too much work. Is there an easier way to apply the Chain Rule? YES:

$$D f(g(x)) = f'(g(x)) \cdot g'(x)$$

derivative of "outside" function

derivative of "inside" function

__Example__: Differentiate each function using the Chain Rule "shortcut."

1.) $y = (x^2 + \sin x)^{20} \xrightarrow{D}$

$$y' = 20\,(x^2 + \sin x)^{19} \cdot (2x + \cos x)$$

D of "outside" D of "inside"

2.) $y = \sqrt{x^2 - 4} \xrightarrow{D} y' = \frac{1}{2}(x^2 - 4)^{-1/2} \cdot (2x)$

3.) $f(x) = x^3 (3x - 5)^{7/2} \xrightarrow{D}$

$$f'(x) = x^3 \cdot \tfrac{7}{2}(3x-5)^{5/2}(3) + (3x^2)(3x-5)^{7/2}$$

4.) $g(x) = \sin(x^2 - 4x) \xrightarrow{D}$

$$g'(x) = \cos(x^2 - 4x) \cdot (2x - 4)$$

5.) $y = \tan^3 x = (\tan x)^3 \xrightarrow{D}$

$$y' = 3(\tan x)^2 \cdot (\sec^2 x)$$

6.) $f(x) = \sin^7 x \xrightarrow{D}$

$$f'(x) = 7 \sin^6 x \cdot (\cos x)$$

<u>Question</u>: How do we differentiate a composition of more than two functions?

<u>answer</u>: Differentiate the functions ("layers") from "outside" to "inside."

7.) $y = \cos^4(x^3)$

("layers": $(\ )^4$, $\cos$, x^3) $\xrightarrow{D}$

$y' = 4\cos^3(x^3) \cdot (-\sin(x^3)) \cdot 3x^2$

8.) $f(x) = \left(3 + \sqrt{2 + \dfrac{x-1}{x^2+1}}\right)^{3/2}$

("layers": $(\ )^{3/2}$, $3 + \sqrt{\cdot}$, $2 + \div$) $\xrightarrow{D}$

$$f'(x) = \frac{3}{2}\left(3 + \sqrt{2 + \frac{x-1}{x^2+1}}\right)^{1/2}$$

$$\cdot \frac{1}{2}\left(2 + \frac{x-1}{x^2+1}\right)^{-1/2}$$

$$\cdot \frac{(x^2+1)(1) - (x-1)(2x)}{(x^2+1)^2}$$

9.) $f(x) = \sin^5(\tan^4(x^3+x))$

("layers": $(\ )^5$, sin, $(\ )^4$, tan, x^3+x) $\xrightarrow{D}$

$$f'(x) = 5\sin^4(\tan^4(x^3+x))$$
$$\cdot \cos(\tan^4(x^3+x))$$
$$\cdot 4(\tan^3(x^3+x)) \cdot \sec^2(x^3+x) \cdot (3x^2+1)$$

<u>Example</u>: Solve $y' = 0$ for x and set up a Sign Chart for y'.

1.) $y = x^2(4-x)^3 \xrightarrow{D}$

$$y' = x^2 \cdot 3(4-x)^2(-1) + (2x)(4-x)^3$$
$$= x(4-x)^2[-3x + 2(4-x)]$$
$$= x(4-x)^2[-3x + 8 - 2x]$$
$$= x(4-x)^2[8-5x] = 0 \longrightarrow$$

$$x = 0,\ x = 4,\ x = \frac{8}{5}$$

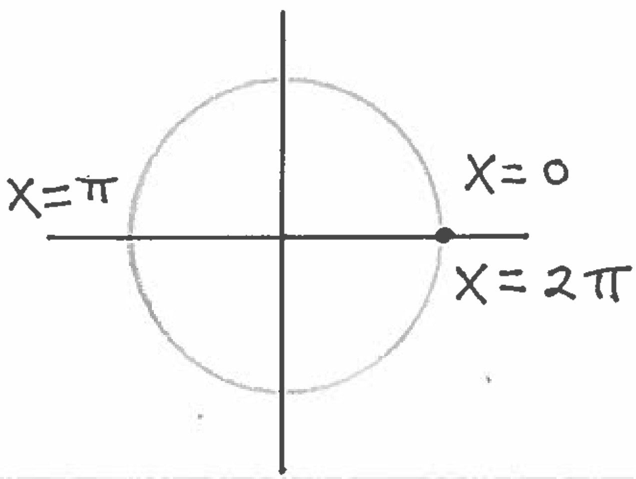
$-$	0	$+$	0	$-$	0	$-$	y'
-1	$x=0$	1	$x=\frac{8}{5}$	3	$x=4$	5	

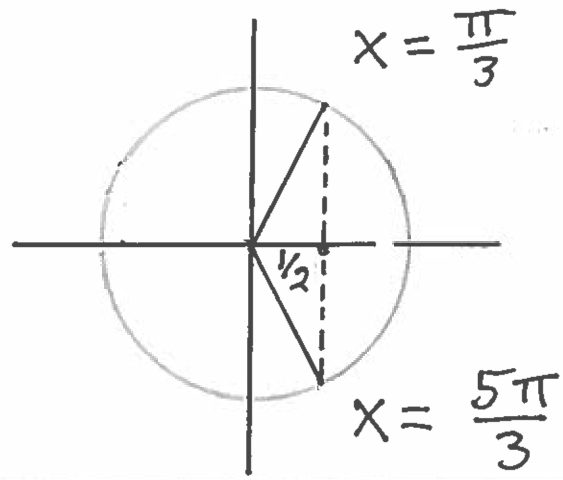
2.) $f(x) = \sin^2 x + \cos x$, $0 \le x \le 2\pi$;

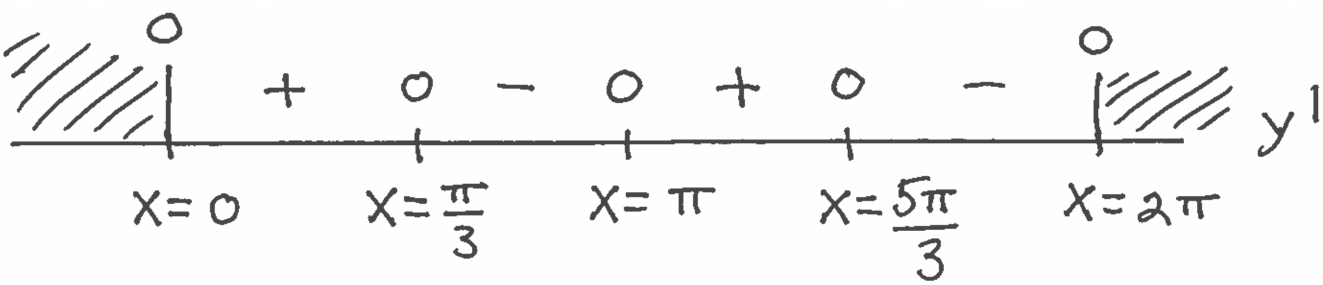
$\xrightarrow{D}$ $f'(x) = 2 \sin x \cos x - \sin x$

$= \sin x \, (2 \cos x - 1) = 0 \rightarrow$

$\sin x = 0$ OR $\cos x = \frac{1}{2}$







Example : assume that $g(x) = x^2 - x$ and $f'(2) = 4$. Find $D\, f(g(x))$ at $x = -1$: $\xrightarrow{D}$

$g'(x) = 2x - 1$, so $D f(g(x)) = f'(g(x)) \cdot g'(x)$

(Let $x = -1$) $= f'(g(-1)) \cdot g'(-1)$

$= f'((-1)^2 - (-1))\,(2(-1) - 1) = f'(2)\,(-3) = (4)(-3) = -12$