Midterm 1 Solutions

1. (24 points.) For each series below, if the series converges then find its sum; otherwise, state that it diverges.

(a) $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots$

$$\frac{1}{1 - \frac{1}{3}} = \frac{3}{2}$$

(b) $\sum_{n=1}^{\infty} \left(\frac{4}{3}\right)^n$

diverges since $\frac{4}{3} > 1$

(c) $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right) = \sum_{n=1}^{\infty} (\ln n - \ln(n+1))$

$$s_n = (\ln 1 - \cancel{\ln 2}) + (\cancel{\ln 2} - \cancel{\ln 3}) + \cdots + (\cancel{\ln n} - \ln(n+1))$$

$$= -\ln(n+1)$$ diverges

2. (24 points.) For each series below, state whether it converges or diverges.

(a) $\sum_{n=1}^{\infty} \frac{\sin n}{e^n}$

$$\sum_{n=1}^{\infty} \left|\frac{\sin n}{e^n}\right| \leq \sum_{n=1}^{\infty} \frac{1}{e^n},$$

which converges because it's a geometric series with $r = \frac{1}{e} < 1$

So $\sum_{n=1}^{\infty} \frac{\sin n}{e^n}$ converges (absolutely)

(b) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{3/2}}$

Integral test:

compare with

$$\int_2^{\infty} \frac{1}{x(\ln x)^{3/2}}\,dx =$$

$$\lim_{b\to\infty} \int_2^{b} \frac{1}{x(\ln x)^{3/2}}\,dx =$$

$$\lim_{b\to\infty} \left[-2(\ln x)^{-1/2}\right]_2^b = \frac{2}{\sqrt{\ln 2}}$$

converges

3. (24 points.) For each series below, state whether it converges or diverges.

(a) $\sum_{n=2}^{\infty} \frac{\sqrt{n}}{n^2-1}$

converges by limit comparison with $\sum \frac{1}{n^{3/2}}$

(b) $\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$

diverges by limit comparison with $\sum \frac{1}{n}$

(c) $\sum_{n=1}^{\infty} e^{1/n}$

diverges by n^{th} term test

(d) $\sum_{n=1}^{\infty} \frac{1}{n^2}$

convergent p-series $p>1$

4. (24 points.) For each series below, state whether it converges absolutely, converges conditionally, or diverges

(a) $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n(n+1)}$

CA, since

$\sum \frac{1}{n(n+1)}$ converges by direct comparison with

$\sum \frac{1}{n^2}$

(b) $\sum_{n=1}^{\infty}(-1)^n \frac{n}{n^2+n+1}$

CC

- converges by alt. series test
- $\sum \frac{n}{n^2+n+1}$ DIV by lim comparison with $\sum \frac{1}{n}$

(c) $\sum_{n=2}^{\infty}(-1)^n \frac{n^2}{n^2-n-1}$

diverges by n^{th} term test

(d) $\sum_{n=2}^{\infty}(-1)^n \frac{1}{\ln n}$

CC

- converges by alt. series test
- $\sum \frac{1}{\ln n}$ diverges by direct comparison with $\sum \frac{1}{n}$

5. (12 points.) Does the following series converge absolutely, converge conditionally, or diverge?

$$\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{\sqrt{n+1}}-\frac{1}{\sqrt{n}}\right)$$

converges absolutely

The series $\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n+1}}-\frac{1}{\sqrt{n}}\right)$

is telescoping with

$$s_n=\left(-\frac{1}{\sqrt{1}}+\cancel{\frac{1}{\sqrt{2}}}\right)+\left(\cancel{-\frac{1}{\sqrt{2}}}+\cancel{\frac{1}{\sqrt{3}}}\right)+\cdots+\left(\cancel{-\frac{1}{\sqrt{n}}}+\frac{1}{\sqrt{n+1}}\right)$$

$$=\frac{1}{\sqrt{n+1}}-1,$$ which

converges.

6. (24 points.) Consider the power series

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2 4^n}. \text{ — } a_n$$

(a) Find the interval of convergence.

$$|a_n|^{1/n} = \frac{1}{n^{2/n}} \left|\frac{x-2}{4}\right| \to \left|\frac{x-2}{4}\right|$$

$$\left|\frac{x-2}{4}\right| < 1 \quad \text{if} \quad -2 < x < 6$$

$x = -2$: $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$ converges

$x = 6$: $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

Answer: $[-2, 6]$

(b) Find the radius of convergence.

$R = 4$

(c) Find all values of x such that the series is conditionally convergent.

Never