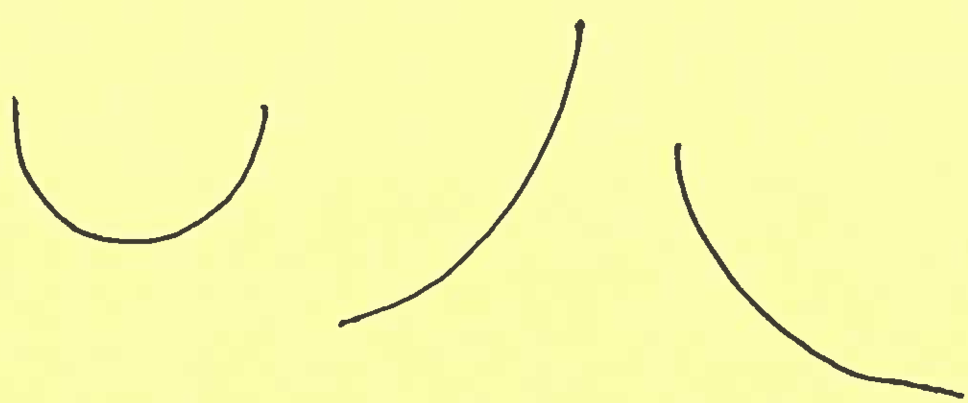
Math 16A
Sections 3.3, 3.7

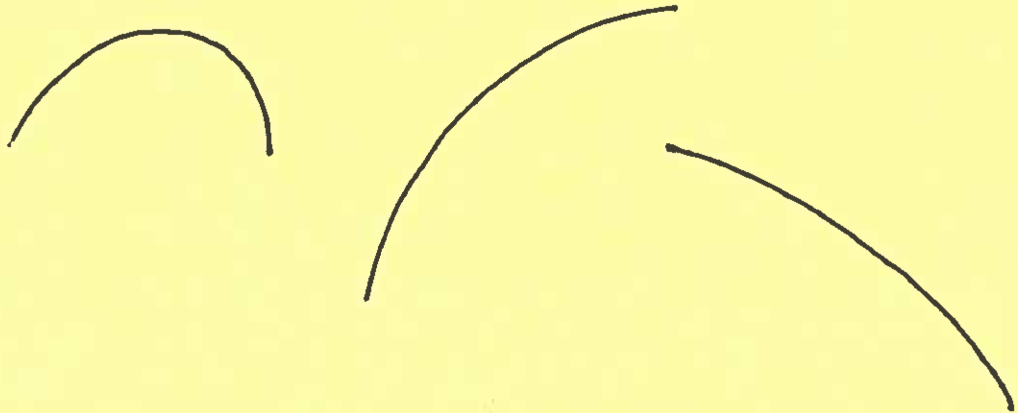
Detailed Graphing Using Both First Derivative Sign Charts and Second Derivative Sign Charts

Concavity and Inflection Points

Examples of Concave Up graphs:



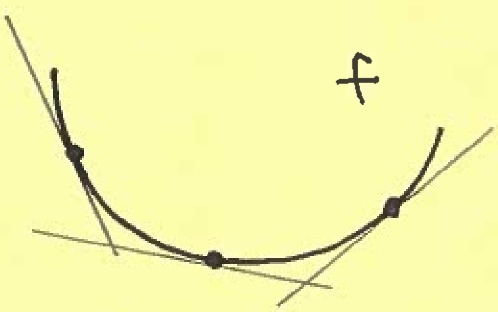
Examples of Concave Down graphs:



How can we determine if a graph is CONCAVE UP (∪) or CONCAVE DOWN (∩)?

Analysis:

I.) CONCAVE UP:

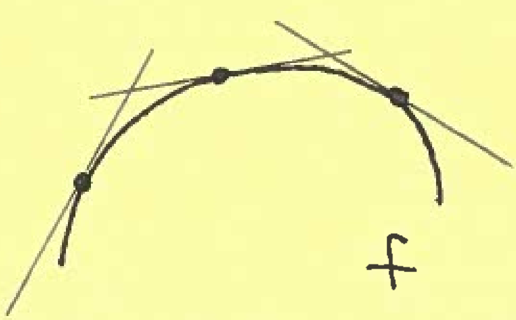


SLOPES f' are (↑), so

$D(f') = \boxed{f'' \text{ is } (+)}$

II.) CONCAVE DOWN:

SLOPES f' are (↓),



so $D(f') = \boxed{f'' \text{ is } (-)}$

III.) SECOND DERIVATIVE SIGN CHART

$$\underset{x=a}{\overset{+ \quad 0 \quad -}{\overline{\quad\quad\quad|\quad\quad\quad}}} \quad f''$$

$$\left.\begin{array}{l} x=a \\ y=f(a) \end{array}\right\} \underline{\text{Inflection Point}}$$

$$\underset{x=a}{\overset{- \quad 0 \quad +}{\overline{\quad\quad\quad|\quad\quad\quad}}} \quad f''$$

$$\left.\begin{array}{l} x=a \\ y=f(a) \end{array}\right\} \underline{\text{Inflection Point}}$$

Math 16A
Kouba
Instructions for DETAILED GRAPHING

1.) State the DOMAIN of the function.

2.) Take the FIRST derivative and set up a SIGN CHART for $f'(x)$. Clearly mark the solutions to $f'(x) = 0$ and their y-values, and identify all RELATIVE and ABSOLUTE maximum and minimum values.

3.) State the OPEN INTERVALS on which f is INCREASING and DECREASING.

4.) Take the SECOND derivative and set up a SIGN CHART for $f''(x)$. Clearly mark the solutions to $f''(x) = 0$ and their y-values, and identify all INFLECTION POINTS.

5.) State the OPEN INTERVALS on which f is CONCAVE UP and CONCAVE DOWN.

6.) Determine all x-INTERCEPTS and y-INTERCEPTS.

7.) If appropriate, determine all HORIZONTAL ASYMPTOTES (H.A.).

8.) If appropriate, determine all VERTICAL ASYMPTOTES (V.A.).

9.) DRAW a rough SKETCH of the graph of $y = f(x)$ and CLEARLY identify the coordinates of all important points on the graph.

__Example__ : Do DETAILED GRAPHING for each function.

1.) $y = 3x^2 - x^3$, Domain: all x-values

$\xrightarrow{D}$ $y' = 6x - 3x^2 = 3x(2-x) = 0$

$-\quad 0 \quad +\quad 0 \quad -\qquad y'$

REL. MIN. $\left\{ \begin{array}{l} x=0 \\ y=0 \end{array} \right.$ $\qquad \left. \begin{array}{l} x=2 \\ y=4 \end{array} \right\}$ REL. MAX.

$\xrightarrow{D}$ $y'' = 6 - 6x = 6(1-x) = 0$

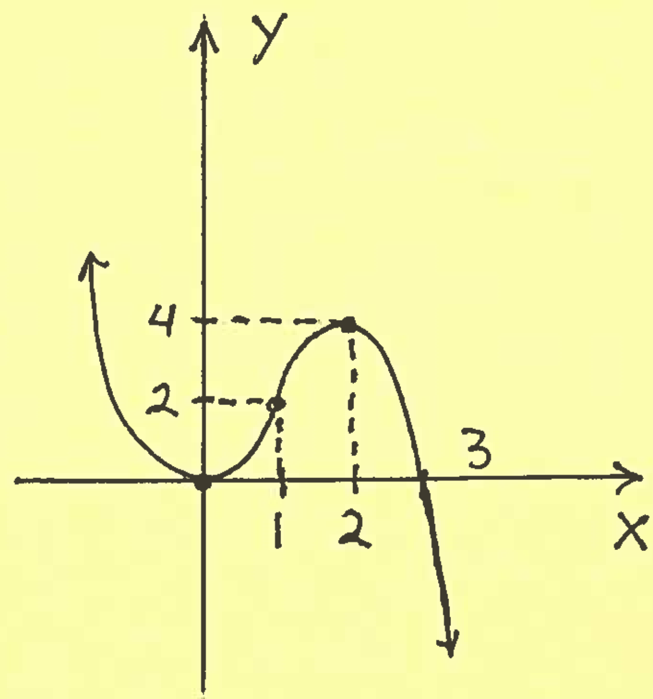
$+\quad 0 \quad -\qquad y''$

$\left. \begin{array}{l} x=1 \\ y=2 \end{array} \right\}$ INFL. PT.

$x=0 : y=0$

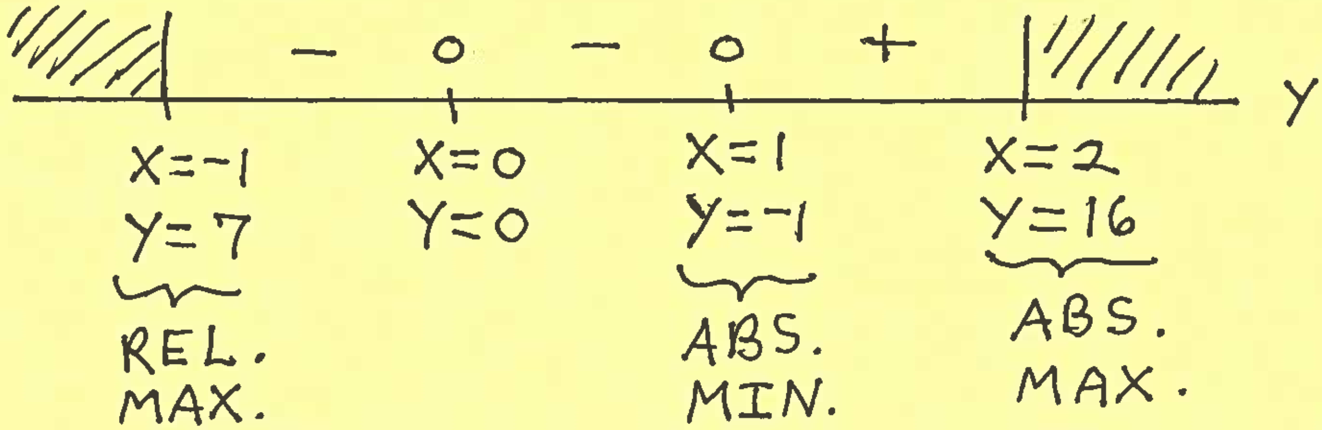
$y=0 : 0 = 3x^2 - x^3$
$= x^2(3-x)$
$\rightarrow x=0, x=3 ;$

y is $(\uparrow)$ for $0<x<2$,
y is $(\downarrow)$ for $x<0, x>2$,
y is $(\cup)$ for $x<1$,
y is $(\cap)$ for $x>1$

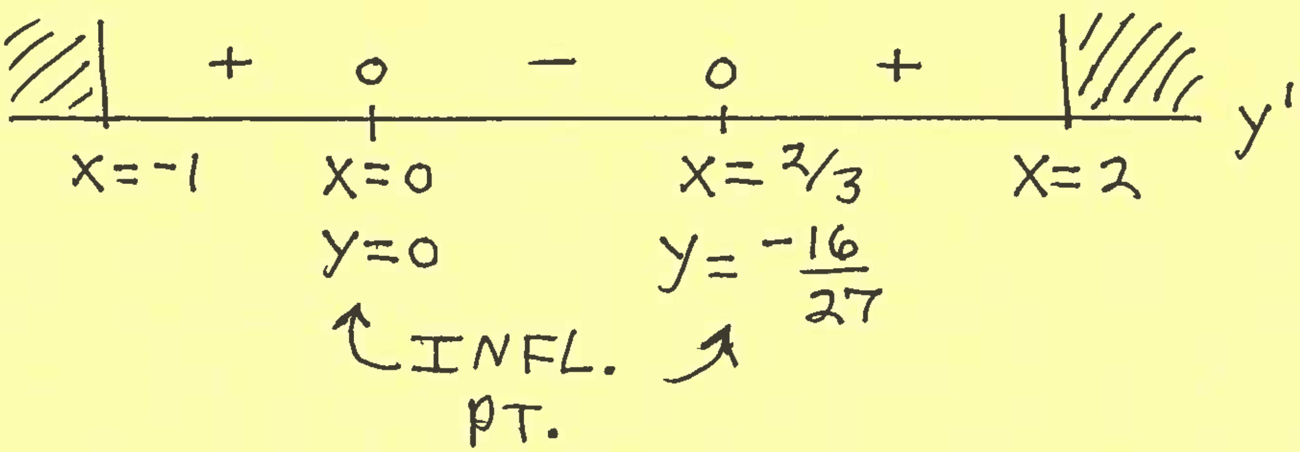


2.) $y = 3x^4 - 4x^3$ on $[-1, 2]$ $\xrightarrow{D}$

$y' = 12x^3 - 12x^2 = 12x^2(x-1) = 0$



$\xrightarrow{D}$ $y'' = 36x^2 - 24x = 12x(3x-2) = 0$



y is $(\uparrow)$ for $1 < x < 2$,

y is $(\downarrow)$ for $-1 < x < 0$, $0 < x < 1$,

y is $(\cup)$ for $-1 < x < 0$, $2/3 < x < 2$,

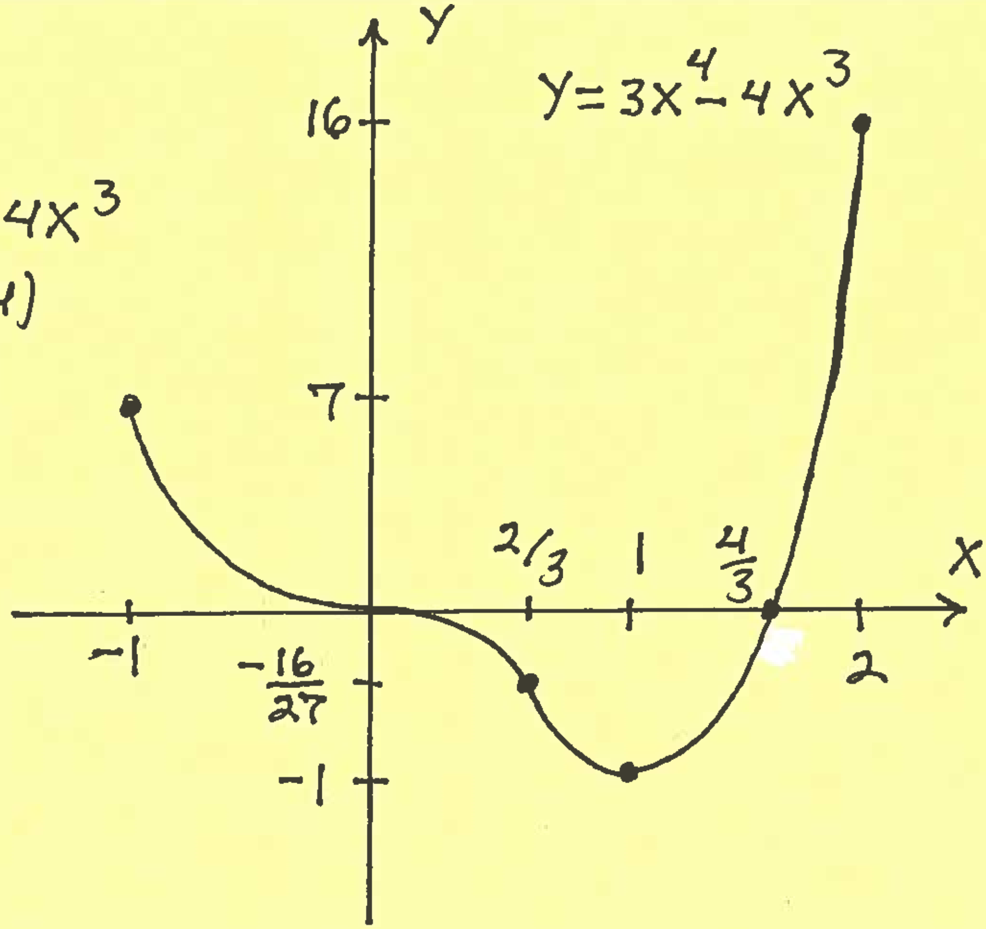
y is $(\cap)$ for $0 < x < 2/3$

$x=0:\ y=0$

$y=0:\ 0=3x^4-4x^3$

$= x^3(3x-4)$

$\rightarrow x=0\ ,\ x=\frac{4}{3}$



3.) $y = 3x^{2/3} - x \xrightarrow{D}$

$y' = \cancel{3}\cdot\frac{2}{\cancel{3}}x^{-1/3} - 1 = \frac{2}{x^{1/3}} - \frac{x^{1/3}}{x^{1/3}} = \frac{2-x^{1/3}}{x^{1/3}} = 0$

$\rightarrow 2 - x^{1/3} = 0 \rightarrow x^{1/3} = 2 \rightarrow x = 8$, and

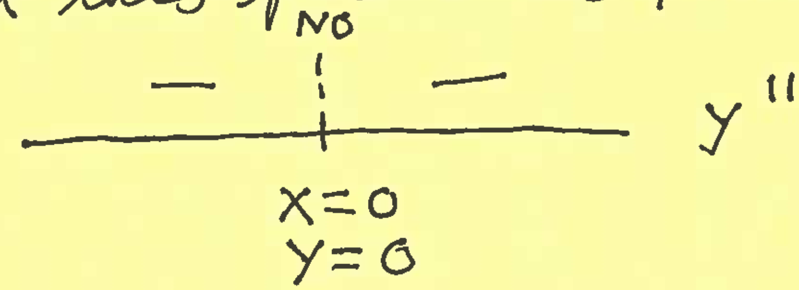
$x^{1/3} \neq 0$, so $x \neq 0$

	NO				
$-$	⋮	$+$	0	$-$	y'

REL. MIN. $\begin{cases} x=0 \\ y=0 \end{cases}$ $\left.\begin{matrix} x=8 \\ y=4 \end{matrix}\right\}$ REL. MAX.

$\xrightarrow{D}\ y'' = 2\cdot\left(\frac{-1}{3}\right)x^{-4/3} = \frac{-2}{3x^{4/3}} = 0,$

but this fraction $\neq 0$!



$x=0 : y=0$

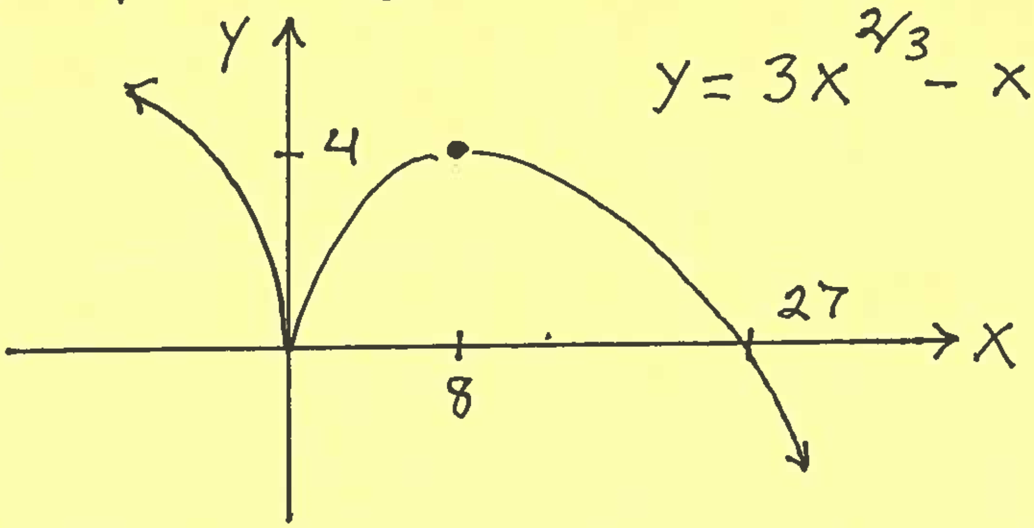
$y=0 : 0 = 3x^{2/3} - x = x^{2/3}(3 - x^{1/3}) = 0$

$\rightarrow x=0, \; x=27$;

y is $(\uparrow)$ for $0<x<8$,

y is $(\downarrow)$ for $x<0$, $x>8$,

y is $(\cap)$ for $x<0$, $x>0$.

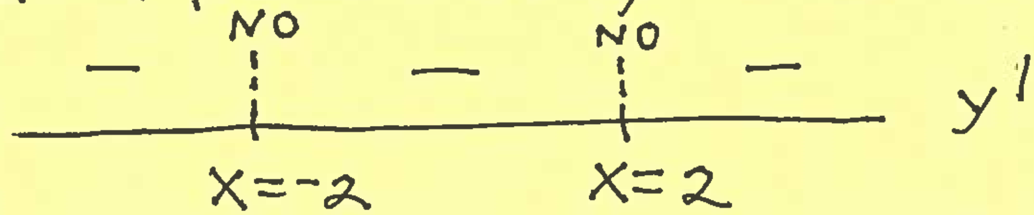


4.) $y = \dfrac{x}{x^2-4}$; Domain : $x \neq 2$, $x \neq -2$

$$\xrightarrow{D} \quad y' = \frac{(x^2-4)(1) - x(2x)}{(x^2-4)^2}$$

$$= \frac{x^2-4-2x^2}{(x^2-4)^2} = \frac{-4-x^2}{(x^2-4)^2} = \frac{-(4+x^2)}{(x^2-4)^2} \neq 0$$

$$\to x^2-4 \neq 0 \to x \neq 2,\ x \neq -2$$



$$\xrightarrow{D} y'' = \frac{(x^2-4)^2(-2x) - {}^{-}(4+x^2)\cdot 2(x^2-4)(2x)}{(x^2-4)^4}$$

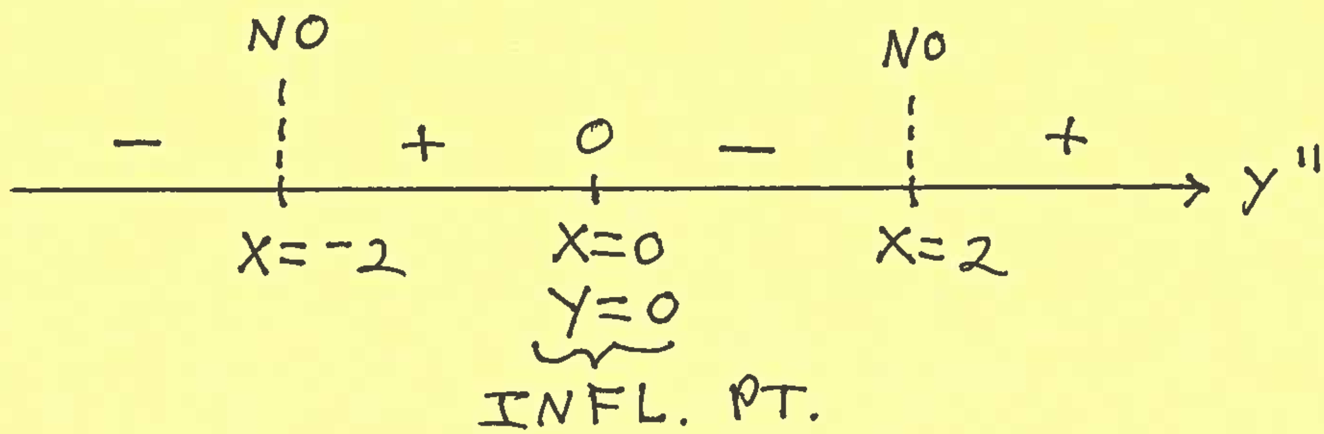
$$= \frac{-2x(x^2-4)\left[(x^2-4) - 2(4+x^2)\right]}{(x^2-4)^4}$$

$$= \frac{-2x\left[x^2-4-8-2x^2\right]}{(x^2-4)^3}$$

$$= \frac{-2x\left[-12-x^2\right]}{(x^2-4)^3}$$

$$= \frac{2x\left[12+x^2\right]}{(x^2-4)^3} = 0 \to$$

$x=0$ and $x \neq 2,\ x \neq -2$



$x=0: y=0$; $y=0: x=0$

y is ($\downarrow$) for $x<0$, $x>0$,

y is ($\cup$) for $-2<x<0$, $x>2$,

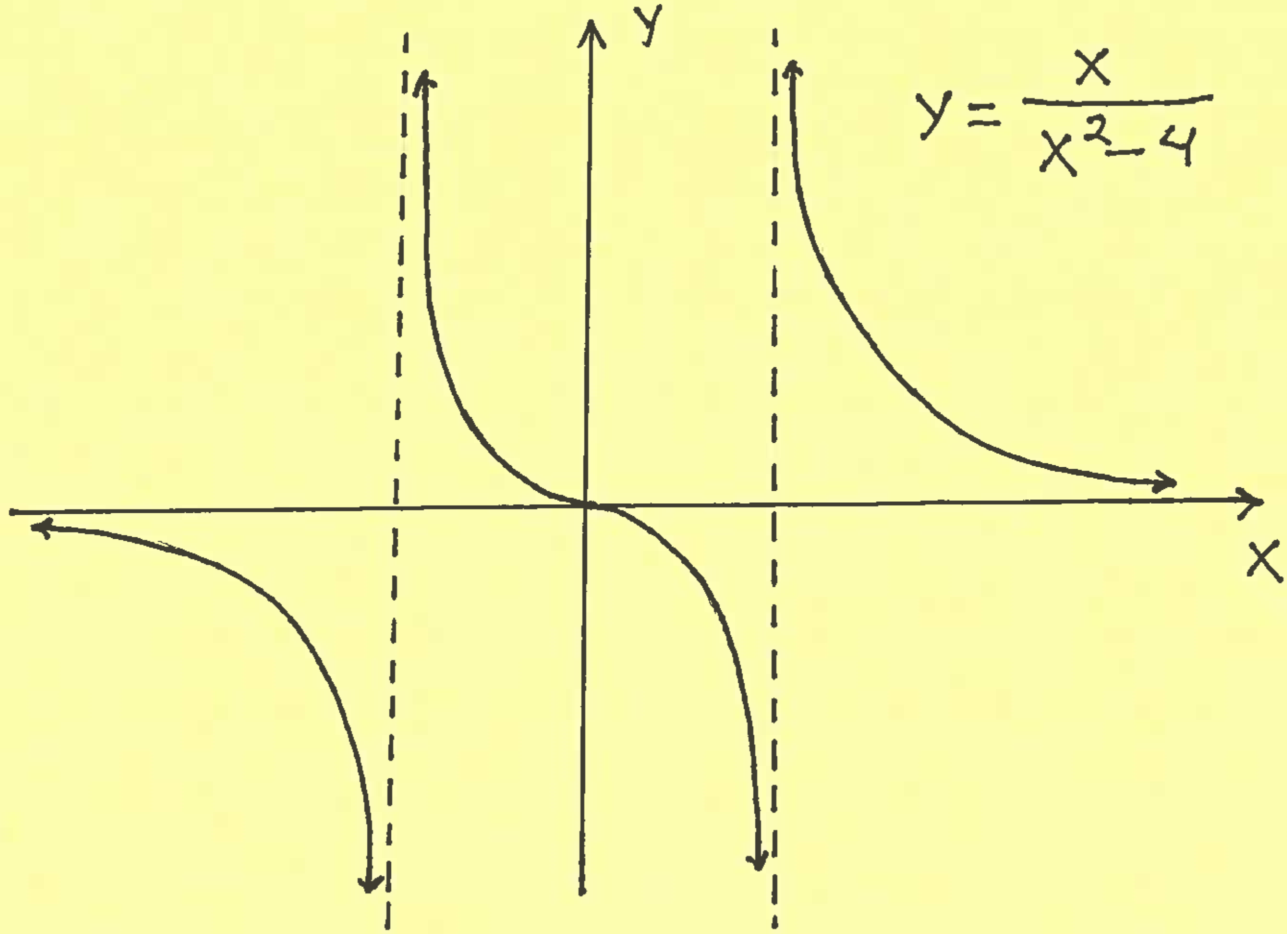
y is ($\cap$) for $x<-2$, $0<x<2$;

$$\lim_{x\to\pm\infty} \frac{x}{x^2-4}\cdot\frac{1/x^2}{1/x^2} = \lim_{x\to\pm\infty}\frac{1/x}{1-\frac{4}{x^2}} = \frac{0}{1-0} = 0,$$

so $\boxed{y=0}$ is a H.A. ;

$$\left.\begin{aligned} \lim_{x\to 2^+} y &= \text{"}\frac{2}{0^+}\text{"} = +\infty \\ \lim_{x\to 2^-} y &= \text{"}\frac{2}{0^-}\text{"} = -\infty \end{aligned}\right\} \boxed{x=2} \text{ is V.A.}$$

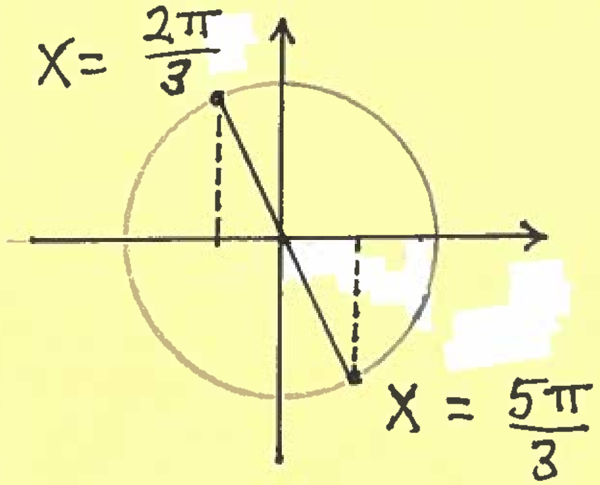
$$\left.\begin{aligned} \lim_{x\to -2^+} y &= \text{"}\frac{-2}{0^-}\text{"} = +\infty \\ \lim_{x\to -2^-} y &= \text{"}\frac{-2}{0^+}\text{"} = -\infty \end{aligned}\right\} \boxed{x=-2} \text{ is V.A.}$$

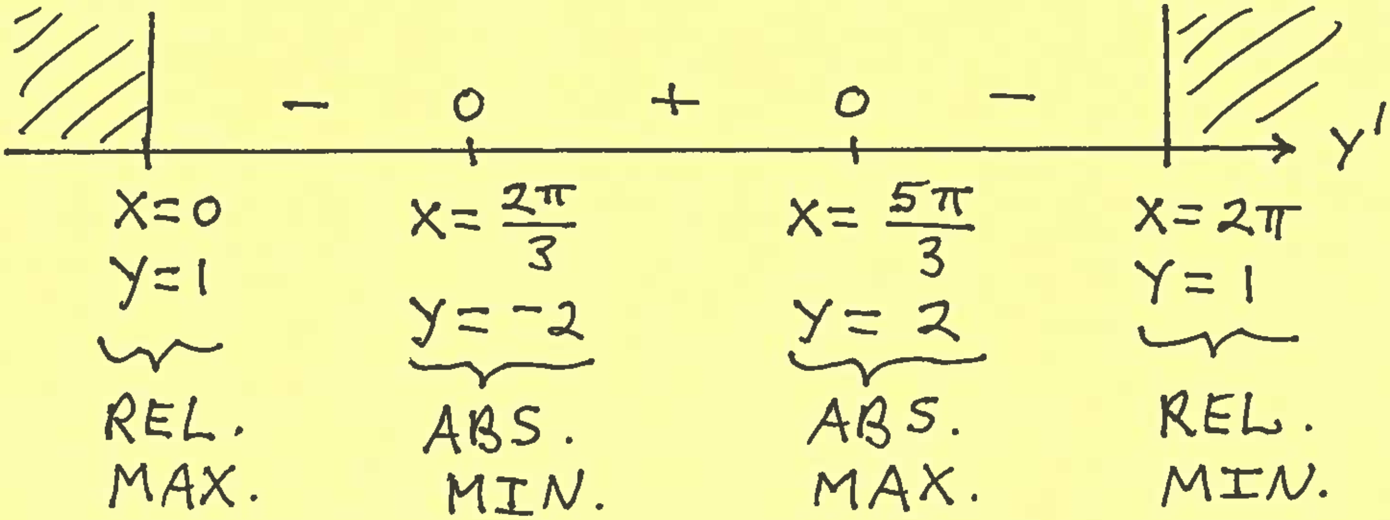


5.) $y = \cos x - \sqrt{3}\sin x$ on $[0, 2\pi]$

$\xrightarrow{D} y' = -\sin x - \sqrt{3}\cos x = 0 \rightarrow$

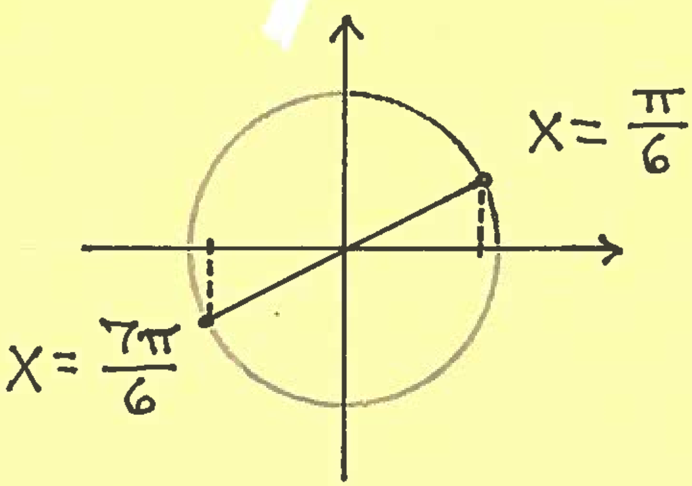
$-\sqrt{3}\cos x = \sin x \longrightarrow \frac{\sin x}{\cos x} = -\sqrt{3} = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \frac{\frac{\sqrt{3}}{2}}{\frac{-1}{2}}$

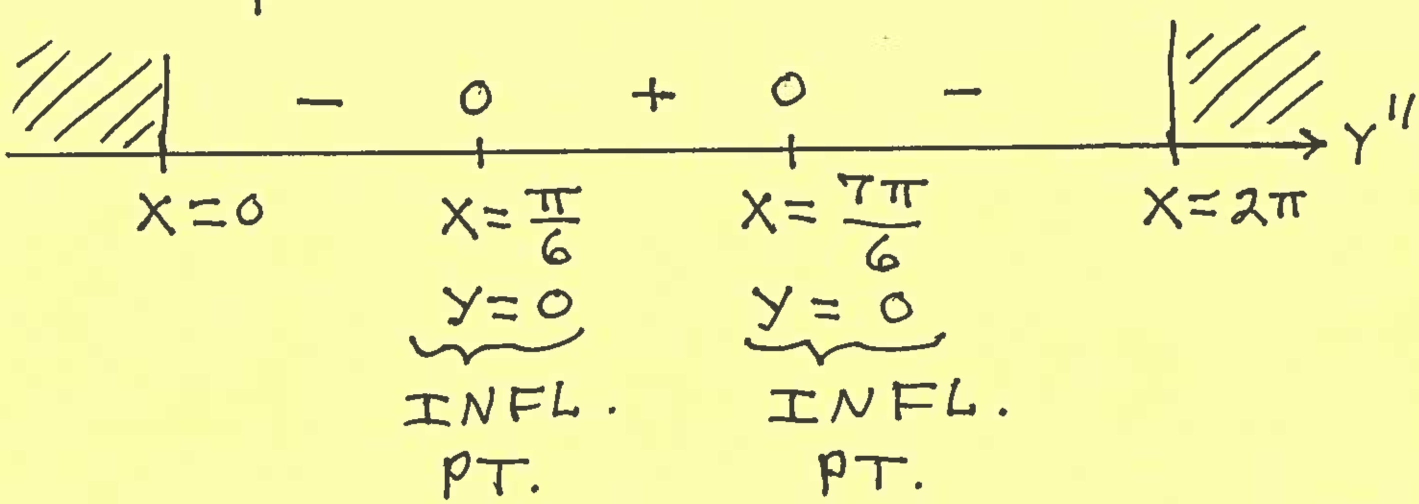


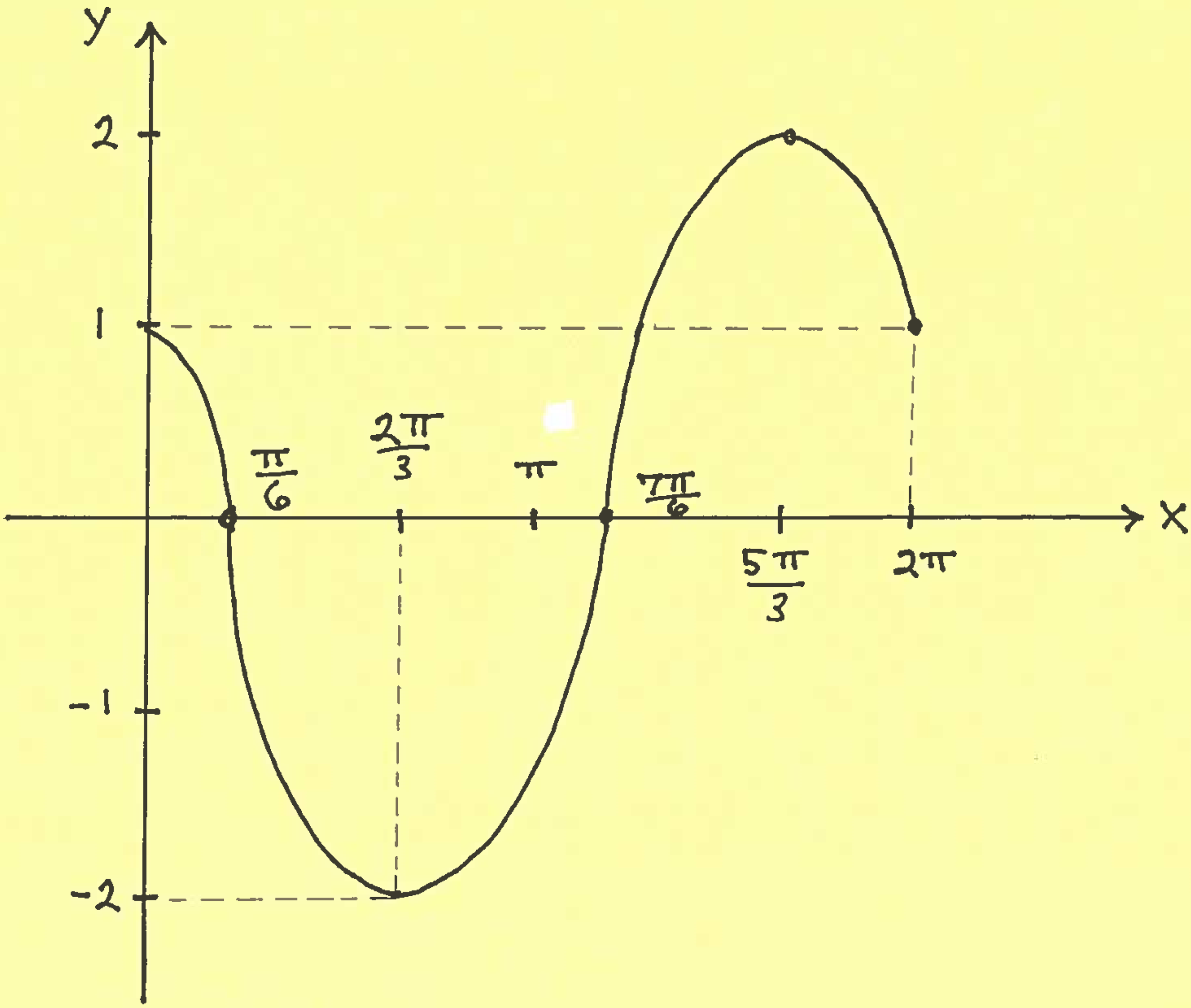


$\xrightarrow{D}$ $y'' = -\cos x + \sqrt{3}\sin x = 0 \rightarrow$

$\sqrt{3}\sin x = \cos x \rightarrow \dfrac{\sin x}{\cos x} = \dfrac{1}{\sqrt{3}} = \dfrac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \dfrac{-\frac{1}{2}}{-\frac{\sqrt{3}}{2}}$






y
2
1
-1
-2
$\frac{\pi}{6}$
$\frac{2\pi}{3}$
π
$\frac{7\pi}{6}$
$\frac{5\pi}{3}$
2π
x