Math 16A
Section 2.8

Related Rates Problems

Recall: (Implicit Differentiation)

I.) assume that y is a function of x:

Example: $D\, x^5 = 5x^4$;

$$D\, y^5 = 5y^4 \cdot y'$$

II.) assume that both x and y are functions of time t:

Example: $D\, t^5 = 5t^4$;

$$D\, x^5 = 5x^4 \cdot \frac{dx}{dt} \ ;$$

$$D\, y^5 = 5y^4 \cdot \frac{dy}{dt} \ .$$

Notation: Preferred notation is $\frac{dy}{dt}$, since y' is ambiguous; it could mean $\frac{dy}{dx}$ or $\frac{dy}{dt}$.

Example: The radius r of a circle is increasing at the rate of 30in./hr. at what rate is the circle's Area changing when $r = 10$ in.?

Given: $\frac{dr}{dt} = 30$ in./hr.

assume: Area of a Circle is

$$A = \pi r^2$$

Find: $\frac{dA}{dt}$ when $r = 10$ in.

$$A = \pi r^2 \xrightarrow{D} \frac{dA}{dt} = \pi \cdot 2r \frac{dr}{dt} \longrightarrow$$

$$\frac{dA}{dt} = 2\pi(10)(30) \longrightarrow$$

$$\frac{dA}{dt} = 600\pi \text{ in.}^2/\text{hr.}$$

Math 16A
Kouba
How to Approach Related Rates Problems

Here are steps which may help you be successful in mastering Related Rates Problems.

1.) Read the problem carefully. Read it several times.

2.) Draw a picture representing the problem.

3.) Label quantities in your picture with variables (if they are changing) and with constants (if they are not changing).

4.) Write down information which is given in the problem.

5.) Write down what is to be found.

6.) Begin with a main equation.

7.) Differentiate the main equation with respect to time t.

8.) Plug in given numbers.

9.) Solve for the unknown quantity.

10.) Don't forget to put units on your final answer.

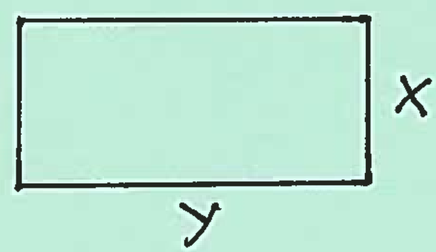
<u>Example</u>: The width x of a rectangle is <u>increasing</u> at the rate of 5 cm./sec. and the length y is <u>decreasing</u> at the rate of 4 cm./sec. At what rate is the rectangle's

1.) Perimeter changing

2.) area changing

when $x = 3$ cm. and $y = 2$ cm.?

<u>Given</u>: $\frac{dx}{dt} = 5$ cm./sec.,



$\frac{dy}{dt} = -4$ cm./sec.

<u>assume</u>: Perimeter $P = 2x + 2y$ and Area $A = xy$.

1.) Find $\frac{dP}{dt}$ when $x = 3,\ y = 2$:

$$\xrightarrow{D} \quad \frac{dP}{dt} = 2 \cdot \frac{dx}{dt} + 2 \cdot \frac{dy}{dt}$$

$$= 2(5) + 2(-4) = 2 \text{ cm./sec.}$$

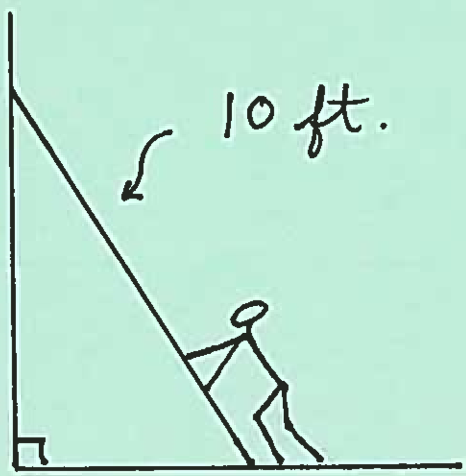
2.) Find $\frac{dA}{dt}$ when $x=3$, $y=2$:

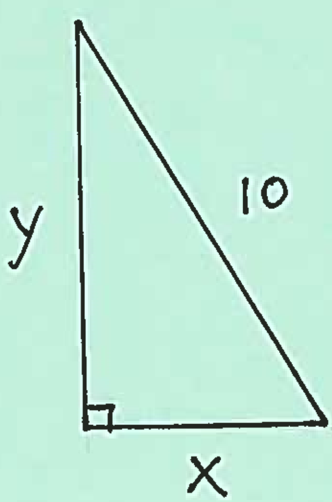
$\xrightarrow{D}$ $\frac{dA}{dt} = x \cdot \frac{dy}{dt} + \frac{dx}{dt} \cdot y$

$= (3)(-4) + (5)(2)$

$= -2$ cm.²/sec.

<u>Example</u>: If the bottom of a 10-ft. ladder is pushed toward the wall at the rate of 2 ft./sec., how fast is the top of the ladder moving up the wall when the bottom of the ladder is 6 ft. from the wall?





<u>Given</u>: $\frac{dx}{dt} = -2$ ft./sec.

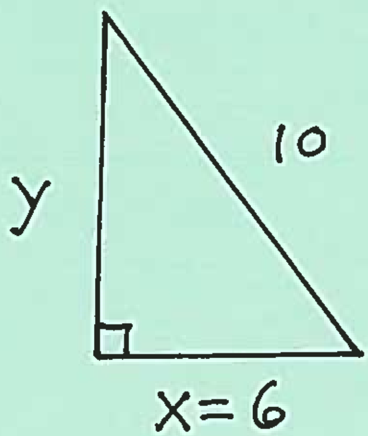
Find $\frac{dy}{dt}$ when $x = 6$ ft. ;

What equation should we start with ?

Pythagoren Theorem : $x^2 + y^2 = 10^2 \xrightarrow{D}$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \longrightarrow$$

$$* \quad \boxed{x\frac{dx}{dt} + y\frac{dy}{dt} = 0}$$; what is y ?



$6^2 + y^2 = 10^2 \longrightarrow y^2 = 64$

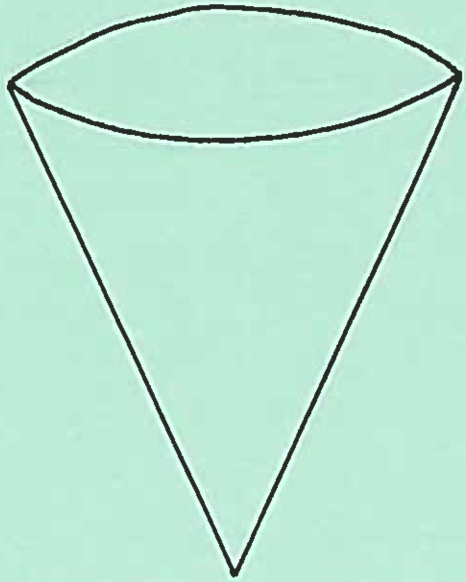
$\longrightarrow y = 8$ ft. ;

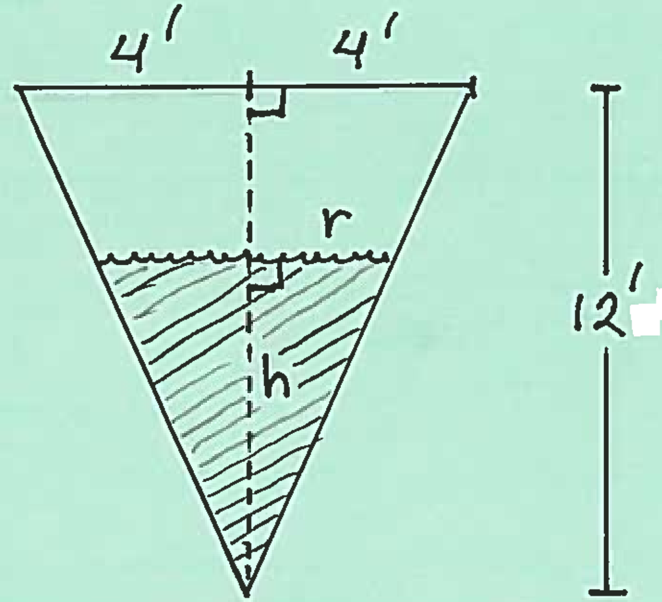
now sub #'s into equation * :

$$(6)(-2) + (8)\frac{dy}{dt} = 0 \longrightarrow$$

$$8\frac{dy}{dt} = 12 \longrightarrow \frac{dy}{dt} = \frac{12}{8} = \frac{3}{2} \text{ ft/sec.}$$

(*) <u>Example</u>: a tank is in the shape of a right circular cone of height 12 ft. and with circular diameter 8 ft.





assume that water fills the tank in such a way that the depth of water h increases at the rate of $\frac{1}{2}$ ft./min. at what rate does the volume V change when the depth of water is h = 10 ft.?

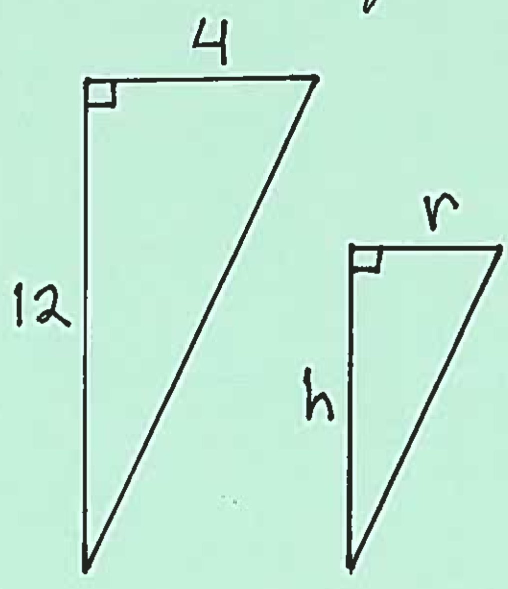
Assume that the Volume of a cone of base radius r and height h is.

$$\boxed{V = \frac{1}{3}\pi r^2 h}\ .$$

<u>Given</u>: $\frac{dh}{dt} = \frac{1}{2}$ ft./min.

<u>Find</u>: $\frac{dV}{dt}$ when $h = 10$ ft.

<u>NOTE</u>: No information is given about r. What should we do about that? Let's use Similar Triangles:



$$\frac{r}{h} = \frac{4}{12} = \frac{1}{3} \rightarrow$$

$$\boxed{r = \frac{1}{3}h}$$; now

rewrite the Volume formula and take its derivative:

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{1}{3}h\right)^2 h \longrightarrow$$

$$\boxed{V = \frac{\pi}{27}h^3} \xrightarrow{D} \frac{dV}{dt} = \frac{\pi}{27}\cdot 3h^2\frac{dh}{dt} \longrightarrow$$

$$\frac{dV}{dt} = \frac{\pi}{9}(10)^2\cdot\left(\frac{1}{2}\right) = \frac{50}{9}\pi \ \frac{ft.^3}{min.}$$