21C: Midterm 2

2014-07-22

Last name ....Key.....

First name .....

Email .....

1. (24 points.) Suppose that $|x| < 1$. Find the sum of the series

$$1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots$$

$$= \frac{d}{dx}\left(1 + x + x^2 + x^3 + \cdots\right)$$

$$= \frac{d}{dx}\left(\frac{1}{1-x}\right)$$

$$= \frac{1}{(1-x)^2}$$

2. (24 points.) Use series to estimate the following integrals to within 10^{-8}.

a) $\int_0^{0.1} \sin x^2\, dx$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\int_0^{0.1} \sin x^2\, dx = \int_0^{0.1} \left[x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \cdots\right] dx$$

$$= \left[\frac{1}{3}x^3 - \frac{1}{3!}\frac{1}{7}x^7 + \frac{1}{5!}\frac{1}{11}x^{11} - \cdots\right]_0^{0.1}$$

$$= \frac{1}{3}(0.1)^3 - \underbrace{\frac{1}{3!}\frac{1}{7}(0.1)^7}_{\text{less than } 10^{-8}} + \cdots \approx \boxed{\frac{1}{3}(0.1)^3}$$

b) $\int_0^{0.1} e^{-x^2}\, dx$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots$$

$$\int_0^{0.1} e^{-x^2} dx = \int_0^{0.1} \left[1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots\right] dx$$

$$= \left[x - \frac{1}{3}x^3 + \frac{1}{2!}\frac{1}{5}x^5 - \frac{1}{3!}\frac{1}{7}x^7 + \cdots\right]_0^{0.1}$$

$$= 0.1 - \frac{1}{3}(0.1)^3 + \frac{1}{2!}\frac{1}{5}(0.1)^5 - \underbrace{\frac{1}{3!}\frac{1}{7}(0.1)^7 + \cdots}_{< 10^{-8}}$$

$$\approx \left[0.1 - \frac{1}{3}(0.1)^3 + \frac{1}{2!}\frac{1}{5}(0.1)^5\right)$$

3. (24 points.)

a) Find the Maclaurin polynomial of order 6 for e^{-x}.

$$P_6(x) = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \frac{x^6}{6!}$$

b) Use your answer from part (a) to estimate $1/e$.

$$P_6(-1) = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!}$$

c) Find an upper bound for the error of your estimate in part (b). Use the alternating series approximation.

$$\frac{1}{7!}$$

4. (24 points.) Let $f(x) = x^2 e^{x^2}$.

a) Find the MacLauren series for $f(x)$.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots$$

$$x^2 e^{x^2} = x^2 + x^4 + \frac{x^6}{2!} + \frac{x^8}{3!} + \frac{x^{10}}{4!} + \cdots$$

b) Find $f^{(7)}(0)$.

0

c) Find $f^{(8)}(0)$.

$$\frac{1}{3!} = \frac{f^{(8)}(0)}{8!}$$

$$\Rightarrow f^{(8)}(0) = \frac{8!}{3!}$$

d) Find $f^{(9)}(0)$.

0

5. (24 points.) Consider the plane that passes through the points $(0,1,0), (1,2,0)$, and $(0,2,1)$.

P Q R

a) Find a vector normal to the plane.

$\overrightarrow{PQ} = (1,1,0)$

$\overrightarrow{PR} = (0,1,1)$

$$\vec{n} = \begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = i - j + k$$

b) Find an equation for the plane.

$x - y + z = D$

$0 - 1 + 0 = D \Rightarrow D = -1$

$x - y + z = -1$

6. (12 points.) Find parametric equations for the line containing $(0, 2, 0)$ and $(1, 3, 5)$.

Q P

$$v = P + t\overrightarrow{PQ}$$

$$\overrightarrow{PQ} = \langle 1, 1, .5 \rangle$$

$$v = (0, 2, 0) + t(1, 1, .5)$$

$$x = 0 + t = t$$

$$y = 2 + t = 2 + t$$

$$z = 0 + (.5)t = .5t$$