MATH 16B

SOLUTIONS – 6.6, CH. 6 RE, AND PINK SHEET

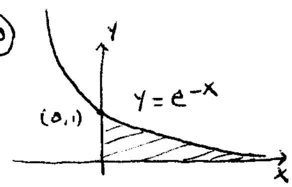
6.6 – (15) $$\int_0^4 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0^+}\int_t^4 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0^+}\int_t^4 x^{-1/2}\,dx = \lim_{t\to 0^+}\left[2x^{1/2}\right]_t^4$$

$$= \lim_{t\to 0^+} 2\left(2 - t^{1/2}\right) = 2(2-0) = \boxed{4}$$

(23) $$\int_0^1 \frac{1}{x^2}\,dx = \lim_{t\to 0^+}\int_t^1 \frac{1}{x^2}\,dx = \lim_{t\to 0^+}\int_t^1 x^{-2}\,dx = \lim_{t\to 0^+}\left[-\frac{1}{x}\right]_t^1$$

$$= \lim_{t\to 0^+}\left(-1-\left(-\frac{1}{t}\right)\right) = \lim_{t\to 0^+}\left(-1+\frac{1}{t}\right) = \infty,$$ so the integral $\boxed{\text{DIVERGES}}$

(30)



a) $$A = \int_0^\infty e^{-x}\,dx = \lim_{t\to\infty}\int_0^t e^{-x}\,dx = \lim_{t\to\infty}\left[-e^{-x}\right]_0^t$$

$$= \lim_{t\to\infty}\left(-e^{-t}-(-1)\right) = \lim_{t\to\infty}\left(-\frac{1}{e^t}+1\right) = 0+1 = \boxed{1}$$

b) (REVOLVE AROUND THE X-AXIS)

$$V = \int_0^\infty \pi\left(e^{-x}\right)^2 dx = \lim_{t\to\infty}\int_0^t \pi\left(e^{-2x}\right)dx = \lim_{t\to\infty}\pi\left[-\frac{1}{2}e^{-2x}\right]_0^t$$

$$= \lim_{t\to\infty} -\frac{\pi}{2}\left(e^{-2t}-1\right) = \lim_{t\to\infty} -\frac{\pi}{2}\left(\frac{1}{e^{2t}}-1\right) = -\frac{\pi}{2}(0-1) = \boxed{\frac{\pi}{2}}$$

CH. 6 RE – (79) $$\int_{-\infty}^0 \frac{1}{3x^2}\,dx = \int_{-\infty}^{-1}\frac{1}{3x^2}\,dx + \int_{-1}^0 \frac{1}{3x^2}\,dx$$ (OR SPLIT AT ANOTHER NEGATIVE NUMBER, INSTEAD OF -1)

a) $$\int_{-1}^0 \frac{1}{3x^2}\,dx = \lim_{t\to 0^-}\int_{-1}^t \frac{1}{3x^2}\,dx = \lim_{t\to 0^-}\frac{1}{3}\int_{-1}^t x^{-2}\,dx = \lim_{t\to 0^-}\frac{1}{3}\left[-\frac{1}{x}\right]_{-1}^t$$

$$= \lim_{t\to 0^-}\frac{1}{3}\left(-\frac{1}{t}-\left(-\frac{1}{-1}\right)\right) = \lim_{t\to 0^-}\frac{1}{3}\left(-\frac{1}{t}-1\right) = \infty,$$

so $\displaystyle\int_{-\infty}^0 \frac{1}{3x^2}\,dx$ $\boxed{\text{DIVERGES}}$

P.S. – (15) $$\int_2^\infty \frac{e^{6/x}}{x^2}\,dx = \lim_{t\to\infty}\int_2^t \frac{e^{6/x}}{x^2}\,dx$$

Let $u = \frac{6}{x} = 6x^{-1}$

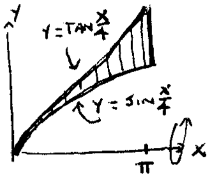
$du = -6x^{-2}\,dx = -\frac{6}{x^2}\,dx$

$$= \lim_{t\to\infty}\left(-\frac{1}{6}\right)\int_2^t e^{6/x}\left(\frac{-6}{x^2}\right)dx = \lim_{t\to\infty} -\frac{1}{6}\int_{x=2}^{x=t} e^u\,du = \lim_{t\to\infty} -\frac{1}{6}\left[e^u\right]_{x=2}^{x=t}$$

$$= \lim_{t\to\infty} -\frac{1}{6}\left[e^{6/x}\right]_2^t = \lim_{t\to\infty} -\frac{1}{6}\left(e^{6/t}-e^3\right) = -\frac{1}{6}\left(e^0-e^3\right) = \boxed{\frac{1}{6}\left(e^3-1\right)}$$

MATH 168 SOLUTIONS - PINK SHEET

(12)



$\tan u = \frac{\sin u}{\cos u} > \sin u$ FOR $0 < u < \frac{\pi}{2}$ (SINCE $0 < \cos u < 1$), FOR $0 < x < \pi/2$

SO $\tan\frac{x}{4} > \sin\frac{x}{4}$ FOR $0 < x < \pi$ (SINCE $0 < \frac{x}{4} < \frac{\pi}{4} < \frac{\pi}{2}$)

$$V = \int_0^{\pi} \pi\left(\left(\tan\frac{x}{4}\right)^2 - \left(\sin\frac{x}{4}\right)^2\right)dx$$

LET $u = \frac{x}{4}$; $du = \frac{1}{4}\,dx$ IF $x = 0,\ u = 0$; $x = \pi,\ u = \pi/4$

$$= \pi\cdot 4\int_0^{\pi}\left(\left(\tan\frac{x}{4}\right)^2 - \left(\sin\frac{x}{4}\right)^2\right)\cdot\left(\frac{1}{4}\right)dx = 4\pi\int_0^{\pi/4}\left(\tan^2 u - \sin^2 u\right)du$$

$$= 4\pi\int_0^{\pi/4}\left(\sec^2 u - 1 - \frac{1}{2}(1-\cos 2u)\right)du = 4\pi\left[\tan u - u - \frac{1}{2}\left(u - \frac{1}{2}\sin 2u\right)\right]_0^{\pi/4}$$

$$= 4\pi\left[\tan u - \frac{3}{2}u + \frac{1}{4}\sin 2u\right]_0^{\pi/4} = 4\pi\left(1 - \frac{3}{2}\cdot\frac{\pi}{4} + \frac{1}{4}\cdot 1 - 0\right) = 4\pi\left(\frac{5}{4} - \frac{3\pi}{8}\right) = \boxed{5\pi - \frac{3\pi^2}{2}}$$

(SINCE $\tan\frac{\pi}{4} = 1$ AND $\sin\frac{\pi}{2} = 1$) $= \boxed{\frac{\pi}{2}(10 - 3\pi)}$

(16)

$$\int_0^3 \frac{15x^2}{x^3-8}\,dx = \int_0^2 \frac{15x^2}{x^3-8}\,dx + \int_2^3 \frac{15x^2}{x^3-8}\,dx$$

a) $\displaystyle\int_0^2 \frac{15x^2}{x^3-8}\,dx = \lim_{T\to 2^-}\int_0^T \frac{15x^2}{x^3-8}\,dx$ LET $u = x^3 - 8$, $du = 3x^2\,dx$

$$= \lim_{T\to 2^-}\frac{15}{3}\int_0^T \frac{3x^2}{x^3-8}\,dx = \lim_{T\to 2^-} 5\left[\ln|x^3-8|\right]_0^T = \lim_{T\to 2^-} 5\left(\ln|T^3-8| - \ln 8\right) = -\infty,$$

(AS $T \to 2^-$, $|T^3 - 8| \to 0^+$ SO $\ln|T^3-8| \to -\infty$)

SO $\displaystyle\int_0^3 \frac{15x^2}{x^3-8}\,dx$ $\boxed{\text{DIVERGES}}$.

(OR SHOW THAT $\displaystyle\int_2^3 \frac{15x^2}{x^3-8}\,dx$ DIVERGES.)

(17)

$$\int_e^{\infty} \frac{1}{x(\ln x)^3}\,dx = \lim_{T\to\infty}\int_e^T \frac{1}{x(\ln x)^3}\,dx$$

LET $u = \ln x$, $du = \frac{1}{x}\,dx$

$$= \lim_{T\to\infty}\int_e^T \frac{1}{(\ln x)^3}\cdot\frac{1}{x}\,dx = \lim_{T\to\infty}\int_{x=e}^{x=T}\frac{1}{u^3}\,du = \lim_{T\to\infty}\int_{x=e}^{x=T} u^{-3}\,du = \lim_{T\to\infty}\left[-\frac{1}{2}u^{-2}\right]_{x=e}^{x=T}$$

$$= \lim_{T\to\infty}\left[-\frac{1}{2}(\ln x)^{-2}\right]_e^T = \lim_{T\to\infty} -\frac{1}{2}\left[\frac{1}{(\ln T)^2} - \frac{1}{1^2}\right] = -\frac{1}{2}\left[0 - 1\right] = \boxed{\frac{1}{2}}$$

(AS $T \to \infty$, $(\ln T)^2 \to \infty$ SO $\frac{1}{(\ln T)^2} \to 0$)

(18)

$$\int_0^{1/e^2} \frac{1}{x(\ln x)^3}\,dx = \lim_{T\to 0^+}\int_T^{1/e^2} \frac{1}{x(\ln x)^3}\,dx$$

LET $u = \ln x$, $du = \frac{1}{x}\,dx$

$$= \lim_{T\to 0^+}\int_T^{1/e^2} \frac{1}{(\ln x)^3}\cdot\frac{1}{x}\,dx = \lim_{T\to 0^+}\int_{x=T}^{x=e^{-2}}\frac{1}{u^3}\,du = \lim_{T\to 0^+}\int_{x=T}^{x=e^{-2}} u^{-3}\,du = \lim_{T\to 0^+}\left[-\frac{1}{2}u^{-2}\right]_{x=T}^{x=e^{-2}}$$

$$= \lim_{T\to 0^+}\left[-\frac{1}{2}(\ln x)^{-2}\right]_T^{e^{-2}} = \lim_{T\to 0^+} -\frac{1}{2}\left(\frac{1}{(-2)^2} - \frac{1}{(\ln T)^2}\right) = -\frac{1}{2}\left(\frac{1}{4} - 0\right) = \boxed{-\frac{1}{8}}$$

(SINCE $\ln e^{-2} = -2$ AND AS $T \to 0^+$, $\ln T \to -\infty$ SO $(\ln T)^2 \to \infty$ AND $\frac{1}{(\ln T)^2} \to 0$)

MATH 16B SOLUTIONS – Sec. 9.2

(18) $f(x) = K\sqrt{x}(1-x)$, $[0,1]$

$\int_0^1 K\sqrt{x}(1-x)\,dx = 1 \qquad K\int_0^1 (x^{1/2} - x^{3/2})\,dx = 1 \qquad K\left[\frac{2}{3}x^{3/2} - \frac{2}{5}x^{5/2}\right]_0^1 = 1 \qquad K\left(\frac{2}{3} - \frac{2}{5}\right) = 1$

$K\left(\frac{4}{15}\right) = 1 \qquad \boxed{K = \frac{15}{4}}$

(19) $f(x) = Ke^{-x/2}$, $[0,\infty)$

$\int_0^\infty Ke^{-x/2}\,dx = 1 \qquad \lim_{T\to\infty}\int_0^T Ke^{-x/2}\,dx = 1 \qquad K\lim_{T\to\infty}\left[-2e^{-x/2}\right]_0^T = 1 \qquad -2K\lim_{T\to\infty}(e^{-T/2} - 1) = 1$

$-2K\lim_{T\to\infty}\left(\frac{1}{e^{T/2}} - 1\right) = 1 \qquad -2K(0-1) = 1 \qquad 2K = 1 \qquad \boxed{K = \frac{1}{2}}$

(27) $f(t) = \frac{1}{30}$, $[0,30]$

a) $P(T \le 5) = \int_0^5 \frac{1}{30}\,dt = \left[\frac{1}{30}t\right]_0^5 = \frac{1}{30}(5) = \boxed{\frac{1}{6}}$

(29) $f(t) = \frac{1}{3}e^{-t/3}$, $[0,\infty)$

a) $P(T < 2) = \int_0^2 \frac{1}{3}e^{-t/3}\,dt = \frac{1}{3}\left[-3e^{-t/3}\right]_0^2 = -(e^{-2/3} - 1) = \boxed{1 - e^{-2/3}} \approx .49$

c) $P(T \ge 2) = 1 - P(T < 2) = 1 - (1 - e^{-2/3}) = \boxed{e^{-2/3}}$ (using part a) $\approx .51$

OR $P(T \ge 2) = \int_2^\infty \frac{1}{3}e^{-t/3}\,dt = \lim_{b\to\infty}\int_2^b \frac{1}{3}e^{-t/3}\,dt = \lim_{b\to\infty}\frac{1}{3}\left[-3e^{-t/3}\right]_2^b$

$= \lim_{b\to\infty} -(e^{-b/3} - e^{-2/3}) = \lim_{b\to\infty} -\left(\frac{1}{e^{b/3}} - e^{-2/3}\right) = -(0 - e^{-2/3}) = \boxed{e^{-2/3}} \approx .51$

(31) $f(t) = \frac{4}{3}e^{-\frac{4}{3}t}$, $[0,\infty)$

Let $u = -\frac{4}{3}t$, $du = -\frac{4}{3}\,dt$

$P(T < 1) = \int_0^1 \frac{4}{3}e^{-4/3\,t}\,dt = (-)\int_0^1 e^{-4/3\,t}\left((-)\frac{4}{3}\,dt\right)$

$= -\left[e^{-4/3\,t}\right]_0^1 = -(e^{-4/3} - 1) = \boxed{1 - e^{-4/3}} \approx .74$

(OR use $\int_0^1 \frac{4}{3}e^{-4/3\,t}\,dt = \frac{4}{3}\left[-\frac{3}{4}e^{-4/3\,t}\right]_0^1$)

(33) $f(x) = \frac{1}{36}xe^{-x/6}$, $[0,\infty)$

c) $P(X > 12) = 1 - P(X \le 12)$

$= 1 - \int_0^{12}\frac{1}{36}xe^{-x/6}\,dx = 1 - \frac{1}{36}\int_0^{12} xe^{-x/6}\,dx \longleftarrow$

u		dv
x	$+$	$e^{-x/6}\,dx$
1	$-$	$-6e^{-x/6}$
0		$36e^{-x/6}$

$= 1 - \frac{1}{36}\left[-6xe^{-x/6} - 36e^{-x/6}\right]_0^{12}$

$= 1 + \left[\frac{1}{6}xe^{-x/6} + e^{-x/6}\right]_0^{12}$

$= 1 + (2e^{-2} + e^{-2} - (0+1)) = 1 + 3e^{-2} - 1 = \boxed{3e^{-2}} = \boxed{\frac{3}{e^2}} \approx .41$