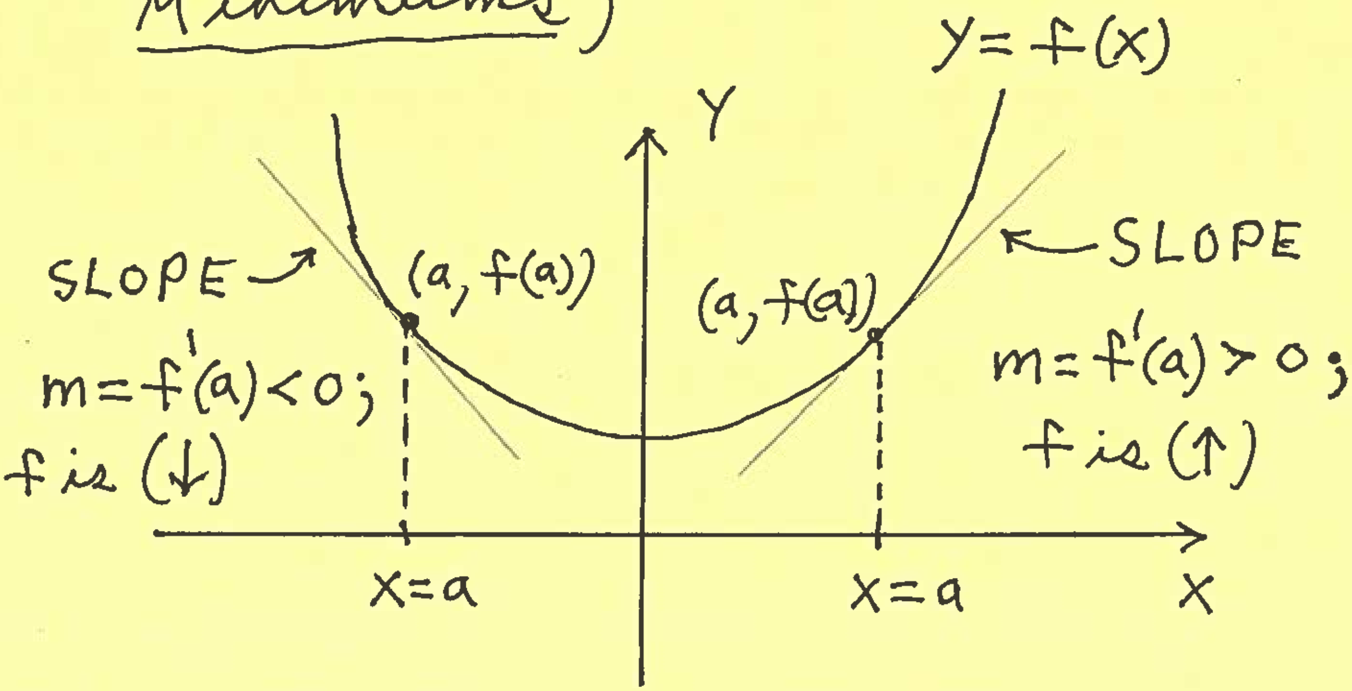
Math 16A
Sections 3.1, 3.2

Using the First Derivative to Find Relative and Absolute Extrema (Maximums and Minimums)



Recall: If $y = f(x)$, then $f'(a)$ is the SLOPE of the line tangent to the graph of $y = f(x)$ at the point $(a, f(a))$.

FACTS: I.) If $f'(a)$ is $(+)$, then the graph of $y = f(x)$ is increasing $(\uparrow)$ at $x = a$.

II.) If $f'(a)$ is $(-)$, then the graph of $y=f(x)$ is decreasing (↓) at $x=a$.

III.) Sign Chart Analysis

$+ \quad 0 \quad -$

$x=a$

$\underbrace{y=f(a)}$

Relative Maximum

$- \quad 0 \quad +$

$x=a$

$\underbrace{y=f(a)}$

Relative Minimum

IV.) I.) absolute Maximum Value: Largest y-value on the graph of $y=f(x)$

II.) absolute Minimum Value: Smallest y-value on the graph of $y=f(x)$

Example: Set up a Sign Chart for $f'(x)$. Determine all Relative and Absolute Extrema. State Open Intervals on which the graph of $y = f(x)$ is $(\uparrow)$ and $(\downarrow)$.

1.) $f(x) = 4x^2 - 8x \xrightarrow{D}$

$$f'(x) = 8x - 8 = 8(x-1) = 0$$

$-$	0	$+$	
	$x=1$		f'

$\left.\begin{array}{l} x = 1 \\ y = -4 \end{array}\right\}$ ABS. MIN.

f is $(\uparrow)$ for $x > 1$;

f is $(\downarrow)$ for $x < 1$

2.) $y = x^2(x-8)^2$ on $[-1, 10] \xrightarrow{D}$

$$y' = x^2 \cdot 2(x-8) + 2x \cdot (x-8)^2$$

$$= 2x(x-8) \cdot [x + (x-8)]$$

$$= 2x(x-8)[2x-8] = 0 \longrightarrow$$

$$x=0,\ x=8,\ x=4$$

y': $-\quad 0 \quad + \quad 0 \quad - \quad 0 \quad +$

$x=-1$	$x=0$	$x=4$	$x=8$	$x=10$
$y=81$	$y=0$	$y=256$	$y=0$	$y=400$
REL. MAX.	ABS. MIN.	REL. MAX.	ABS. MIN.	ABS. MAX.

f is ($\uparrow$) for $0 < x < 4$, $8 < x < 10$;

f is ($\downarrow$) for $-1 < x < 0$, $4 < x < 8$

Example: Set up a Sign Chart for $f'(x)$. Determine all Relative and absolute Extrema. State Open Intervals on which the graph of $y = f(x)$ is increasing ($\uparrow$) and decreasing ($\downarrow$). Sketch the graph of f indicating x- and y-intercepts.

1.) $y = x^4 - 4x^3 \xrightarrow{D}$

$y' = 4x^3 - 12x^2 = 4x^2(x-3) = 0 \rightarrow$

$x = 0, \ x = 3$

−	0	−	0	+	y'
	$x=0$		$x=3$, $y=-27$		ABS. MIN.

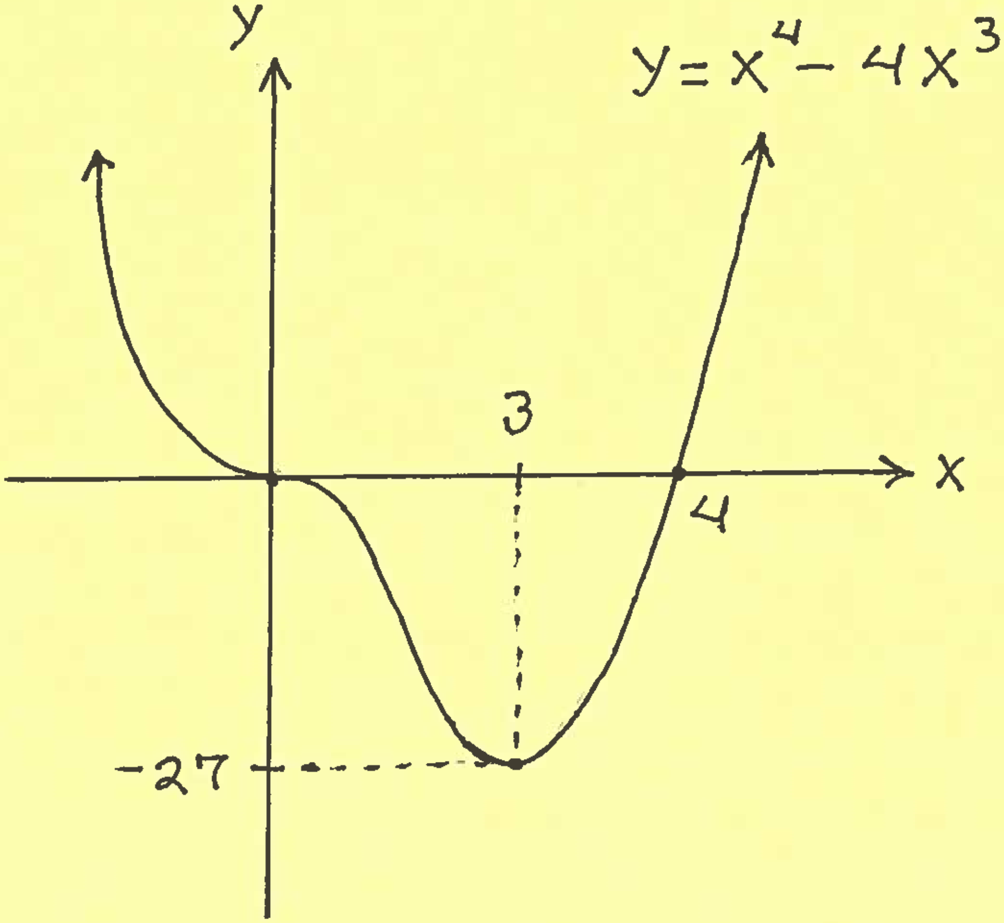
f is ($\uparrow$) for $x > 3$;

f is ($\downarrow$) for $x < 0$, $0 < x < 3$;

$x = 0$: $y = 0$

$y = 0$: $0 = x^4 - 4x^3 = x^3(x-4) = 0$

$\rightarrow x = 0, \ x = 4$



2.) $y = x - 4\sqrt{x}$

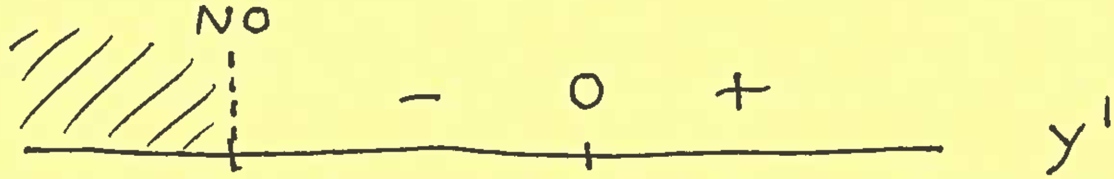
NOTE: Domain is $x \geq 0$.

$\xrightarrow{D} y = 1 - 4 \cdot \frac{1}{2} x^{-1/2} = 1 - \frac{2}{\sqrt{x}}$

$$= \frac{\sqrt{x}}{\sqrt{x}} - \frac{2}{\sqrt{x}} = \frac{\sqrt{x} - 2}{\sqrt{x}} = 0 \rightarrow$$

$\sqrt{x} - 2 = 0 \rightarrow \sqrt{x} = 2 \rightarrow x = 4$ AND

$\sqrt{x} \neq 0$, so $x \neq 0$;



REL. MAX. $\begin{cases} x=0 \\ y=0 \end{cases}$ $\quad$ $\left.\begin{matrix} x=4 \\ y=-4 \end{matrix}\right\}$ ABS. MIN.

y is $(\uparrow)$ for $x>4$,

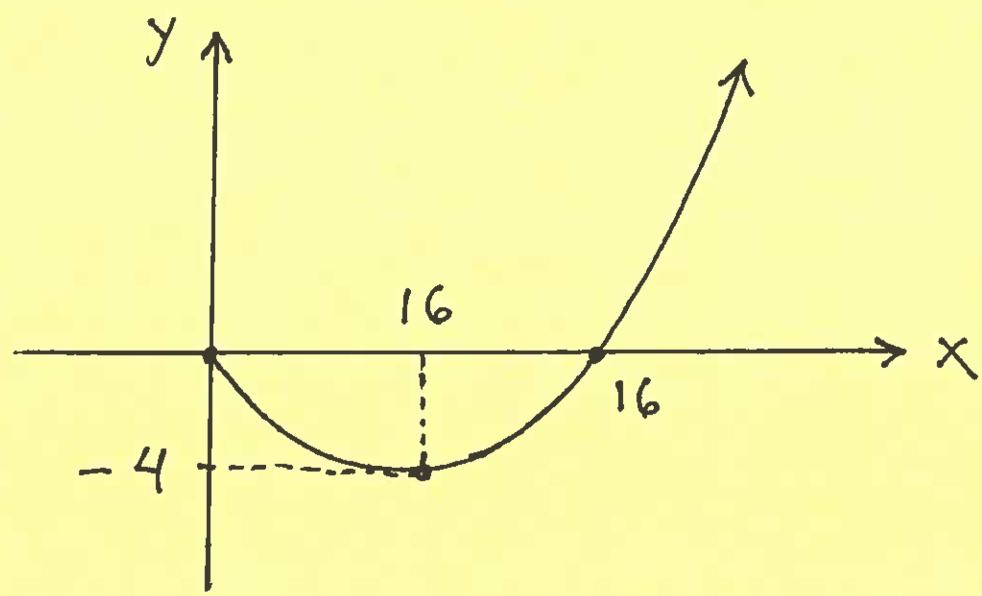
y is $(\downarrow)$ for $0<x<4$;

$x=0: \; y=0$

$y=0: \; 0 = x - 4\sqrt{x} = (\sqrt{x})^2 - 4\sqrt{x}$

$= \sqrt{x}\,(\sqrt{x}-4) \longrightarrow$

$x=0, \; x=16$



3.) $y = \dfrac{4x}{x^2+4} \xrightarrow{D}$

$$y' = \frac{(x^2+4)(4) - (4x)(2x)}{(x^2+4)^2}$$

$$= \frac{4x^2+16-8x^2}{(x^2+4)^2}$$

$$= \frac{16-4x^2}{(x^2+4)^2} = \frac{4(4-x^2)}{(x^2+4)^2} = \frac{4(2-x)(2+x)}{(x^2+4)^2} = 0;$$

$$\begin{array}{ccccc} - & 0 & + & 0 & - \end{array} \quad y'$$

$$\text{ABS. MIN.} \begin{cases} x=-2 \\ y=-1 \end{cases} \qquad \left.\begin{matrix} x=2 \\ y=1 \end{matrix}\right\} \text{ABS. MAX.}$$

y is $(\uparrow)$ for $-2<x<2$,

y is $(\downarrow)$ for $x<-2$, $x>2$;

$x=0$: $y=0$ and $y=0$: $x=0$;

<u>NOTE</u>: $\displaystyle \lim_{x\to\pm\infty} \frac{4x}{x^2+4} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{x\to\pm\infty} \frac{4x}{x^2+4} \cdot \frac{1/x^2}{1/x^2}$

$\displaystyle = \lim_{x\to\pm\infty} \frac{4/x}{1+4/x^2} = \frac{0}{1+0} = 0$, so

$\boxed{y=0}$ is an H.A.