MATH 108 GREEN SHEET

① LET $m \in \mathbb{Z}$ WITH $m \geq 2$, AND LET $a, b, c, d \in \mathbb{Z}$.

SHOW THE FOLLOWING:

A) IF $a \equiv b \pmod{m}$ AND $c \equiv d \pmod{m}$, THEN $a + c \equiv b + d \pmod{m}$.

B) IF $a \equiv b \pmod{m}$, THEN $la \equiv lb \pmod{m}$ FOR EVERY $l \in \mathbb{Z}$.

C) IF $a \equiv b \pmod{m}$ AND $c \equiv d \pmod{m}$, THEN $ac \equiv bd \pmod{m}$.

[HINT: USE PART B).]

② USE #1 TO SHOW THAT THE OPERATIONS OF ADDITION AND MULTIPLICATION DEFINED ON $\mathbb{Z}_m$ BY

$$[a] + [c] = [a + c] \quad \text{AND} \quad [a][c] = [ac]$$

ARE WELL-DEFINED.

③ LET (a_n) BE DEFINED BY $a_1 = 1$, $a_2 = 8$, AND $a_n = a_{n-1} + 2a_{n-2}$ FOR $n \geq 3$. PROVE THAT $a_n = 3(2^{n-1}) + 2(-1)^n$ FOR ALL $n \in \mathbb{N}$ USING INDUCTION.

④ USE INDUCTION TO PROVE THAT IF $a_1, \ldots, a_n > 0$ WITH $a_1 \cdots a_n = 1$, THEN $a_1 + \cdots + a_n \geq n$.

[HINT: IN THE INDUCTION STEP, SHOW THAT IF THE a_i ARE NOT ALL EQUAL, THEN YOU CAN ASSUME WLOG THAT $a_1 < 1$ AND $a_{n+1} > 1$. THEN APPLY THE INDUCTION HYPOTHESIS TO $a_1 a_{n+1}, a_2, \ldots, a_n$.]

⑤ USE #4 TO PROVE THE ARITHMETIC MEAN - GEOMETRIC MEAN INEQUALITY:

IF $a_1, \ldots, a_n > 0$, THEN $\boxed{\dfrac{a_1 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \cdots a_n}}$.

(SEE #32 AND #33 IN CH. 4)

⑥ LET $A_r = \{(x, y) : y = 2x + r\} \subseteq \mathbb{R}^2$ FOR EACH $r \in \mathbb{R}$, AND LET $\mathcal{P}$ BE THE PARTITION OF $\mathbb{R}^2$ GIVEN BY $\mathcal{P} = \{A_r : r \in \mathbb{R}\}$. DEFINE AN EQUIVALENCE RELATION ON $\mathbb{R}^2$ WHICH HAS $\mathcal{P}$ AS ITS SET OF EQUIVALENCE CLASSES.