MATH 16B

SEC. 6.3 - PARTIAL FRACTIONS

TO FIND AN INTEGRAL OF THE FORM $\boxed{\int \frac{P(x)}{Q(x)}\,dx}$ WHERE $P(x)$ AND $Q(x)$ ARE POLYNOMIALS,

① **DIVIDE** $P(x)$ BY $Q(x)$ IF THE DEGREE OF $P(x) \geq$ THE DEGREE OF $Q(x)$.

② **FACTOR $Q(x)$** COMPLETELY OVER THE REAL NUMBERS

③ **WRITE $\frac{P(x)}{Q(x)}$ AS A SUM OF PARTIAL FRACTIONS:**

EACH FACTOR OF $Q(x)$ OF THE FORM $(ax+b)^n$ CORRESPONDS TO A SUM OF TERMS

$$\frac{C_1}{ax+b} + \frac{C_2}{(ax+b)^2} + \frac{C_3}{(ax+b)^3} + \cdots + \frac{C_n}{(ax+b)^n}.$$

EXAMPLES

a) $$\frac{5x+9}{x^2(x+2)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2} + \frac{D}{(x+2)^2} + \frac{E}{(x+2)^3}$$

b) $$\frac{x^2+5}{(x+3)(x^2-8x+16)} = \frac{x^2+5}{(x+3)(x-4)^2} = \frac{A}{x+3} + \frac{B}{x-4} + \frac{C}{(x-4)^2}$$

c) $$\frac{8x-3}{(x^2-4)(x^2-5x+6)} = \frac{8x-3}{(x+2)(x-2)^2(x-3)} = \frac{A}{x+2} + \frac{B}{x-2} + \frac{C}{(x-2)^2} + \frac{D}{x-3}$$

④ **SOLVE FOR THE COEFFICIENTS** BY MULTIPLYING THROUGH BY $Q(x)$ AND THEN

a) SUBSTITUTING VALUES OF x FOR WHICH $Q(x)=0$ AND/OR

b) EQUATING COEFFICIENTS OF LIKE POWERS OF x ON BOTH SIDES.

⑤ **INTEGRATE EACH TERM.**

EXAMPLE FIND $\int \frac{3x-5}{x^2-2x-3}\,dx$

$$\frac{3x-5}{x^2-2x-3} = \frac{3x-5}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1}$$

$$3x-5 = A(x+1) + B(x-3)$$

$x=3$: $\quad 4 = 4A \quad$ $A=1$

$x=-1$: $\quad -8 = -4B \quad$ $B=2$

$$\int \frac{3x-5}{x^2-2x-3}\,dx = \int\left(\frac{1}{x-3} + \frac{2}{x+1}\right)dx = \boxed{\ln|x-3| + 2\ln|x+1| + C}$$

EXAMPLE FIND $\int \frac{5x+8}{x(x-4)^2}\,dx$

$$\frac{5x+8}{x(x-4)^2} = \frac{A}{x} + \frac{B}{x-4} + \frac{C}{(x-4)^2}$$

$$5x+8 = A(x-4)^2 + Bx(x-4) + Cx$$

$x=0$: $\quad 8 = 16A \quad$ $A = 1/2$

$x=4$: $\quad 28 = 4C \quad$ $C = 7$

COEFF. OF x^2: $\quad 0 = A+B \quad$ $B = -1/2$

$$\int \frac{5x+8}{x(x-4)^2}\,dx = \int\left(\frac{1/2}{x} - \frac{1/2}{x-4} + \frac{7}{(x-4)^2}\right)dx = \boxed{\frac{1}{2}\ln|x| - \frac{1}{2}\ln|x-4| - 7(x-4)^{-1} + C}$$