Math 16A
Section 2.7

Implicit Differentiation

Example: 1.) $D\,x^5 = 5x^4$

2.) $D\,(1+3x^2)^5 = 5(1+3x^2)^4 \cdot 6x$

3.) $D\,(\sin x)^5 = 5(\sin x)^4 \cdot \cos x$

4.) $D\,(f(x))^5 = 5(f(x))^4 \cdot f'(x)$

assume that y is a function of x:

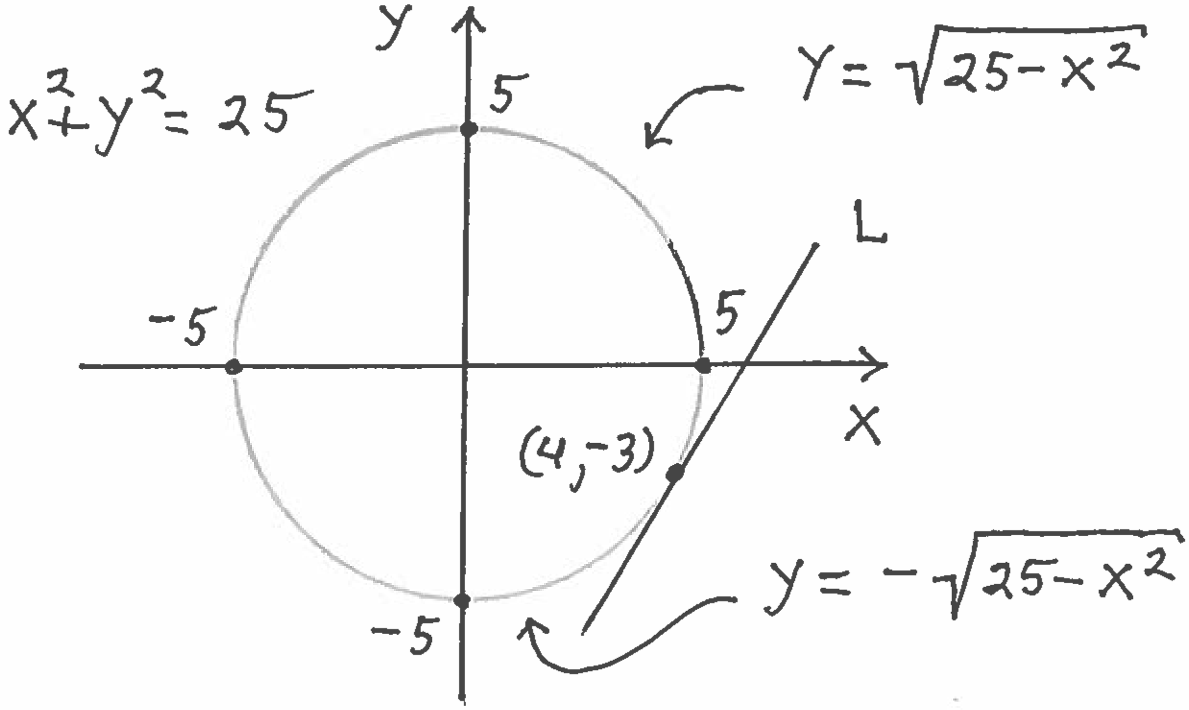
5.) $\underbrace{D(y)^5 = 5y^4 \cdot y'}$, where $y' = \frac{dy}{dx}$

This is IMPLICIT DIFFERENTIATION

NOTE: Implicit Differentiation is the CHAIN RULE applied to the implicit function y.

Example: Find the SLOPE of the line tangent to the graph of the circle $x^2+y^2=25$ at the point $(4,-3)$ using a.) Ordinary Differentiation

b.) Implicit Differentiation



a.) $y=-(25-x^2)^{1/2} \xrightarrow{D}$

$$y'=\frac{-1}{2}(25-x^2)^{-1/2}\cdot(-2x)=\frac{x}{\sqrt{25-x^2}} ;$$

SLOPE of tangent line L at point (4, −3) is

$$m=y'=\frac{(4)}{\sqrt{25-(4)^2}}=\frac{4}{\sqrt{9}}=\boxed{\frac{4}{3}} .$$

b.) $x^2+y^2=25 \xrightarrow{D}$

$D(x^2)+D(y^2)=D(25) \longrightarrow$

$$2X + \underbrace{2Y \cdot Y'}_{\text{Implicit Differentiation}} = 0 \longrightarrow$$

$$2YY' = -2X \rightarrow Y' = \frac{-\cancel{2}X}{\cancel{2}Y} \rightarrow y' = \frac{-X}{Y},$$

so SLOPE of tangent line L at point $X=4, Y=-3$ is

$$m = y' = \frac{-(4)}{(-3)} = \boxed{\frac{4}{3}}.$$

Example: Use Implicit Differentiation to find $y' = \frac{dY}{dx}$ for each equation.

1.) $x^2 + y^3 = 2X \xrightarrow{D}$

$$2X + 3Y^2 \cdot Y' = 2 \rightarrow 3y^2 y' = 2 - 2X \rightarrow$$

$$y' = \frac{2-2X}{3y^2}$$

2.) $x^3 y^2 + 3Y = 4 - X \xrightarrow{D}$

$$(X^3 \cdot 2YY' + 3X^2 \cdot y^2) + 3y' = -1 \rightarrow$$

$$2X^3 YY' + 3Y' = -1 - 3X^2 y^2 \longrightarrow$$

$$y'(2X^3 Y + 3) = -1 - 3X^2 Y^2 \longrightarrow$$

$$y' = \frac{-1-3x^2y^2}{2x^3y+3}$$

3.) $\sin x + \cos y = 5x - 2y \xrightarrow{D}$

$\cos x + -\sin y \cdot y' = 5 - 2y' \rightarrow$

$2y' - y'\sin y = 5 - \cos x \rightarrow$

$y'(2-\sin y) = 5 - \cos x \rightarrow$

$$y' = \frac{5-\cos x}{2-\sin y}$$

4.) $(x-y)^4 = 4x^3 - 2y^4 \xrightarrow{D}$

$4(x-y)^3[1-y'] = 12x^2 - 8y^3 \cdot y' \rightarrow$

$4(x-y)^3 - 4(x-y)^3 y' = 12x^2 - 8y^3y' \rightarrow$

$8y^3y' - 4(x-y)^3y' = 12x^2 - 4(x-y)^3 \rightarrow$

$y'[8y^3 - 4(x-y)^3] = 12x^2 - 4(x-y)^3 \rightarrow$

$$y' = \frac{12x^2 - 4(x-y)^3}{8y^3 - 4(x-y)^3}$$

5.) $\tan(xy) = \sqrt{x} + \sqrt{y} \xrightarrow{D}$

$\sec^2(xy) \cdot [xy' + (1)y] = \frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2} \cdot y' \rightarrow$

$xy'\sec^2(xy) + y\sec^2(xy) = \frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2} \cdot y' \rightarrow$

$$xy' \sec^2(xy) - \frac{1}{2} y^{-1/2} y' = \frac{1}{2} x^{-1/2} - y \sec^2(xy) \rightarrow$$

$$y' \left[x \sec^2(xy) - \frac{1}{2} y^{-1/2} \right] = \frac{1}{2} x^{-1/2} - y \sec^2(xy) \rightarrow$$

$$y' = \frac{\frac{1}{2} x^{-1/2} - y \sec^2(xy)}{x \sec^2(xy) - \frac{1}{2} y^{-1/2}}$$

<u>Example</u>: Find the SLOPE and CONCAVITY of the graph of each equation at the indicated point. Then sketch the graph near the indicated point.

1.) $xy + y^2 = 2$ @ point $(1, -2)$:

$\xrightarrow{D} xy' + (1)y + 2yy' = 0 \longrightarrow$

$(x+2y)\, y' = -y \rightarrow \boxed{y' = \dfrac{-y}{x+2y}}$,

so SLOPE at $x = 1$, $y = -2$ is

$$m = y' = \frac{-(-2)}{(1)+2(-2)} = \frac{-2}{3} \text{ and}$$

graph is $(\downarrow)$; $\xrightarrow{D}$

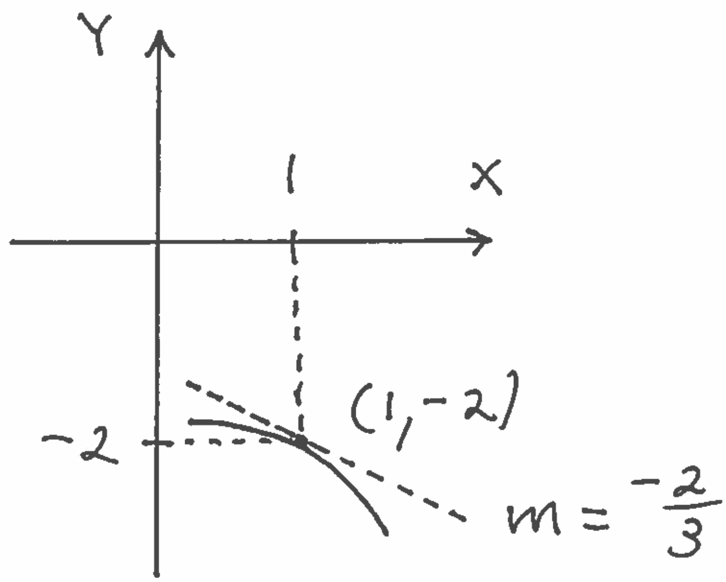
$$y'' = \frac{(x+2y)(-y') - (-y)(1+2y')}{(x+2y)^2} \text{ ; now}$$

let $x=1$, $y=-2$, and $y' = \frac{-2}{3} \rightarrow$

$$y'' = \frac{(1-4)\left(\frac{2}{3}\right) - (2)\left(1-\frac{4}{3}\right)}{(1-4)^2}$$

$$= \frac{(-3)\left(\frac{2}{3}\right) - (2)\left(-\frac{1}{3}\right)}{9} = \left(\frac{-6}{3} + \frac{2}{3}\right)\left(\frac{1}{9}\right) \rightarrow$$

$y'' = \frac{-4}{27}$, so graph is $(\cap)$



2.) $xy^3 + y = 1$ @ $x=0$:

$\xrightarrow{D} (x \cdot 3y^2 y' + (1) y^3) + y' = 0 \longrightarrow$

$(3xy^2+1)\, y' = -y^3 \rightarrow \boxed{y' = \frac{-y^3}{3xy^2+1}}$,

so SLOPE at $x=0$, $y=1$ is

$m = y' = \dfrac{-(1)^3}{3(0)(1)^2+1} = -1$ and

graph is $(\downarrow)$; $\xrightarrow{D}$

$$y'' = \frac{(3xy^2+1)(-3y^2y') - (-y^3)(3x \cdot 2yy' + 3 \cdot y^2)}{(3xy^2+1)^2} \; ;$$

now let $x=0$, $y=1$, and $y' = -1 \longrightarrow$

$y'' = \dfrac{(1)(3) - (-1)(3)}{(1)^2} = 6$, so graph is $(\cup)$