Math 16A (Fall 2005)
Kouba
Exam 1

Please PRINT your name here : KEY

Your HW/Exam ID Number ____________

1. PLEASE DO NOT TURN THIS PAGE UNTIL TOLD TO DO SO.

2. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO, IN ANY WAY, ASSIST ANOTHER PERSON IN THE COMPLETION OF THIS EXAM. PLEASE KEEP YOUR OWN WORK COVERED UP AS MUCH AS POSSIBLE DURING THE EXAM SO THAT OTHERS WILL NOT BE TEMPTED OR DISTRACTED. THANK YOU FOR YOUR COOPERATION.

2. No notes, books, or classmates may be used as resources for this exam. YOU MAY USE A CALCULATOR ON THIS EXAM.

3. Read directions to each problem carefully. Show all work for full credit. In most cases, a correct answer with no supporting work will NOT receive full credit. What you write down and how you write it are the most important means of your getting a good score on this exam. Neatness and organization are also important.

4. Make sure that you have 7 pages, including the cover page.

5. You may NOT use L'Hopital's Rule on this exam.

6. You may NOT use shortcuts for finding limits to infinity.

7. **Using only a calculator to determine limits will receive little credit.**

8. **You will be graded on proper use of limit notation.**

9. You have until 10:50 a.m. sharp to finish the exam.

10. The following trigonometry identities are at your disposal :

$a.)$ $\sin 2\theta = 2\sin\theta\cos\theta$
$b.)$ $\cos 2\theta = 2\cos^2\theta - 1$
$= 1 - 2\sin^2\theta$
$= \cos^2\theta - \sin^2\theta$

1.) (9 pts. each) Determine the following limits.

a.) $\lim_{x \to 3} \frac{x^2 - 9}{x^2 - 5x + 6}$ (HINT: Factor.)

$$\text{"}\tfrac{0}{0}\text{"} = \lim_{x \to 3} \frac{\cancel{(x-3)}(x+3)}{\cancel{(x-3)}(x-2)} = \frac{6}{1} = 6$$

b.) $\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4}$ (HINT: Use a conjugate.)

$$\text{"}\tfrac{0}{0}\text{"} = \lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4} \cdot \frac{\sqrt{x} + 2}{\sqrt{x} + 2}$$

$$= \lim_{x \to 4} \frac{\cancel{x-4}}{\cancel{(x-4)}(\sqrt{x}+2)}$$

$$= \lim_{x \to 4} \frac{1}{\sqrt{x}+2} = \frac{1}{2+2} = \frac{1}{4}$$

c.) $\lim_{x \to 0} \frac{\frac{1}{x+1} + \frac{1}{x-1}}{\frac{x}{1}}$ (HINT: Add fractions first.)

$$\text{"}\tfrac{0}{0}\text{"} = \lim_{x \to 0} \frac{x \cancel{-1} + x \cancel{+1}}{(x+1)(x-1)} \cdot \frac{1}{x}$$

$$= \lim_{x \to 0} \frac{2\cancel{x}}{(x+1)(x-1)\cancel{x}} = \frac{2}{(1)(-1)} = -2$$

d.) $\lim_{x \to \infty} \frac{2x^2 - 3x}{3x^2 + 500}$ (HINT: Divide by the highest power of x.)

$$\text{"}\tfrac{\infty}{\infty}\text{"} = \lim_{x \to \infty} \frac{2x^2 - 3x}{3x^2 + 500} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}}$$

$$= \lim_{x \to \infty} \frac{2 - \frac{3}{x}}{3 + \frac{500}{x^2}} = \frac{2-0}{3+0} = \frac{2}{3}$$

2.) Consider the function $f(x) = 7 - \sqrt{x-3}$.

a.) (4 pts.) Determine the domain of f.

$x - 3 \geq 0 \rightarrow x \geq 3$

Domain : $x \geq 3$

b.) (4 pts.) Determine the range of f.

$\sqrt{x-3} = 0$ when $x = 3$ and $\lim\limits_{x \to \infty} \sqrt{x-3} = \infty$; then $7 \geq 7 - \sqrt{x-3} > -\infty$ so

Range : $y \leq 7$

3.) (8 pts.) Let $f(x) = \dfrac{x+2}{x-1}$. Find a function $g(x)$ so that $f(g(x)) = x$.

$f(g(x)) = x \rightarrow \dfrac{g(x)+2}{g(x)-1} = x$

$\rightarrow g(x) + 2 = x\,g(x) - x$

$\rightarrow 2 + x = x\,g(x) - g(x)$

$\rightarrow 2 + x = (x-1)\,g(x)$

$\rightarrow g(x) = \dfrac{2+x}{x-1}$

4.) a.) (5 pts.) Write the three-step definition for the following statement : Function $y = f(x)$ is continuous at $x = a$.

i.) $f(a)$ is defined (finite).

ii.) $\lim\limits_{x \to a} f(x)$ exists (finite).

iii) $\lim\limits_{x \to a} f(x) = f(a)$.

b.) (5 pts.) Use the definition in part a.) to determine if the following function is continuous at $x = 1$.

$$f(x) = \begin{cases} x^2 + 3x - 1 \ , & \text{if } x < 1 \\ 3 \ , & \text{if } x = 1 \\ \sqrt{x + 8} & \text{if } x > 1 \end{cases}$$

i.) $f(1) = 3$

ii.) $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} \sqrt{x+8} = \sqrt{9} = 3$;

$\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} (x^2 + 3x - 1) = 1 + 3 - 1 = 3$;

then $\lim\limits_{x \to 1} f(x) = 3$.

iii) $\lim\limits_{x \to 1} f(x) = 3 = f(1)$ so f is continuous at $x = 1$.

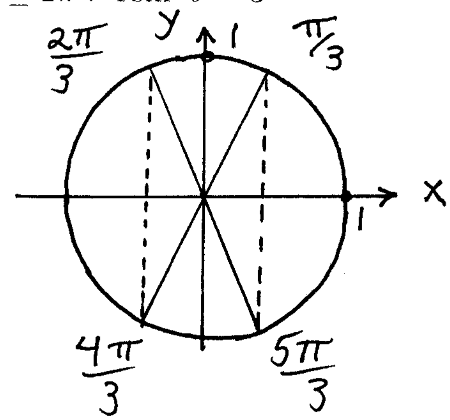
5.) (8 pts.) Solve the following trigonometry equation for θ , $0 \leq \theta \leq 2\pi$: $4 \sin^2 \theta = 3$

$4 \sin^2 \theta = 3 \rightarrow$

$\sin^2 \theta = \frac{3}{4} \rightarrow$

$\sin \theta = \pm \frac{\sqrt{3}}{2} \rightarrow$

$\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$



6.) (8 pts.) *Use limits* to find the equation(s) for all vertical asymptote(s) for $y = \dfrac{x^2+x-6}{(x+1)(x-2)}$. YOU NEED NOT GRAPH THE FUNCTION.

$$\lim_{x\to 2} \frac{x^2+x-6}{(x+1)(x-2)} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x\to 2} \frac{\cancel{(x-2)}(x+3)}{(x+1)\cancel{(x-2)}} = \frac{5}{3} \,;$$

$$\lim_{x\to -1} \frac{x^2+x-6}{(x+1)(x-2)} = \text{"}\frac{-6}{(0^{\pm})(-3)}\text{"} = \pm\infty \quad \text{so}$$

$\boxed{x=-1}$ is a V.A.

7.) (8 pts.) Find all points of intersection, (x, y), for the functions $y = x^2 + x$ and $y = 2x^2 - 2x + 2$.

$$x^2 + x = 2x^2 - 2x + 2 \rightarrow$$
$$0 = x^2 - 3x + 2 \rightarrow$$
$$0 = (x-2)(x-1) \rightarrow$$

$x = 2$
$y = 6$

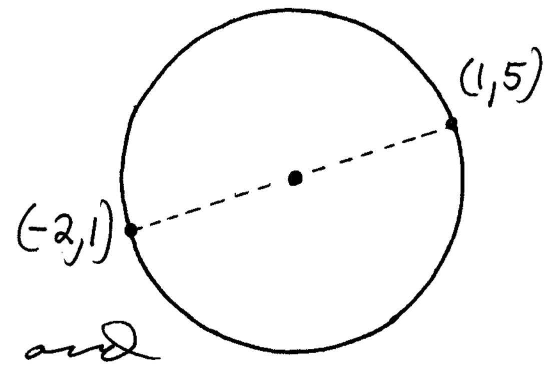
$x = 1$
$y = 2$

8.) (8 pts.) The segment joining the points $(1, 5)$ and $(-2, 1)$ is the diameter of a circle. Determine an equation for this circle.

midpoint :

$\left(\frac{-2+1}{2}, \frac{5+1}{2}\right) = \left(\frac{-1}{2}, 3\right)$

is center of circle ;



Distance between $(-2,1)$ and $(1,5)$ is $L = \sqrt{(-2-1)^2 + (1-5)^2} = \sqrt{9+16} = 5$

so radius is $5/2$; then circle is

$$\left(x - \frac{-1}{2}\right)^2 + (y-3)^2 = \left(\frac{5}{2}\right)^2$$

9.) (6 pts.) Evaluate the following limit : $\lim_{x \to -\infty} \left(x + \sqrt{x^2+4}\right)$ (HINT: Start with a conjugate.)

$$\lim_{x \to -\infty} \underbrace{\left(x + \sqrt{x^2+4}\right)}_{\text{“}\infty - \infty\text{”}} \cdot \frac{\left(x - \sqrt{x^2+4}\right)}{\left(x - \sqrt{x^2+4}\right)}$$

$$= \lim_{x \to -\infty} \frac{x^2 - (x^2+4)}{x - \sqrt{x^2+4}}$$

$$= \lim_{x \to -\infty} \frac{-4}{x - \sqrt{x^2+4}}$$

$$= \text{“} \frac{-4}{-\infty - \infty} \text{”}$$

$$= \text{“} \frac{4}{\infty} \text{”}$$

$$= 0$$

Each of the following EXTRA CREDIT PROBLEMS is worth 10 points. These problems are OPTIONAL.

1.) Determine the domain for the following function : $y = \dfrac{5}{3 - \sqrt{x^2 - 8x}}$

$x^2 - 8x = x(x-8) \geq 0$

$+ \quad 0 \quad - \quad 0 \quad +$

$x=0 \qquad x=8$

$3 - \sqrt{x^2 - 8x} \neq 0 \rightarrow \sqrt{x^2-8x} \neq 3 \rightarrow$

$x^2 - 8x \neq 9 \rightarrow x^2 - 8x - 9 \neq 0 \rightarrow$

$(x-9)(x+1) \neq 0$

$\downarrow \qquad \downarrow$

$x \neq 9 \qquad x \neq -1$

Domain: $x \leq 0$ but $x \neq -1$,
$x \geq 8$ but $x \neq 9$.

2.) Evaluate the following limit : $\displaystyle\lim_{x \to \pi/2} \frac{\sin(2x)}{\cos^2 x - \cos x}$

$$\overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x \to \frac{\pi}{2}} \frac{2 \sin x \cancel{\cos x}}{\cancel{\cos x}(\cos x - 1)}$$

$$= \frac{2 \sin \frac{\pi}{2}}{\cos \frac{\pi}{2} - 1} = \frac{2 \cdot (1)}{0 - 1} = -2$$