MATH 108 ANSWERS - GREEN SHEET

① A) IF $a \equiv b \pmod m$ AND $c \equiv d \pmod m$, THEN $a+c \equiv b+d \pmod m$.

PF SINCE $a \equiv b \pmod m$ AND $c \equiv d \pmod m$, $m \mid (a-b)$ AND $m \mid (c-d)$.
THEN $m \mid [(a-b)+(c-d)]$, SO $m \mid [(a+c)-(b+d)]$ AND
THEREFORE $a+c \equiv b+d \pmod m$ (USING RESULT 4.3).

REMARK SEE RESULT 4.10 FOR A SLIGHTLY DIFFERENT PROOF.

B) IF $a \equiv b \pmod m$, THEN $\ell a \equiv \ell b \pmod m$ FOR EVERY $\ell \in \mathbb{Z}$.

PF SINCE $a \equiv b \pmod m$, $m \mid (a-b)$ SO $a-b = km$ FOR SOME $k \in \mathbb{Z}$.
THEN $\ell a - \ell b = \ell(a-b) = \ell(km) = (\ell k)m$ WHERE $\ell k \in \mathbb{Z}$,
SO $m \mid (\ell a - \ell b)$ AND THEREFORE $\ell a \equiv \ell b \pmod m$.

REMARK SEE ALSO RESULT 4.9.

C) IF $a \equiv b \pmod m$ AND $c \equiv d \pmod m$, THEN $ac \equiv bd \pmod m$.

PF SINCE $a \equiv b \pmod m$, $ac \equiv bc \pmod m$ BY PART B); AND
SINCE $c \equiv d \pmod m$, $bc \equiv bd \pmod m$ BY PART B).
THEREFORE $ac \equiv bd \pmod m$ BY TRANSITIVITY.

REMARK SEE RESULT 4.11 FOR A DIFFERENT PROOF.

② LET $[a]=[b]$ AND $[c]=[d]$ IN $\mathbb{Z}_m$,
SO $a \equiv b \pmod m$ AND $c \equiv d \pmod m$.
THEN 1) $a+c \equiv b+d \pmod m$ BY ①A), SO $[a+c]=[b+d]$ IN $\mathbb{Z}_m$ AND
THEREFORE ADDITION IS WELL-DEFINED;
AND 2) $ac \equiv bd \pmod m$ BY ①C), SO $[ac]=[bd]$ IN $\mathbb{Z}_m$ AND
THEREFORE MULTIPLICATION IS WELL-DEFINED.

REMARK SEE TH. 8.9 FOR A DIFFERENT PROOF OF 1).

③ $a_1 = 1$, $a_2 = 8$, $a_n = a_{n-1} + 2a_{n-2}$ FOR $n \geq 3$.
SHOW THAT $a_n = 3(2^{n-1}) + 2(-1)^n$ FOR ALL $n \in \mathbb{N}$.

PF 1) THIS IS TRUE FOR $n=1$, SINCE $a_1 = 3(2^0) + 2(-1) = 3-2 = 1$.
THIS IS TRUE FOR $n=2$, SINCE $a_2 = 3(2^1) + 2(1) = 6+2 = 8$.

2) ASSUME THAT $a_k = 3(2^{k-1}) + 2(-1)^k$ FOR $k = 1, \ldots, n$ WHERE $n \in \mathbb{N}$ WITH $n \geq 2$.

THEN
$$\begin{aligned} a_{n+1} &= a_n + 2a_{n-1} = \left[3(2^{n-1}) + 2(-1)^n\right] + 2\left[3(2^{n-2}) + 2(-1)^{n-1}\right] \\ &= 3(2^{n-1}) + 3(2^{n-1}) + 2(-1)^n + 4(-1)^{n-1} \\ &= 2\cdot 3(2^{n-1}) + 2(-1)^{n-1}[-1+2] \\ &= 3(2^n) + 2(-1)^{n-1} \\ &= 3(2^n) + 2(-1)^{n+1}, \end{aligned}$$
SO THE FORMULA IS VALID FOR $n+1$.

THEREFORE $a_n = 3(2^{n-1}) + 2(-1)^n$ FOR ALL $n \in \mathbb{N}$ BY THE PSI.

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④ IF $a_1, \ldots, a_n > 0$ WITH $a_1 \cdots a_n = 1$, THEN $a_1 + \cdots + a_n \geq n$.

PF (BY INDUCTION)

1) THIS IS TRUE FOR $n = 1$, SINCE $a_1 = 1 \Rightarrow a_1 \geq 1$.

2) ASSUME THAT THIS STATEMENT IS TRUE FOR AN ARBITRARY $n \in \mathbb{N}$, AND LET $a_1, \ldots, a_{n+1} > 0$ WITH $a_1 \cdots a_{n+1} = 1$.

A) IF $a_1 = a_2 = \cdots = a_{n+1}$, THEN $a_1 \cdots a_{n+1} = 1 \Rightarrow a_1^{n+1} = 1 \Rightarrow a_1 = 1 \Rightarrow a_1 + \cdots + a_{n+1} = n+1$.

B) IF THE a_i ARE NOT ALL EQUAL, WE CAN ASSUME WLOG THAT $a_1 \leq a_2 \leq \cdots \leq a_{n+1}$ WITH $a_1 < a_{n+1}$.

THEN $a_1 < 1$, SINCE OTHERWISE $1 = a_1 \cdots a_{n+1} > a_1^{n+1} \geq 1$, WHICH GIVES A CONTRADICTION.

SIMILARLY, $a_{n+1} > 1$, SINCE OTHERWISE $1 = a_1 \cdots a_{n+1} < a_{n+1}^{n+1} \leq 1$, WHICH GIVES A CONTRADICTION.

THEREFORE $(1 - a_1)(a_{n+1} - 1) > 0$, SO $a_1 + a_{n+1} - a_1 a_{n+1} > 1$.

IF WE APPLY THE INDUCTION HYPOTHESIS TO $a_1 a_{n+1}, a_2, \ldots, a_n$, WE HAVE $(a_1 a_{n+1}) a_2 \cdots a_n = 1$ SO $a_1 a_{n+1} + a_2 + \cdots + a_n \geq n$.

THEREFORE
$$a_1 + a_2 + \cdots + a_n + a_{n+1} = (a_1 a_{n+1} + a_2 + \cdots + a_n) + (a_1 + a_{n+1} - a_1 a_{n+1}) \geq n+1,$$

SO THE ASSERTION IS VALID FOR $n+1$.

THEREFORE THE ASSERTION IS VALID FOR ALL $n \in \mathbb{N}$ BY THE PMI.

⑤ SHOW THAT IF $a_1, \ldots, a_n > 0$, THEN $\dfrac{a_1 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \cdots a_n}$.

PF LET $a_1, \ldots, a_n > 0$, AND LET $P = a_1 \cdots a_n$ AND LET $b_i = \dfrac{a_i}{\sqrt[n]{P}}$ FOR $1 \leq i \leq n$.

THEN $b_1, \ldots, b_n > 0$ AND $b_1 \cdots b_n = \dfrac{a_1 \cdots a_n}{P} = 1$, SO $b_1 + \cdots + b_n \geq n$ BY #4

AND THEREFORE $\dfrac{a_1}{\sqrt[n]{P}} + \cdots + \dfrac{a_n}{\sqrt[n]{P}} \geq n$.

THUS $\dfrac{a_1 + \cdots + a_n}{n} \geq \sqrt[n]{P} = \sqrt[n]{a_1 \cdots a_n}$.

⑥ $A_r = \{(x, y) : y = 2x + r\} \subseteq \mathbb{R}^2$.

(x, y) AND (x', y') ARE IN A_r IFF $y = 2x + r$ AND $y' = 2x' + r$
IFF $y - 2x = r = y' - 2x'$,

SO WE CAN DEFINE

$$\boxed{(x, y) \sim (x', y') \text{ IFF } y - 2x = y' - 2x'}$$

OR

$$\boxed{(x, y) \sim (x', y') \text{ IFF } y - y' = 2(x - x')}$$

(NOTICE THAT THIS GIVES AN EQUIVALENCE RELATION, SINCE $(x, y) \sim (x', y')$ IFF $f(x, y) = f(x', y')$ WHERE $f : \mathbb{R}^2 \to \mathbb{R}$ IS DEFINED BY $f(x, y) = y - 2x$.)