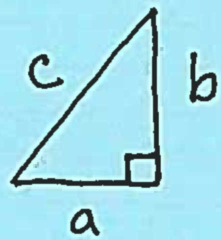
Math 16A
Sections 1.1, 1.2

Pre-Calculus Review

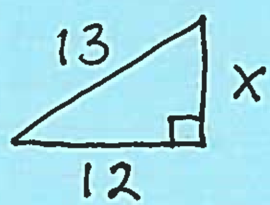
Pythagorean Theorem (for right Δ's):



$$a^2 + b^2 = c^2$$

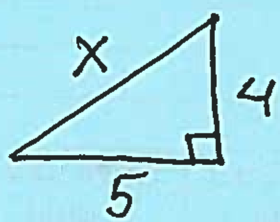
Example: Solve for x.

1.)



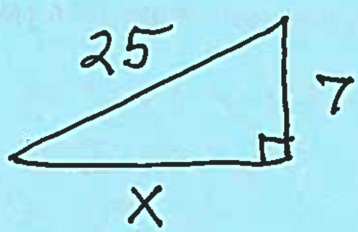
$\rightarrow x^2 + 12^2 = 13^2$
$\rightarrow x^2 + 144 = 169$
$\rightarrow x^2 = 25 \rightarrow x = 5$

2.)



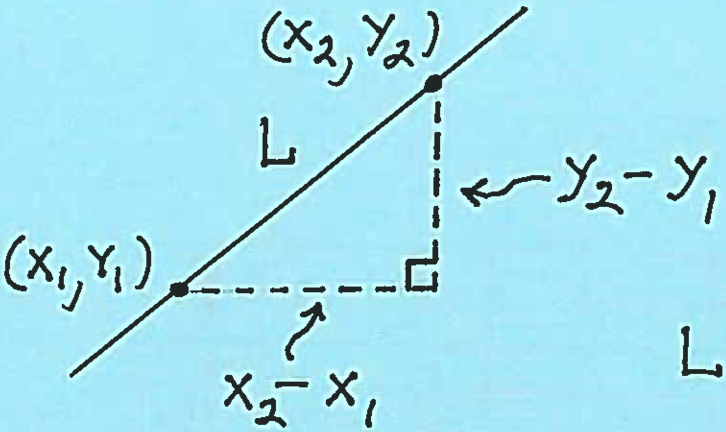
$\rightarrow x^2 = 4^2 + 5^2$
$\rightarrow x^2 = 16 + 25 = 41$
$\rightarrow x = \sqrt{41}$

3.)



$\rightarrow x^2 + 7^2 = 25^2$
$\rightarrow x^2 + 49 = 625$
$\rightarrow x^2 = 576 \rightarrow x = 24$

<u>Rule</u>: The <u>distance</u> L between points (x_1, y_1) and (x_2, y_2) is



$$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

(This follows immediately from the Pythagorean Theorem.)

<u>Example</u>: Find the distance L between the given points.

1.) $(1, -1)$ and $(2, -\frac{1}{2})$:

$$L = \sqrt{(2-1)^2 + \left(-\tfrac{1}{2} - (-1)\right)^2}$$

$$= \sqrt{1^2 + \left(\tfrac{1}{2}\right)^2}$$

$$= \sqrt{\tfrac{4}{4} + \tfrac{1}{4}}$$

$$= \sqrt{\tfrac{5}{4}}$$

$$= \frac{\sqrt{5}}{\sqrt{4}} = \frac{\sqrt{5}}{2}$$

2.) $(4,-1)$ and $(1,3)$:

$$L = \sqrt{(4-1)^2 + (-1-3)^2}$$

$$= \sqrt{3^2 + 4^2}$$

$$= \sqrt{25} = 5$$

3.) $\left(\frac{1}{3}, 0\right)$ and $\left(-1, \frac{1}{2}\right)$:

$$L = \sqrt{\left(-1-\frac{1}{3}\right)^2 + \left(\frac{1}{2}-0\right)^2}$$

$$= \sqrt{\left(-\frac{4}{3}\right)^2 + \left(\frac{1}{2}\right)^2}$$

$$= \sqrt{\frac{16}{9} + \frac{1}{4}}$$

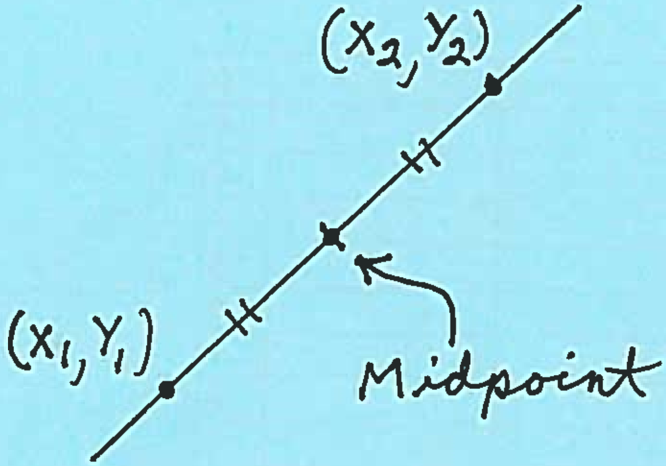
$$= \sqrt{\frac{16}{9} \cdot \frac{4}{4} + \frac{1}{4} \cdot \frac{9}{9}}$$

$$= \sqrt{\frac{64}{36} + \frac{9}{36}}$$

$$= \sqrt{\frac{73}{36}}$$

$$= \frac{\sqrt{73}}{\sqrt{36}} = \frac{\sqrt{73}}{6}$$

Rule: The midpoint of the line segment joining points (x_1, y_1) and (x_2, y_2) is



$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

(This can be verified using the distance formula and slopes.)

Example: Find the midpoint of the line segments joining the given points.

1.) $(1, -2)$ and $(5, 0)$:

midpt. is : $\left(\frac{1+5}{2}, \frac{-2+0}{2}\right) = (3, -1)$

2.) $(4, -3)$ and $(-20, \frac{1}{2})$:

midpt. is: $\left(\frac{4+{}^{-}20}{2}, \frac{-3+\frac{1}{2}}{2}\right) = \left(\frac{-16}{2}, \frac{\frac{-6}{2}+\frac{1}{2}}{\frac{2}{1}}\right)$

$$= \left(-8, \frac{-5}{2}\cdot\frac{1}{2}\right) = \left(-8, -5/4\right)$$

3.) $\left(\frac{2}{3}, \frac{-1}{2}\right)$ and $\left(\frac{3}{4}, \frac{4}{5}\right)$:

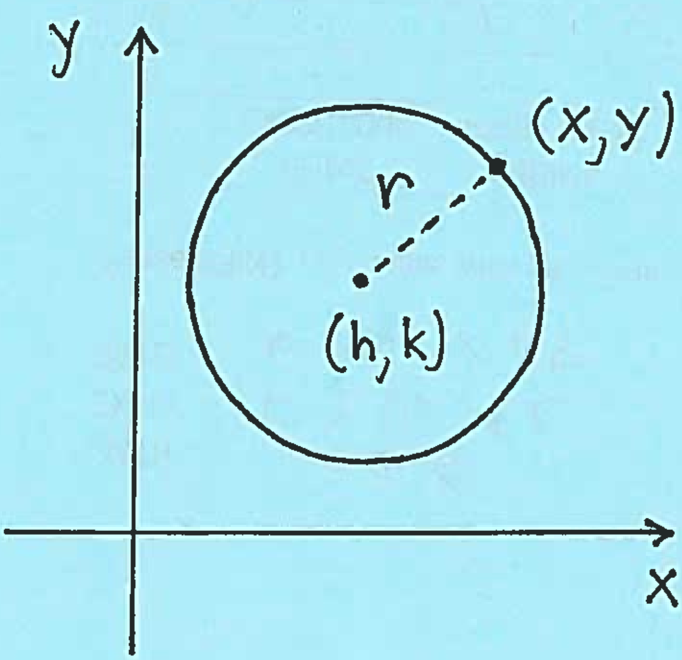
midpt. is: $\left(\dfrac{\frac{2}{3}+\frac{3}{4}}{2}, \dfrac{\frac{-1}{2}+\frac{4}{5}}{2}\right)$

$$= \left(\frac{\frac{2}{3}\cdot\frac{4}{4}+\frac{3}{4}\cdot\frac{3}{3}}{\frac{2}{1}}, \frac{\frac{-1}{2}\cdot\frac{5}{5}+\frac{4}{5}\cdot\frac{2}{2}}{\frac{2}{1}}\right)$$

$$= \left(\left(\frac{8}{12}+\frac{9}{12}\right)\cdot\frac{1}{2}, \left(\frac{-5}{10}+\frac{8}{10}\right)\cdot\frac{1}{2}\right)$$

$$= \left(\frac{17}{12}\cdot\frac{1}{2}, \frac{3}{10}\cdot\frac{1}{2}\right) = \left(\frac{17}{24}, \frac{3}{20}\right)$$

<u>Circles</u>:



<u>Rule</u>: The equation of the <u>circle</u> with center (h, k) and radius r is

$$(x-h)^2 + (y-k)^2 = r^2$$

(This follows immediately from the distance formula:

$$r = \sqrt{(x-h)^2 + (y-k)^2}$$
)

<u>Example</u>: Find the center and radius of each circle.

1.) $x^2 + y^2 = 4 \longrightarrow (x-0)^2 + (y-0)^2 = 2^2$

so center: $(0,0)$, radius $r = 2$

2.) $(x-1)^2 + (y+3)^2 = 100 \longrightarrow$

$(x-1)^2 + (y-(-3))^2 = 10^2$, so

center: $(1,-3)$, radius $r = 10$

3.) $x^2 + y^2 = 4y \longrightarrow x^2 + y^2 - 4y = 0$

$\longrightarrow x^2 + \left(y^2 - 4y + \left(\frac{-4}{2}\right)^2\right) = \left(\frac{-4}{2}\right)^2$

(complete the square ☺)

$\longrightarrow x^2 + (y^2 - 4y + 4) = 4$

$\longrightarrow x^2 + (y-2)^2 = 2^2$, so center : $(0, 2)$, radius $r = 2$

4.) $4x^2 + 4y^2 - 8x + 2y = 2 \longrightarrow$

$(4x^2 - 8x) + (4y^2 + 2y) = 2 \longrightarrow$

$4(x^2 - 2x \quad) + 4(y^2 + \frac{1}{2}y \quad) = 2 \rightarrow$

(complete the square twice)

$$4(x^2 - 2x + \underline{\underline{1}}) + 4(y^2 + \tfrac{1}{2}y + \tfrac{1}{16})$$
$$= 2 + 4(1) + 4\left(\tfrac{1}{16}\right)$$
$$= 6 + \tfrac{1}{4}$$
$$= \frac{24}{4} + \frac{1}{4} = \frac{25}{4} \longrightarrow$$

$$4(x-1)^2 + 4\left(y + \tfrac{1}{4}\right)^2 = \frac{25}{4} \longrightarrow$$

$$(x-1)^2 + \left(y + \tfrac{1}{4}\right)^2 = \frac{25}{4} \cdot \frac{1}{4} = \frac{25}{16} \rightarrow$$

$$(x-1)^2 + \left(y - \left(-\tfrac{1}{4}\right)\right)^2 = \left(\frac{5}{4}\right)^2, \text{ so}$$

center : $\left(1, -\frac{1}{4}\right)$, radius $r = \frac{5}{4}$

Example: Find an equation of the circle passing through the point $(-3,2)$ and having center $(1,-1)$.

circle: $(x-h)^2+(y-k)^2=r^2$ and

$$(x,y)=(-3,2),\ (h,k)=(1,-1) \rightarrow$$

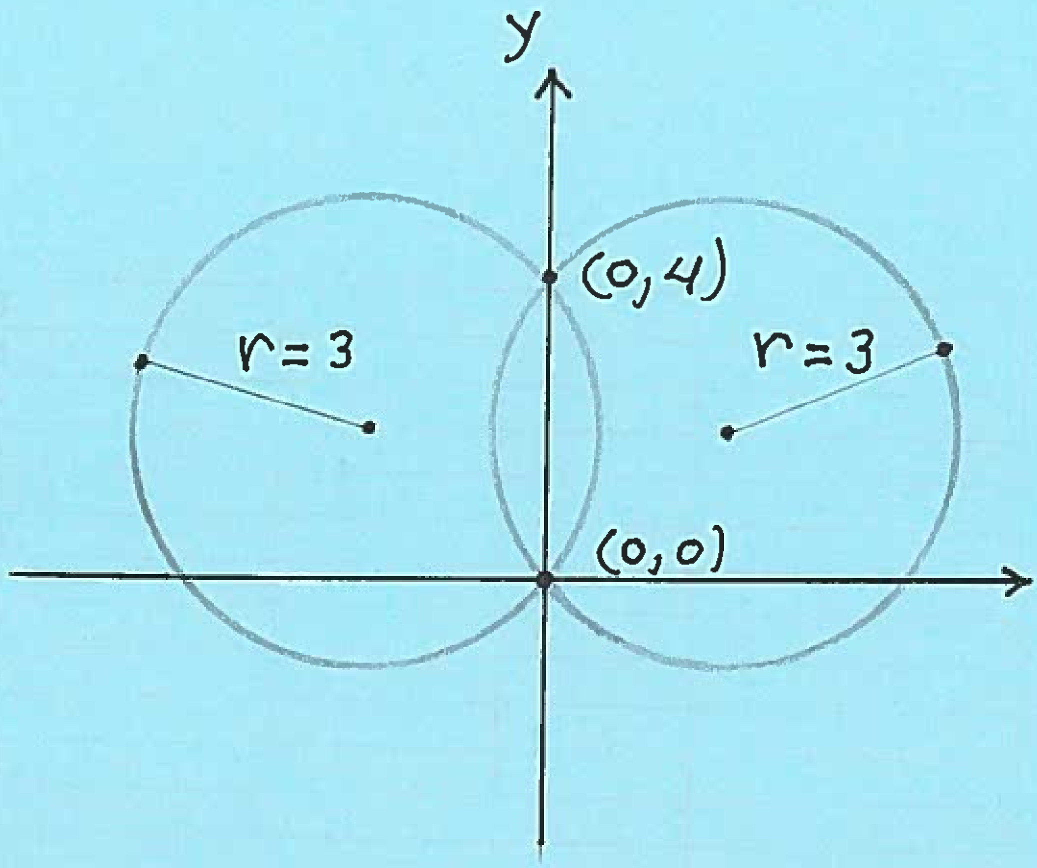
$$(-3-1)^2+(2-(-1))^2=r^2 \longrightarrow$$

$$16+9=25=r^2, \text{ so } r=5$$

and circle is

$$(x-1)^2+(y+1)^2=5^2$$

Example: (challenging) Find equations of all possible circles which pass through the points $(0,4)$ and $(0,0)$ and have radius $r=3$.



There appear to be 2 possible circles. ~~Let's~~ find them. Begin with

$$(x-h)^2+(y-k)^2 = r^2 \quad , \text{ and } \boxed{r=3}$$

$$\rightarrow (x-h)^2+(y-k)^2 = 3^2 = 9 \ ;$$

pt. $(x,y) = (0,0)$:

$$(0-h)^2+(0-k)^2 = 9 \rightarrow$$

$$\boxed{h^2+k^2=9}$$

pt. $(x, y) = (0, 4)$:

$$(0 - h)^2 + (4 - k)^2 = 9 \longrightarrow$$

$$\boxed{h^2 + (4-k)^2 = 9}$$

Let's solve two equations and two unknowns : Then

$$h^2 = 9 - k^2 \longrightarrow (\text{SUB}) \longrightarrow$$

$$(9 - k^2) + (4 - k)^2 = 9 \longrightarrow$$

$$\cancel{9} - \cancel{k^2} + 16 - 8k + \cancel{k^2} = \cancel{9} \longrightarrow$$

$$16 - 8k = 0 \rightarrow 16 = 8k \longrightarrow \boxed{k = 2} ;$$

then $h^2 = 9 - (2)^2 = 5 \longrightarrow$

$$\boxed{h = \pm\sqrt{5}} ;$$

circle 1 : $(x - \sqrt{5})^2 + (y - 2)^2 = 3^2 = 9$

circle 2 : $(x + \sqrt{5})^2 + (y - 2)^2 = 3^2 = 9$

Recall: Given a circle of radius r:

1.) Area: $A = \pi r^2$

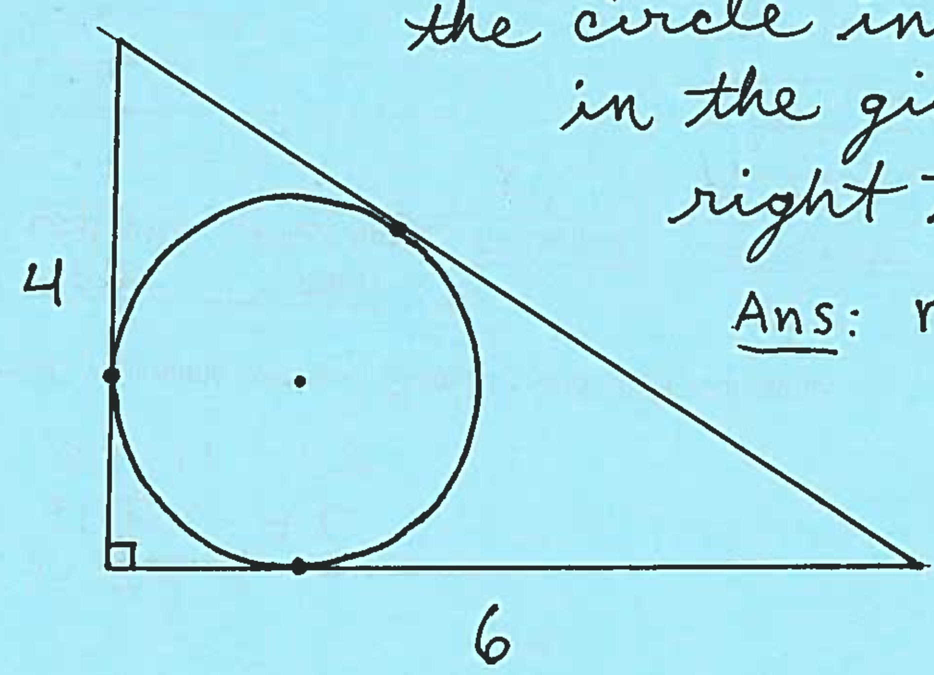
2.) Circumference: $C = 2\pi r$

Recall: (Quadratic Formula)

If $ax^2 + bx + c = 0$, then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example: (Optional Practice, Challenging) Find the Area and Circumference of the circle inscribed in the given right triangle.



Ans: $r = 5 - \sqrt{13}$

Some Basic Facts about Numbers

I.) If $A \cdot B = 0$, then $A = 0$ or $B = 0$.

II.) If $\dfrac{A}{B} = 0$, then $A = 0$ and $B \neq 0$.

III.) $|Z| = \begin{cases} Z, & \text{if } Z \geq 0 \\ -Z, & \text{if } Z < 0 \end{cases}$

IV.) $\sqrt{Z^2} = |Z|$

Sign Charts

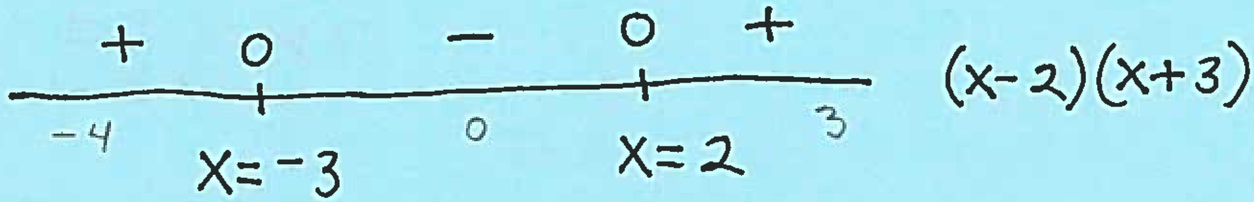
Example: Solve for X.

1.) $X^2 + X - 6 = 0 \longrightarrow$

$(X-2)(X+3) = 0 \longrightarrow X = 2,\ X = -3$

2.) $X^2 + X > 6 \longrightarrow X^2 + X - 6 > 0 \longrightarrow$

$(X-2)(X+3) > 0$



Answer: $X < -3,\ X > 2$

3.) $x(x+1) \le 3x+8 \longrightarrow$

$x^2 + x \le 3x + 8 \longrightarrow$

$x^2 - 2x - 8 \le 0 \longrightarrow$

$(x-4)(x+2) \le 0 \longrightarrow$

$+ \quad 0 \quad - \quad 0 \quad +$ $\qquad (x-4)(x+2)$

$-3 \quad x=-2 \quad 0 \quad x=4 \quad 6$

Answer: $-2 \le x \le 4$

4.) $\dfrac{x^2-4}{x} < 0 \longrightarrow \dfrac{(x-2)(x+2)}{x} < 0$

NO

$- \quad 0 \quad + \quad | \quad - \quad 0 \quad +$ $\qquad \dfrac{(x-2)(x+2)}{x}$

$-3 \quad x=-2 \quad -1 \quad x=0 \quad 1 \quad x=2 \quad 3$

Answer: $x < -2, \; 0 < x < 2$

Polynomial Division

Example: Rewrite each expression using polynomial division.

1.) $\dfrac{x^3 - 2x}{x+1}$

$$
\begin{array}{r|l}
 & x^2 - x - 1 \\
\hline
x+1 & x^3 - 2x \\
 & -(x^3 + x^2) \\
\cline{2-2}
 & -x^2 - 2x \\
 & -(-x^2 - x) \\
\cline{2-2}
 & -x \\
 & -(-x - 1) \\
\cline{2-2}
 & +1
\end{array}
$$

$$= x^2 - x - 1 + \frac{1}{x+1}$$

2.) $\dfrac{3x^3 - x^2 + 2x + 5}{x-1}$

$$= 3x^2 + 2x + 4 + \frac{9}{x-1} \quad)$$

$$
\begin{array}{r|l}
 & \quad 3x^2 + 2x + 4 \\
\hline
x-1 & \quad 3x^3 - x^2 + 2x + 5 \\
 & -(3x^3 - 3x^2) \\
\cline{2-2}
 & \qquad\quad 2x^2 + 2x + 5 \\
 & \qquad -(2x^2 - 2x) \\
\cline{2-2}
 & \qquad\qquad\quad 4x + 5 \\
 & \qquad\qquad -(4x - 4) \\
\cline{2-2}
 & \qquad\qquad\qquad\quad 9
\end{array}
$$