Using a Differential to Estimate the Propagated Absolute Percentage Error

SEE the definitions for absolute percentage errors and the worked out example on the next page.

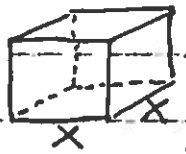
Math 16A
Kouba

Finding % Errors Using a Differential

Let $y = f(x)$ be a differentiable function; and Δx: change (error) in x, Δy: change (error) in f; then

$\frac{|\Delta x|}{x}$: absolute relative (%) error in x,

$\frac{|\Delta y|}{y}$: absolute relative (%) error in f;

Example: 

The edge of a cube is measured with possible % error of at most 2%. Use a differential to estimate the maximum % error in computing the cube's volume:

$V = x^3 \xrightarrow{D} V' = 3x^2$, and $\frac{|\Delta x|}{x} \leq 2\%$; estimate the value of $\frac{|\Delta V|}{V}$. Then

$$\frac{|\Delta V|}{V} \approx \frac{|dV|}{V} = \frac{|V' \cdot \Delta x|}{V} = \frac{|3x^2 \cdot \Delta x|}{x^3}$$

$$= \frac{3x^2 \cdot |\Delta x|}{x^3} = 3 \cdot \frac{|\Delta x|}{x} \leq 3 \cdot (2\%)$$

$$= 6\%, \text{ i.e., } \frac{|\Delta V|}{V} \leq 6\%.$$

Example: The radius of a sphere is measured with an absolute % error of at most 5%. Use a Differential to estimate the maximum absolute % error in computing the sphere's Surface area.

The Surface Area of a sphere of radius r is given by

$$S = 4\pi r^2.$$

Assume that

$$\frac{|\Delta r|}{r} \leq 5\%.$$

Estimate the value of

$$\frac{|\Delta S|}{S}$$

using a Differential. Then

$$|\Delta S| \approx |dS| \,,$$

so that

$$\frac{|\Delta S|}{S} \approx \frac{|dS|}{S}$$

$$= \frac{|S' \cdot \Delta r|}{S}$$

$$= \frac{|8\pi r \cdot \Delta r|}{4\pi r^2}$$

$$= 2 \cdot \frac{|\Delta r|}{r}$$

$$\leq 2 \cdot (5\%)$$

$$= 10\% \,, \quad \text{i.e.,}$$

$$\frac{|\Delta S|}{S} \leq 10\% \,.$$