MATH 17B

SOLUTIONS - SEC. 7.4

① $$\int_0^\infty 3e^{-6x}\,dx = \lim_{T\to\infty}\int_0^T 3e^{-6x}\,dx = \lim_{T\to\infty} 3\left[-\frac{1}{6}e^{-6x}\right]_0^T = \lim_{T\to\infty} -\frac{1}{2}\left(e^{-6T}-1\right)$$
$$= \lim_{T\to\infty} -\frac{1}{2}\left(\frac{1}{e^{6T}}-1\right) = -\frac{1}{2}(0-1) = \boxed{\frac{1}{2}}$$

② $$\int_0^\infty xe^{-x}\,dx = \lim_{T\to\infty}\int_0^T xe^{-x}\,dx$$
LET $u = x,\ dv = e^{-x}dx$
$du = dx,\ v = -e^{-x}$
$$= \lim_{T\to\infty}\left[-xe^{-x} - \int -e^{-x}\,dx\right]_0^T = \lim_{T\to\infty}\left[-xe^{-x} - e^{-x}\right]_0^T = \lim_{T\to\infty}\left((-Te^{-T} - e^{-T}) - (0-1)\right)$$
$$= \lim_{T\to\infty}\left(-\frac{T}{e^T} - \frac{1}{e^T} + 1\right) = \left(\lim_{T\to\infty}\frac{-T}{e^T}\right) - 0 + 1 = \left(\lim_{T\to\infty}\frac{-1}{e^T}\right) + 1 = 0 + 1 = \boxed{1}$$
(USING L'HOSPITAL'S RULE)

③ $$\int_0^\infty \frac{2}{1+x^2}\,dx = \lim_{T\to\infty}\int_0^T \frac{2}{1+x^2}\,dx = \lim_{T\to\infty}\left[2\tan^{-1}x\right]_0^T = \lim_{T\to\infty} 2\left(\tan^{-1}T - \tan^{-1}0\right)$$
$$= 2\left(\frac{\pi}{2} - 0\right) = \boxed{\pi}$$
(SINCE $\tan^{-1}T \to \frac{\pi}{2}$ AS $T\to\infty$)

④ $$\int_e^\infty \frac{dx}{x(\ln x)^2}\,dx = \lim_{T\to\infty}\int_e^T \frac{1}{(\ln x)^2}\cdot\frac{1}{x}\,dx$$
LET $u = \ln x$
$du = \frac{1}{x}\,dx$
WHEN $x = e,\ u = \ln e = 1$
$x = T,\ u = \ln T$
$$= \lim_{T\to\infty}\int_1^{\ln T}\frac{1}{u^2}\,du = \lim_{T\to\infty}\left[-\frac{1}{u}\right]_1^{\ln T} = \lim_{T\to\infty}\left(-\frac{1}{\ln T} - (-1)\right) = 0 + 1 = \boxed{1}$$

⑤ $$\int_1^\infty \frac{1}{x^{3/2}}\,dx = \lim_{T\to\infty}\int_1^T x^{-3/2}\,dx = \lim_{T\to\infty}\left[-2x^{-1/2}\right]_1^T = \lim_{T\to\infty} -2\left(T^{-1/2}-1\right)$$
$$= \lim_{T\to\infty} -2\left(\frac{1}{T^{1/2}}-1\right) = -2(0-1) = \boxed{2}$$

⑥ $$\int_{-\infty}^{-1}\frac{1}{1+x^2}\,dx = \lim_{T\to-\infty}\int_T^{-1}\frac{1}{1+x^2}\,dx = \lim_{T\to-\infty}\left[\tan^{-1}x\right]_T^{-1} = \lim_{T\to-\infty}\left(\tan^{-1}(-1) - \tan^{-1}T\right)$$
$$= -\frac{\pi}{4} - \left(-\frac{\pi}{2}\right) = \boxed{\frac{\pi}{4}}$$
(SINCE $\tan^{-1}T \to -\pi/2$ AS $T\to-\infty$)

⑨ $$\int_0^\infty \frac{x}{(1+x^2)^2}\,dx = \lim_{T\to\infty}\int_0^T \frac{x}{(1+x^2)^2}\,dx$$
LET $u = 1+x^2$
$du = 2x\,dx$
$$= \lim_{T\to\infty}\frac{1}{2}\int_0^T \frac{1}{(1+x^2)^2}\cdot 2x\,dx = \lim_{T\to\infty}\frac{1}{2}\int_{x=0}^{x=T}\frac{1}{u^2}\,du = \lim_{T\to\infty}\frac{1}{2}\left[-\frac{1}{u}\right]_{x=0}^{x=T} = \lim_{T\to\infty}\frac{1}{2}\left[-\frac{1}{1+x^2}\right]_0^T$$
$$= \lim_{T\to\infty}\frac{1}{2}\left(-\frac{1}{1+T^2} - (-1)\right) = \frac{1}{2}(0+1) = \boxed{\frac{1}{2}}$$

⑩ $$\int_0^\infty x^3e^{-x^4}\,dx = \lim_{T\to\infty}\int_0^T x^3e^{-x^4}\,dx$$
LET $u = -x^4$
$du = -4x^3\,dx$
$$= \lim_{T\to\infty}\left(-\frac{1}{4}\right)\int_0^T e^{-x^4}(-4)x^3\,dx = \lim_{T\to\infty} -\frac{1}{4}\left[e^{-x^4}\right]_0^T = \lim_{T\to\infty} -\frac{1}{4}\left(e^{-T^4}-1\right)$$
$$= \lim_{T\to\infty} -\frac{1}{4}\left(\frac{1}{e^{T^4}}-1\right) = -\frac{1}{4}(0-1) = \boxed{\frac{1}{4}}$$

(13) $\displaystyle\int_0^{\pi/2} \frac{\cos x}{\sqrt{\sin x}}\,dx = \lim_{t\to 0^+}\int_t^{\pi/2}\frac{\cos x}{\sqrt{\sin x}}\,dx$ LET $\underline{u=\sin x}$, $du=\cos x\,dx$; IF $x=t$, $u=\sin t$; $x=\pi/2$, $u=\sin\frac{\pi}{2}=1$

$\displaystyle= \lim_{t\to 0^+}\int_{\sin t}^{1}\frac{1}{\sqrt{u}}\,du = \lim_{t\to 0^+}\int_{\sin t}^{1}u^{-1/2}\,du = \lim_{t\to 0^+}\left[2u^{1/2}\right]_{\sin t}^{1} = \lim_{t\to 0^+}2\left(1-(\sin t)^{1/2}\right) = 2(1-0) = \boxed{2}$

(17) $\displaystyle\int_1^{\infty}\frac{1}{x^3}\,dx = \lim_{t\to\infty}\int_1^t x^{-3}\,dx = \lim_{t\to\infty}\left[-\tfrac{1}{2}x^{-2}\right]_1^t = \lim_{t\to\infty}-\tfrac{1}{2}\left(\tfrac{1}{t^2}-1\right) = -\tfrac{1}{2}(0-1) = \boxed{\tfrac{1}{2}}$

(18) $\displaystyle\int_1^{\infty}\frac{1}{x^{1/3}}\,dx = \lim_{t\to\infty}\int_1^t x^{-1/3}\,dx = \lim_{t\to\infty}\left[\tfrac{3}{2}x^{2/3}\right]_1^t = \lim_{t\to\infty}\tfrac{3}{2}\left(t^{2/3}-1\right) = \underline{\infty}$,

SO THE INTEGRAL $\boxed{\text{DIVERGES}}$.

(22) $\displaystyle\int_0^2\frac{1}{(x-1)^4}\,dx = \int_0^1\frac{1}{(x-1)^4}\,dx + \int_1^2\frac{1}{(x-1)^4}\,dx$

i) $\displaystyle\int_1^2\frac{1}{(x-1)^4}\,dx = \lim_{t\to 1^+}\int_t^2 (x-1)^{-4}\,dx$ LET $\underline{u=x-1}$, $du=dx$

$\displaystyle= \lim_{t\to 1^+}\int_{x=t}^{x=2}u^{-4}\,du = \lim_{t\to 1^+}\left[-\tfrac{1}{3}u^{-3}\right]_{x=t}^{x=2} = \lim_{t\to 1^+}\left[-\tfrac{1}{3}(x-1)^{-3}\right]_t^2 = \lim_{t\to 1^+}-\tfrac{1}{3}\left(1^{-3}-(t-1)^{-3}\right)$

$\displaystyle= \lim_{t\to 1^+}-\tfrac{1}{3}\left(1-\frac{1}{(t-1)^3}\right) = \underline{\infty}$ (SINCE AS $t\to 1^+$, $(t-1)^3\to 0^+$ SO $\frac{1}{(t-1)^3}\to\infty$),

SO $\displaystyle\int_0^2\frac{1}{(x-1)^4}\,dx$ $\boxed{\text{DIVERGES}}$.

(25) $\displaystyle\int_e^{\infty}\frac{dx}{x\ln x} = \lim_{t\to\infty}\int_e^t\frac{1}{x\ln x}\,dx$ LET $\underline{u=\ln x}$, $du=\frac{1}{x}\,dx$; IF $x=e$, $u=\ln e=1$; $x=t$, $u=\ln t$

$\displaystyle= \lim_{t\to\infty}\int_1^{\ln t}\frac{1}{u}\,du = \lim_{t\to\infty}\left[\ln u\right]_1^{\ln t} = \lim_{t\to\infty}\left(\ln(\ln t)-\underset{\downarrow\,0}{\ln 1}\right) = \lim_{t\to\infty}\ln(\ln t) = \underline{\infty}$,

SO THE INTEGRAL $\boxed{\text{DIVERGES}}$.

(26) $\displaystyle\int_1^e\frac{dx}{x\ln x} = \lim_{t\to 1^+}\int_t^e\frac{1}{x\ln x}\,dx$ LET $\underline{u=\ln x}$, $du=\frac{1}{x}\,dx$; IF $x=t$, $u=\ln t$; $x=e$, $u=\ln e=1$

$\displaystyle= \lim_{t\to 1^+}\int_{\ln t}^1\frac{1}{u}\,du = \lim_{t\to 1^+}\left[\ln u\right]_{\ln t}^1 = \lim_{t\to 1^+}\left(\underset{\downarrow\,0}{\ln 1}-\ln(\ln t)\right) = \lim_{t\to 1^+}-\ln(\ln t) = \underline{\infty}$

(SINCE AS $t\to 1^+$, $\ln t\to 0^+$ SO $\ln(\ln t)\to -\infty$), SO THE INTEGRAL $\boxed{\text{DIVERGES}}$.

[$\underline{\text{OR}}$ USE $\displaystyle\lim_{t\to 1^+}(\ln 1-\ln(\ln t)) = \lim_{t\to 1^+}\ln\left(\frac{1}{\ln t}\right) = \infty$ SINCE $\ln t\to 0^+$ AS $t\to 1^+$, SO $\frac{1}{\ln t}\to\infty$]

(31) $\displaystyle\int_0^{\infty}Ce^{-3x}\,dx = 1$

SINCE $\displaystyle\int_0^{\infty}e^{-3x}\,dx = \lim_{t\to\infty}\int_0^t e^{-3x}\,dx = \lim_{t\to\infty}\left[-\tfrac{1}{3}e^{-3x}\right]_0^t = \lim_{t\to\infty}-\tfrac{1}{3}\left(e^{-3t}-1\right)$

$\displaystyle= \lim_{t\to\infty}-\tfrac{1}{3}\left(\frac{1}{e^{3t}}-1\right) = -\tfrac{1}{3}(0-1) = \tfrac{1}{3}$,

$C\cdot\frac{1}{3} = 1$ SO $\boxed{C=3}$

(32) $\displaystyle\int_{-\infty}^{\infty} \frac{C}{1+x^2}\,dx = 1$

$\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$ WHERE

1) $\displaystyle\int_{0}^{\infty} \frac{1}{1+x^2}\,dx = \lim_{T\to\infty}\int_{0}^{T} \frac{1}{1+x^2}\,dx = \lim_{T\to\infty}\Big[\tan^{-1}x\Big]_0^T = \lim_{T\to\infty}(\tan^{-1}T - \tan^{-1}0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$ AND

2) $\displaystyle\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx = \lim_{T\to-\infty}\int_{T}^{0} \frac{1}{1+x^2}\,dx = \lim_{T\to-\infty}\Big[\tan^{-1}x\Big]_T^0 = \lim_{T\to-\infty}(\tan^{-1}0 - \tan^{-1}T) = 0 - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$,

SO $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \pi$ AND $C\pi = 1$ GIVES $C = \boxed{\frac{1}{\pi}}$

(35) a) SINCE $x^2 \ge x$ FOR $x \ge 1$, $-x^2 \le -x$ AND THUS $\underline{0 < e^{-x^2} \le e^{-x}}$ FOR $x \ge 1$.

b) SINCE $\displaystyle\int_1^{\infty} e^{-x}\,dx$ CONVERGES BECAUSE $\displaystyle\int_1^{\infty} e^{-x}\,dx = \lim_{T\to\infty}\int_1^T e^{-x}\,dx = \lim_{T\to\infty}\Big[-e^{-x}\Big]_1^T$

$\displaystyle = \lim_{T\to\infty} -(e^{-T} - e^{-1}) = -(0 - e^{-1}) = e^{-1}$,

$\displaystyle\int_1^{\infty} e^{-x^2}\,dx$ CONVERGES BY THE COMPARISON TEST SINCE $\underline{0 < e^{-x^2} \le e^{-x}}$ FOR $x \ge 1$.

(39) a) IF $x \ge 1$, THEN $x^2 \ge 1 \Rightarrow 3x^2 \ge 1 \Rightarrow 4x^2 \ge 1 + x^2 \Rightarrow 2x \ge \sqrt{1+x^2} \Rightarrow \underline{0 < \frac{1}{2x} \le \frac{1}{\sqrt{1+x^2}}}$

b) SINCE $\underline{\dfrac{1}{\sqrt{1+x^2}} \ge \dfrac{1}{2x} > 0}$ FOR $x \ge 1$ AND

$\displaystyle\int_1^{\infty} \frac{1}{2x}\,dx = \lim_{T\to\infty}\int_1^T \frac{1}{2x}\,dx = \lim_{T\to\infty}\frac{1}{2}\Big[\ln x\Big]_1^T = \lim_{T\to\infty}\frac{1}{2}(\ln T - \underbrace{\ln 1}_{0}) = \infty$,

$\displaystyle\int_1^{\infty} \frac{1}{\sqrt{1+x^2}}\,dx$ DIVERGES BY THE COMPARISON TEST.

(40) $\displaystyle\int_1^{\infty} \frac{1}{\sqrt{1+x^6}}\,dx$

SINCE $\underline{\dfrac{1}{\sqrt{1+x^6}} \le \dfrac{1}{\sqrt{x^6}} = \dfrac{1}{x^3}}$ FOR $x \ge 1$ AND

$\displaystyle\int_1^{\infty} \frac{1}{x^3}\,dx$ CONVERGES (AS IN #17),

$\displaystyle\int_1^{\infty} \frac{1}{\sqrt{1+x^6}}\,dx$ $\boxed{\text{CONVERGES}}$ BY THE COMPARISON TEST.