Homework due: Tuesday 2/21/12

Problems

1. Recall that if $X_1, X_2, \ldots$ is a sequence of random variables, an event A is called a **tail event** for the sequence $(X_n)_n$ if for any $n \geq 1$, $A \in \sigma(X_{n+1}, X_{n+2}, \ldots)$ (i.e., the question of whether A occurred depends on the entire sequence but does not depend on any fixed initial number of terms in the sequence). **Kolmogorov's 0-1 law** is the famous result we proved last quarter that says that if the random variables $X_1, X_2, \ldots$ are independent, then the probability of any tail event is 0 or 1.

   Use Lévy's 0-1 law (see Theorem 3.25 in the lecture notes) to give a new proof of Kolmogorov's 0-1 law.

2. Compute the probability $p_{\text{extinction}}$ that a Galton-Watson process will become extinct if the distribution of the number of descendants is $p_0 = 1/8$, $p_1 = 3/8$, $p_2 = 3/8$, $p_3 = 1/8$. (Note that in the genealogical tree interpretation of the process, this distribution corresponds to the assumptions that each family has exactly 3 children, each child is male with probability 1/2, and only males pass on the family name.)

3. Let $1 < p \leq \infty$, and let $(X_i)_{i \in I}$ be a family of random variables. Assume that the family is bounded in $L_p(\Omega, \mathcal{F}, \mathbf{P})$, i.e., there exists some $C > 0$ such that $\|X_i\|_p \leq C$ for all $i \in I$.

   (a) Prove that the family $(X_i)_{i \in I}$ is uniformly integrable.

   (b) Give a counterexample demonstrating that the same assumption in the case $p = 1$ does not imply uniform integrability.

4. In this problem we discuss a version of the **optional stopping theorem**, an important result in martingale theory.

   Recall that a stopping time (relative to a given filtration $(\mathcal{G}_n)_{n=0}^{\infty}$) is a random variable $T : \Omega \to \mathbb{N} \cup \{\infty\}$ that has the property that for each $n \geq 0$, the event $\{T \leq n\}$ is in $\mathcal{G}_n$. Here, we will consider only almost surely finite stopping times, i.e., we assume that $\mathbf{P}(T = \infty) = 0$. If $(X_n)_{n=0}^{\infty}$ is a martingale and T is a stopping time, the goal is to understand when is it true that

   $$\mathbf{E}(X_T) = \mathbf{E}(X_0). \tag{1}$$

   Here X_T represents the value of the martingale stopped at time T. This equality is a version of the "you can't beat the system" principle. (But it's not always true — there are situations when you can beat the system, see below.)

   (a) Let $(S_n)_{n=0}^{\infty}$ denote the simple symmetric random walk on $\mathbb{Z}$, and let the stopping time T be defined by $T = \inf\{n \geq 1 : S_n = 1\}$. We proved in class that $T < \infty$ a.s. Does the equation (1) above hold? If not, why would this be a poor method for a gambler to use a gamble on a symmetric random walk as a guaranteed way of making money? (You might want to read the next part of the question before attempting to answer this.)

   (b) The formulation of the optional stopping theorem is as follows.

   Theorem (Optional stopping theorem)**.** *Let $(X_n)_{n=0}^{\infty}$ be a supermartingale, and let T be an a.s. finite stopping time.* If *any of the following assumptions holds:*

(i) T is bounded (i.e., there is an $M > 0$ such that $T \le M$);

(ii) The sequence X_n is uniformly bounded (i.e., there is an $M > 0$ such that $|X_n| \le M$ for all n);

(iii) $\mathbf{E}(T) < \infty$ and the martingale differences $X_{n+1} - X_n$ are uniformly bounded;

then *X_T is integrable and we have $\mathbf{E}(X_T) \le \mathbf{E}(X_0)$. If instead of a supermartingale we assume $(X_n)_n$ is a martingale, then under the same conditions, the conclusion is that $\mathbf{E}(X_T) = \mathbf{E}(X_0)$.*

Prove the theorem, using the following hints for each of the conditions (i)–(iii).

Hint for (i). We showed in class that $(X_{T\wedge n})_{n=0}^{\infty}$ is a supermartingale. Consider what that means when $n = M$ where M is the bound for T.

Hints for (ii) and (iii). Again consider the implications of $(X_{T\wedge n})_{n=0}^{\infty}$ being a supermartingale, this time when $n \to \infty$.

5. Use the optional stopping theorem formulated above to prove the following version of **Wald's equation** for random walks: if $S_n = \sum_{k=1}^{n} X_k$ is a random walk with i.i.d. steps satisfying $|X_n| \le M$ a.s. and mean value $\mu = \mathbf{E}(X_1)$, and T is a stopping time satisfying $\mathbf{E}(T) < \infty$, then $\mathbf{E}(S_T) = \mu \mathbf{E}(T)$.

   Notes. 1. The formula is still true if one assumes that $\mathbf{E}|X_1| < \infty$, without requiring that the X_n be uniformly bounded — for the (easy) proof see Theorem 4.1.5 (pages 185–186) in Durrett's book.

   2. Wald's equation is frequently applied in a situation in which the stopping time T is actually independent of the random walk. (The theorem is still valid in that case, since we can incorporate the additional randomness that goes into T into the filtration $(\mathcal{G}_n)_{n=1}^{\infty}$, defining $\mathcal{G}_n = \sigma(X_1, \ldots, X_n, T)$.) For example, if $X_1, X_2, \ldots$ is an i.i.d. sequence of Bernoulli(p) random variables, and $N \sim \text{Poisson}(\lambda)$ is independent of the X_n's, we can consider the sum

$$S_N = \sum_{k=1}^{N} X_k$$

   (a sum of a Poisson-random number of i.i.d. Bernoulli r.v.'s). This has the following intuitive interpretation known as "Poisson splitting": a stream of customers walk into a bank (say) at random times, such that the number of customers who came in during a certain hour of the day is a Poisson-distributed random variable N. Once at the bank, each customer either turns left or right (since they need to do business with one of two departments). The choice of whether to turn left or right is a Bernoulli variable with bias p, with each customer making the choice independently of the others. The number of customers who went into the department on the left is therefore represented by the random sum S_N above. Wald's equation gives the unsurprising result that

$$\mathbf{E}(S_N) = \mathbf{E}(N)\mathbf{E}(X_1) = \lambda p.$$

   As a bonus problem, try showing that S_N in fact has the Poisson distribution with mean λp, that $N - S_N$ has the Poisson distribution with mean $\lambda(1-p)$, and that S_N and $N - S_N$ are independent of each other.