Homework due: Thursday 3/8/12 (note extended deadline)

Problems

1. (a) Let $T : (0,1) \to (0,1)$ be a piecewise-smooth and piecewise strictly monotonic interval map. Let μ be a probability measure on $(0,1)$ (with the Borel sets) defined in terms of a density $f(x)$, i.e., $\mu(dx) = f(x)\,dx$. Show that T preserves the measure μ if and only if the following equation holds:

$$f(x) = \sum_{y \in T^{-1}(x)} \frac{1}{|T'(y)|} f(y),$$

where the sum ranges over all preimages y of x.

Hint: For a measure on $(0,1)$ (or more generally on $\mathbb{R}$), the measure preserving condition is clearly equivalent to the statement that if X is a random variable with distribution μ, then the r.v. $T(X)$ also has distribution μ. So, it is enough to know how to compute the density function of $T(X)$. This is the standard "density of a function of a random variable" formula from basic probability theory — see for example (for the special case in which T is one-to-one and increasing) exercise 6 on page 91 of my 235A lecture notes posted on the course web page.

(b) Show that the logistic map $L_r(x) = rx(1-x)$, in the case $r = 4$ (see pages 59–60 in the lecture notes), preserves the measure $\mu(dx) = \left(\pi\sqrt{x(1-x)}\right)^{-1} dx$ on $(0,1)$.

(c) Show that the Gauss map $G(x) = \left\{\frac{1}{x}\right\}$ (the fractional value of $1/x$) preserves the Gauss measure $\gamma(dx) = \frac{1}{\log 2} \cdot \frac{dx}{1+x}$.

(d) (Optional) Use a similar idea to show that **Boole's transformation** $T : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined by

$$T(x) = x - \frac{1}{x}$$

preserves Lebesgue measure λ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

Note. This is an example of an **infinite measure preserving system** — i.e., the measure space is not a probability space since it is equipped with a measure such that $\lambda(\Omega) = \infty$. However, the definition of a measure preserving system remains the same. The study of such systems is a sub-branch of ergodic theory known as **infinite ergodic theory**.

2. Let $(\Omega, \mathcal{F}, \mathbf{P}, T)$ be a measure preserving system. Prove that the set

$$\mathcal{I} = \{A \in \mathcal{F} \,:\, A \text{ is } T\text{-invariant}\}$$

is a σ-algebra.

3. Let $(\Omega, \mathcal{F}, \mathbf{P}, T)$ be a measure preserving system. Prove that a random variable X is T-invariant (meaning that $X = X \circ T$ a.s.) if and only if it is measurable with respect to the σ-algebra $\mathcal{I}$ of invariant events. Use this to show that the system is ergodic if and only if any invariant random variable is almost surely constant.

4. (a) Prove that a measure preserving system $(\Omega, \mathcal{F}, \mathbf{P}, T)$ is ergodic if and only if the probability measure $\mathbf{P}$ cannot be represented in the form

$$\mathbf{P} = \alpha Q_1 + (1-\alpha) Q_2, \tag{1}$$

where $0 < \alpha < 1$ and Q_1, Q_2 are two T-invariant probability measures on the measurable space $(\Omega, \mathcal{F})$ which are **singular**, i.e., such that there exist disjoint sets $A, B \in \mathcal{F}$ such that $\Omega = A \cup B$, $Q_1(B) = 0$ and $Q_2(A) = 0$.

(b) (Optional) Prove the following stronger version of the above result: $(\Omega, \mathcal{F}, \mathbf{P}, T)$ is ergodic if and only if the probability measure $\mathbf{P}$ cannot be represented in the form (1) where $0 < \alpha < 1$ and Q_1, Q_2 are two distinct T-invariant probability measures on the measurable space $(\Omega, \mathcal{F})$.

Hint. The easy "if" direction is similar to part (a) above. The other direction is more difficult. It can be done with the help of Von Neumann's L_2 ergodic theorem. Argue as follows: assume that $\mathbf{P}$ is ergodic and has a representation of the form (1) where $0 < \alpha < 1$ where the probability measures Q_1, Q_2 are T-invariant. The goal is to prove that $Q_1 = Q_2 = \mathbf{P}$. Take an arbitrary r.v. X on $(\Omega, \mathcal{F}, \mathbf{P})$, and denote $\mu_X^{(n)} = \frac{1}{n}\sum_{k=0}^{n-1} X \circ T^k$ (the nth ergodic average of X).

$$\mu_X^{(n)} \xrightarrow[n\to\infty]{L_2(\mathbf{P})} \mathbf{E}_{\mathbf{P}} X,$$

which more explicitly means that

$$\mathbf{E}_P \left(\mu_X^{(n)} - \mathbf{E}_{\mathbf{P}} X \right)^2 \to 0 \text{ as } n \to \infty.$$

What can you deduce from this regarding the limits of $\mu_X^{(n)}$ in the spaces $L_2(Q_1)$ and $L_2(Q_2)$?

5. Let $(\Omega, \mathcal{F}, \mathbf{P}, T)$ be a measure preserving system.

(a) Prove that if T^2 is ergodic then T is ergodic.

(b) Give a counterexample showing that the converse is not true: if T is ergodic then T^2 is not necessarily ergodic.

Hint: Start with some ergodic system; build a new system whose sample space consists of two disjoint copies of the sample space of the original system, and whose measure preserving map is defined in a clever way in terms of the map of the original system.