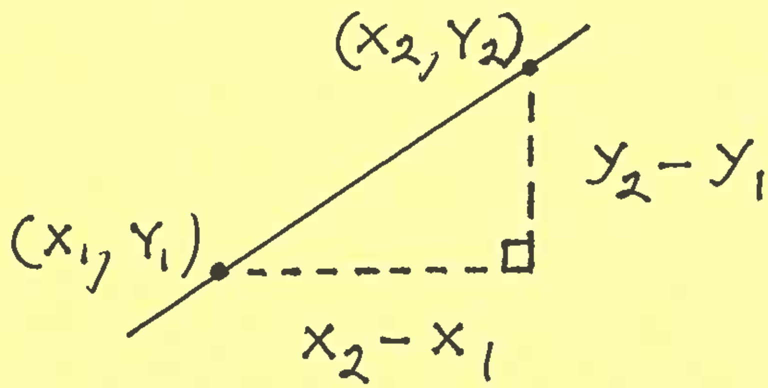
Math 16A
Sections 1.3, 1.4

Pre-Calculus Review (cont'd)

Lines

Definition: The *slope* of the line passing through points (x_1, y_1) and (x_2, y_2) is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x} = \frac{\text{rise}}{\text{run}}$$



Example: Find the slope of the line passing through the given points.

1.) $(1, -1/3)$ and $(\frac{1}{2}, -2)$ $\longrightarrow$

$$m = \frac{-2-(-1/3)}{1/2 - 1} = \frac{\frac{-6}{3}+\frac{1}{3}}{\frac{1}{2}-\frac{2}{2}}$$

$$= \frac{\frac{-5}{3}}{\frac{-1}{2}} = \frac{5}{3}\cdot\frac{2}{1} = \frac{10}{3}$$

2.) $(3,-1)$ and $(-2,7)$ (Opt. Prac.)

3.) $\left(\frac{-1}{2},\frac{3}{4}\right)$ and $\left(\frac{2}{3},-1\right)$ (Opt. Prac.)

<u>Slope-Intercept Form</u>: Line with slope m and y-intercept b is

$$\boxed{y = mx + b}$$

<u>Point-Slope Form</u>: Line with slope m and passing through the point (x_1, y_1) is

$$\boxed{y - y_1 = m(x - x_1)}$$

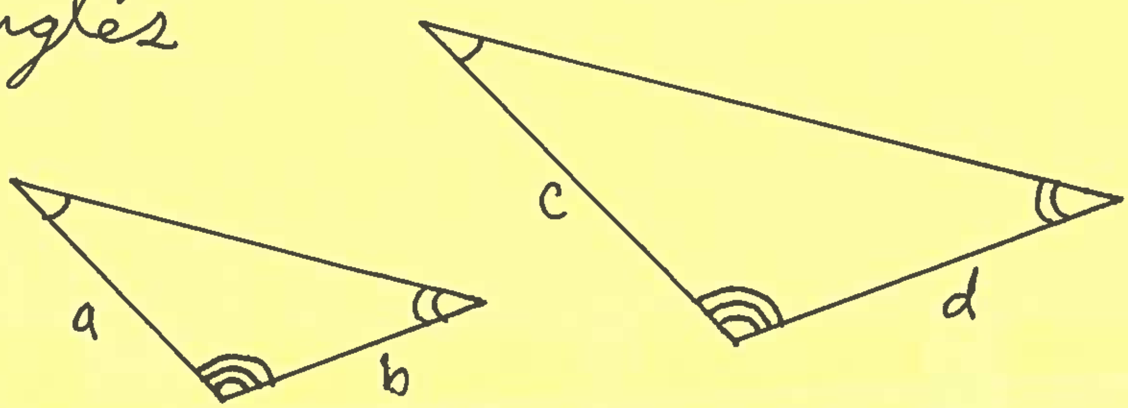
<u>Parallel Lines</u> : Slopes are equal

$$\left.\begin{array}{l} y = 3x + 2 \\ y = 3x - 7 \end{array}\right\} \text{parallel lines}$$

<u>Perpendicular Lines</u> : Slopes are negative reciprocals

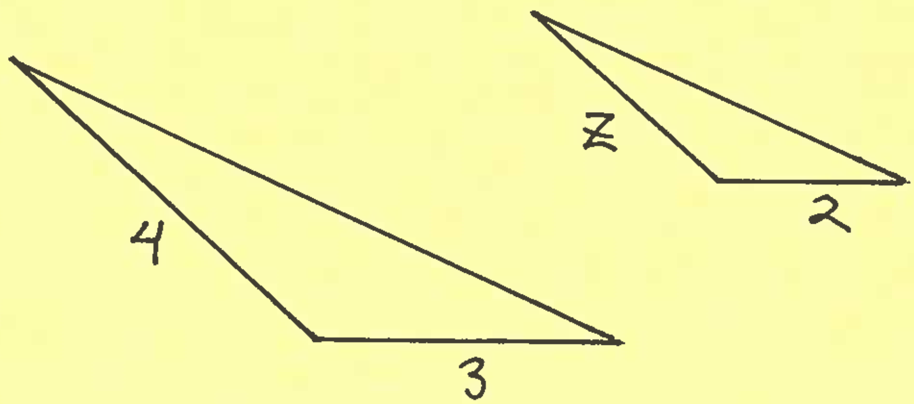
$$\left.\begin{array}{l} y = \frac{2}{3}x + 4 \\ y = \frac{-3}{2}x - 5 \end{array}\right\} \text{perpendicular lines}$$

<u>Similar Triangles</u> : Two triangles with the same angles



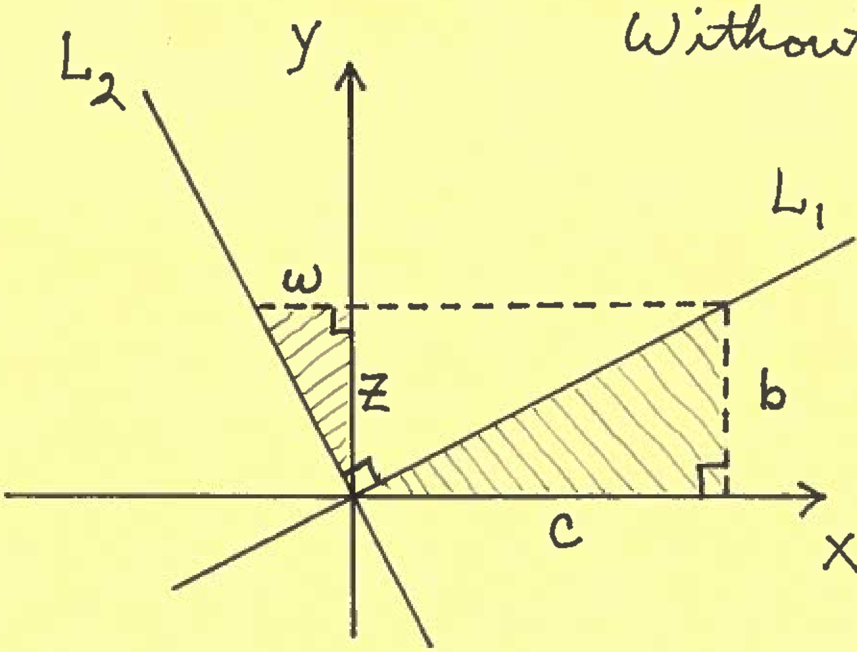
<u>Fact</u>: $\frac{a}{b} = \frac{c}{d}$

Example: Solve for z where the triangles are similar.



$$\frac{4}{3} = \frac{z}{2} \longrightarrow z = 2\left(\frac{4}{3}\right) = \frac{8}{3}$$

Question: How can we prove that the slopes of perpendicular lines are negative reciprocals?



Without loss of generality, assume that lines L_1 and L_2 are perpendicular and intersect at the origin. Assume that the slope of L_1 is $\boxed{\frac{b}{c}}$ and the slope of L_2 is $\boxed{-\frac{z}{w}}$. It is easy to show that the two shaded right triangles are <u>similar</u>, so that

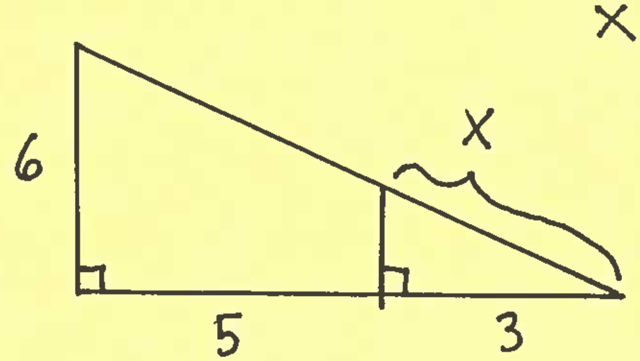
$$\frac{w}{z} = \frac{b}{c} \longrightarrow w = \frac{b}{c} z \quad ; \text{ then}$$

the slope of L_2 is

$$m = \frac{-z}{w} = \frac{-\cancel{z}}{\frac{b}{c}\cancel{z}} = \boxed{\frac{-c}{b}} \text{ , i.e.,}$$

the slopes are negative reciprocals.

<u>Example</u>: (Opt. Prac.) Solve for x in the following diagram.



<u>ANS</u>: $x = \frac{15}{4}$

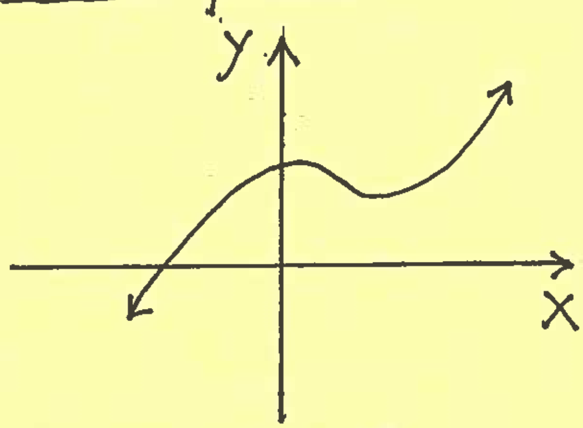
Functions

__Definition__: In an equation with variables x and y, variable y is a __function__ of x if

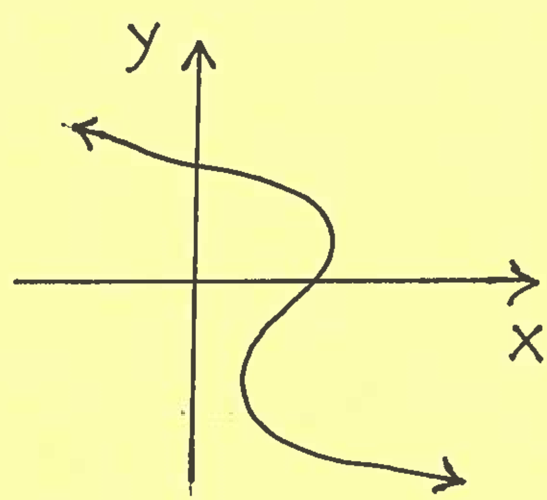
1.) each x-value has exactly one y-value

2.) the graph of the equation passes the vertical line test.

__Example__:



IS a function



NOT a function

__Example__: Show that the following equation represents y as a function of x, or conclude that y is not a function of x.

1.) $xy = \frac{x-y}{x+3} \longrightarrow$

$xy(x+3) = x - y \longrightarrow$

$x^2y + 3xy = x - y \longrightarrow$

$x^2y + 3xy + y = x \longrightarrow$

$y(x^2+3x+1) = x \longrightarrow$

$y = \frac{x}{x^2+3x+1}$, so clearly each x-value has exactly one y-value, and y IS a function of x.

2.) $xy = x^2 - y^2 + 9 \longrightarrow$

$y^2 + xy = x^2 + 9$; <u>NOTE</u>: $x=0$

$\longrightarrow y^2 + (0)y = (0)^2 + 9$

$\longrightarrow y^2 = 9 \longrightarrow y = 3, y = -3,$

so $x=0$ has TWO y-values,

and y is NOT a function of x.

<u>Function Notation</u>: In the previous example 1.)

$$y = \frac{x}{x^2+3x+1}$$ is a function of x,

so write $$f(x) = \frac{x}{x^2+3x+1};$$

then for example,

$$f(1) = \frac{(1)}{(1)^2+3(1)+1} = \frac{1}{5};$$

$$f(-2) = \frac{(-2)}{(-2)^2+3(-2)+1} = \frac{-2}{-1} = 2;$$

$$f\left(\frac{1}{2}\right) = \frac{\left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)^2+3\left(\frac{1}{2}\right)+1} = \frac{\frac{1}{2}}{\frac{1}{4}+\frac{6}{4}+\frac{4}{4}}$$

$$= \frac{\frac{1}{2}}{\frac{11}{4}} = \frac{1}{2}\cdot\frac{4}{11} = \frac{2}{11}$$

<u>Definition</u>: Function $y = f(x)$ is one-to-one if

1.) each y-value has a unique (exactly one) x-value.

2.) the graph of f passes the horizontal line test.

<u>algebraic Definition of one-to-one:</u>

Function $y = f(x)$ is 1-1:

if $f(x_1) = f(x_2)$, then $x_1 = x_2$.

<u>FACT</u>: If function $y = f(x)$ is 1-1, then f has an <u>inverse function</u>, f^{-1}.

<u>Example</u>: Use the algebraic definition of 1-1 to show that the following function is 1-1. Then find its inverse function:

$$f(x) = \frac{x-3}{2x+5}$$

$$f(x_1) = f(x_2) \rightarrow \frac{x_1 - 3}{2x_1 + 5} = \frac{x_2 - 3}{2x_2 + 5}$$

$$\rightarrow (x_1 - 3)(2x_2 + 5) = (x_2 - 3)(2x_1 + 5)$$

$$\rightarrow \cancel{2x_1x_2} + 5x_1 - 6x_2 - \cancel{15}$$

$$= \cancel{2x_1x_2} + 5x_2 - 6x_1 - \cancel{15}$$

$$\rightarrow 5x_1 + 6x_1 = 5x_2 + 6x_2$$

$$\rightarrow 11x_1 = 11x_2$$

$\rightarrow \quad x_1 = x_2$, i.e., f is 1-1

Find inverse function:

$$y = \frac{x - 3}{2x + 5}$$ (switch x and y)

$$\rightarrow x = \frac{y - 3}{2y + 5}$$ (now solve for y)

$$\rightarrow x(2y + 5) = y - 3$$

$$\rightarrow 2xy + 5x = y - 3$$

$$\rightarrow 2xy - y = -5x - 3$$

$\rightarrow \quad y(2x-1) = -5x-3$

$$\rightarrow \quad y = \frac{-5x-3}{2x-1} = f^{-1}(x) .$$

Functional Composition

If $y=f(x)$ and $y=g(x)$ are functions, then

$$(f \circ g)(x) = f(g(x))$$

and $(g \circ f)(x) = g(f(x))$

Example: Find and simplify $f \circ g$ and $g \circ f$ for

$$f(x) = \frac{x+1}{x-1} \text{ and } g(x) = \frac{2-x}{x+3} .$$

$$(f \circ g)(x) = f(g(x))$$

$$= f\left(\frac{2-x}{x+3}\right)$$

$$= \frac{\left(\frac{2-x}{x+3}\right)+1}{\left(\frac{2-x}{x+3}\right)-1}$$

$$= \frac{\frac{2-x}{x+3}+1}{\frac{2-x}{x+3}-1} \cdot \frac{x+3}{x+3}$$

$$= \frac{(2-x)+(x+3)}{(2-x)-(x+3)} = \frac{5}{5-2x} \ ;$$

$$(g \circ f)(x) = g\left(f(x)\right)$$

$$= g\left(\frac{x+1}{x-1}\right)$$

$$= \frac{2-\left(\frac{x+1}{x-1}\right)}{\left(\frac{x+1}{x-1}\right)+3}$$

$$= \frac{2-\frac{x+1}{x-1}}{\frac{x+1}{x-1}+3} \cdot \frac{x-1}{x-1}$$

$$= \frac{2(x-1)-(x+1)}{(x+1)+3(x-1)}$$

$$= \frac{2x-2-x-1}{x+1+3x-3} = \frac{x-3}{4x-2}.$$

Domain and Range of a Function

Definition: Let $y=f(x)$ be a function.

I.) The Domain of f is the set of admissable x-values.

II.) The Range of f is the set of corresponding y-values.

Example: Find the Domain and Range for each function.

1.) $f(x) = 4 + 3\sin x$

Domain : all x-values

Range : We know $-1 \le \sin x \le 1$

$\rightarrow -3 \le 3\sin x \le 3$

$\rightarrow 4+{}^-3 \le 4+3\sin x \le 4+3$

$\rightarrow 1 \le \underbrace{4+3\sin x}_{y} \le 7$, so

Range is $1 \le y \le 7$.

2.) $f(x) = 2+\sqrt{3-x}$; we need $3-x \ge 0 \rightarrow x \le 3$ so

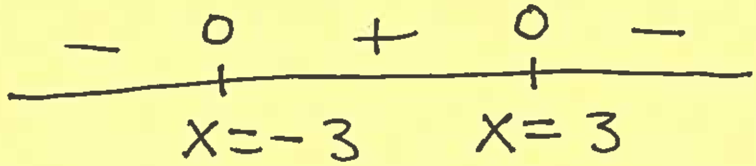
Domain : all $x \le 3$

Range : $\sqrt{3-x} = 0$ when $x = 3$ and $\sqrt{3-x} \ge 0$ for $x \le 3$. If $x \rightarrow -\infty$, then $\sqrt{3-x} \rightarrow +\infty$, so

Range is all $y \ge 2$

3.) $f(x) = \sqrt{9-x^2}$; we need

$9 - x^2 = (3-x)(3+x) \geq 0$, so



$(3-x)(3+x)$

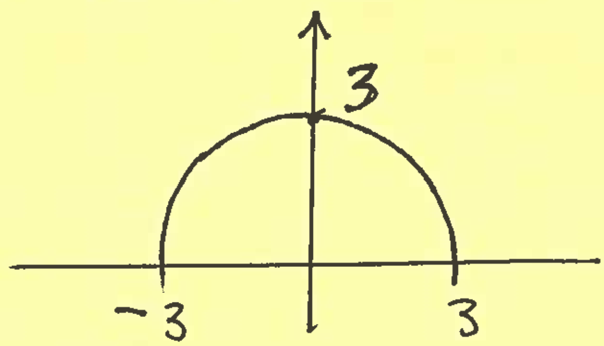
<u>Domain</u> : $-3 \leq x \leq 3$

<u>Range</u> : $y = \sqrt{9-x^2} \geq 0 \rightarrow$

$y^2 = \left(\sqrt{9-x^2}\right)^2 = 9 - x^2 \rightarrow$

$x^2 + y^2 = 9$,

so f is TOP half of circle of radius 3 :



<u>Range</u> : $0 \leq y \leq 3$