MATH 108 REVIEW PROBLEMS

1. PROVE THAT IF A IS DENUMERABLE, THEN $A \cup \{x\}$ IS DENUMERABLE. (GIVE A FORMAL PROOF.)
2. PROVE THAT $6 \mid (n^3 + 5n)$ FOR ALL $n \in \mathbb{N}$.
3. IF $A_n = \left(2 - \frac{5}{n}, 7 - \frac{3}{n}\right]$ $\forall n \in \mathbb{N}$, FIND a) $\bigcup_{n \in \mathbb{N}} A_n$ b) $\bigcap_{n \in \mathbb{N}} A_n$
4. PROVE THAT $\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} \leq \frac{2n}{n+1}$ $\forall n \in \mathbb{N}$.
5. FOR $0 \leq x < 1$, LET $A_x = \{t \in \mathbb{R} : t - x \in \mathbb{Z}\}$.
SHOW THAT $\{A_x : 0 \leq x < 1\}$ IS A PARTITION OF $\mathbb{R}$.
6. PROVE THAT $|A| < |\mathcal{P}(A)|$ FOR ANY SET A.
7. SHOW THAT $\mathbb{R} \times \mathbb{N} \approx \mathbb{R}$.
8. PROVE THAT $(0,1) \approx \mathcal{P}(\mathbb{N})$.
9. DEFINE $f : \mathbb{R} \to \mathbb{R}$ BY $f(x) = \begin{cases} 4x - 5, & \text{IF } x < 2 \\ x^2 - 1, & \text{IF } x \geq 2. \end{cases}$
A) PROVE OR DISPROVE THAT f IS 1-1.
B) PROVE OR DISPROVE THAT f IS ONTO.
10. PROVE THAT $\sqrt{6}$ IS IRRATIONAL.
11. SHOW THAT $\mathbb{Z}$ IS DENUMERABLE. (GIVE A FORMAL PROOF.)
12. DEFINE A BIJECTION $f : (0,1) \to (-\infty, 8)$, AND SHOW THAT f IS BIJECTIVE.
13. PROVE OR DISPROVE: IF $f : A \to A$ AND $g : A \to A$ AND $f \circ g$ IS BIJECTIVE, THEN f AND g ARE BIJECTIVE.
14. DEFINE A RELATION S ON $\mathbb{R}$ BY $x S y$ IFF $x \leq 0$ OR $y \geq 2x$.
SHOW WHETHER OR NOT S SATISFIES EACH OF THE 3 PROPERTIES OF AN EQUIVALENCE RELATION.
15. LET $f : A \to B$ AND LET C AND D BE SUBSETS OF B.
SHOW THAT $f^{-1}(C - D) = f^{-1}(C) - f^{-1}(D)$.
16. PROVE THAT IF $m, n \in \mathbb{N}$, THEN $n = qm + r$ FOR SOME $q, r \in \mathbb{Z}$ WITH $0 \leq r < m$.
17. LET $f : A \to B$, AND LET $\mathcal{P}$ BE A PARTITION OF B.
PROVE OR DISPROVE THAT $\mathcal{Q} = \{f^{-1}(x) : x \in \mathcal{P}\}$ IS A PARTITION OF A.
18. PROVE THAT $\log_2 5$ IS IRRATIONAL.
19. LET f BE THE RELATION FROM $\mathbb{Z}_4$ TO $\mathbb{Z}_8$ GIVEN BY $f([x]) = [x^2 + 6x]$.
PROVE OR DISPROVE THAT f IS A WELL-DEFINED FUNCTION.
20. PROVE THAT IF $U_1, \ldots, U_n$ ARE OPEN SETS IN A METRIC SPACE (X, d), THEN $U_1 \cap \cdots \cap U_n$ IS OPEN.
21. DEFINE A RELATION R ON $\mathbb{Z}$ BY $m R n$ IFF $4 \mid (m+n)$ OR $3 \mid (m-n)$.
SHOW WHETHER OR NOT R SATISFIES EACH OF THE 3 PROPERTIES OF AN EQUIVALENCE RELATION.
22. LET A BE A SUBSET OF A METRIC SPACE (X, d), AND LET $x \in X$.
PROVE THAT $x \in \overline{A}$ IFF $B_\delta(x) \cap A \neq \emptyset$ FOR EVERY $\delta > 0$.
23. PROVE OR DISPROVE: IF x IS A POSITIVE RATIONAL AND y IS IRRATIONAL, THEN x^y IS IRRATIONAL.
24. PROVE THAT EVERY INTEGER $n > 1$ HAS A PRIME FACTOR.