Homework Set No. 2 – Probability Theory (235A), Fall 2011

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1. (a) Let X be a random variable with distribution function F_X and piecewise continuous density function f_X. Let $[a,b] \subset \mathbb{R}$ be an interval (possibly infinite) such that

$$\mathbf{P}(X \in [a,b]) = 1,$$

and let $g : [a,b] \to \mathbb{R}$ be a monotone (strictly) increasing and differentiable function. Prove that the random variable $Y = g(X)$ (this is the function on Ω defined by $Y(\omega) = g(X(\omega))$, in other words the composition of the two functions g and X) has density function

$$f_Y(x) = \begin{cases} \frac{f_X(g^{-1}(x))}{g'(g^{-1}(x))} & x \in (g(a), g(b)), \\ 0 & \text{otherwise.} \end{cases}$$

(b) If $\lambda > 0$, we say that a random variable has the exponential distribution with parameter λ if

$$F_X(x) = \begin{cases} 0 & x < 0, \\ 1 - e^{-\lambda x} & x \geq 0, \end{cases}$$

and denote this $X \sim \mathrm{Exp}(\lambda)$. Find an algorithm to produce a random variable with $\mathrm{Exp}(\lambda)$ distribution using a random number generator that produces uniform random numbers in $(0,1)$. In other words, if $U \sim U(0,1)$, find a function $g : (0,1) \to \mathbb{R}$ such that the random variable $X = g(U)$ has distribution $\mathrm{Exp}(\lambda)$.

(c) We say that a non-negative random variable $X \geq 0$ has the **lack of memory property** if it satisfies that

$$\mathbf{P}(X \geq t \mid X \geq s) = \mathbf{P}(X \geq t - s) \qquad \text{for all } 0 < s < t.$$

Prove that exponential random variables have the lack of memory property.

(d) Prove that any non-negative random variable that has the lack of memory property has the exponential distribution with some parameter $\lambda > 0$. (This is easier if one assumes that the function $G(x) = \mathbf{P}(X \geq x)$ is differentiable on $[0,\infty)$, so you can make this assumption if you fail to find a more general argument).

2. (a) Prove the **inclusion-exclusion principle**: If $A_1, \ldots, A_n$ are events in a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, then

$$\mathbf{P}\left(\bigcup_{k=1}^{n} A_k\right) = s_1 - s_2 + s_3 - s_4 + s_5 - \ldots + (-1)^{n-1} s_n,$$

where

$$\begin{aligned}
s_1 &= \mathbf{P}(A_1) + \mathbf{P}(A_2) + \ldots + \mathbf{P}(A_n) = \sum_{k=1}^{n} \mathbf{P}(A_k), \\
s_2 &= \sum_{1 \le k_1 < k_2 \le n} \mathbf{P}(A_{k_1} \cap A_{k_2}), \\
s_3 &= \sum_{1 \le k_1 < k_2 < k_3 \le n} \mathbf{P}(A_{k_1} \cap A_{k_2} \cap A_{k_3}), \\
&\vdots \\
s_d &= \sum_{1 \le k_1 < \ldots < k_d \le n} \mathbf{P}(A_{k_1} \cap A_{k_2} \cap \ldots \cap A_{k_d}), \\
&\vdots \\
s_n &= \mathbf{P}(A_1 \cap A_2 \cap \ldots \cap A_n).
\end{aligned}$$

(b) N letters addressed to different people are inserted at random into N envelopes that are labelled with the names and addresses of the N recipients, such that all $N!$ possible matchings between the letters and envelopes are equally likely. What is the probability of the event that no letter will arrive at its intended destination? Compute this probability for any N, and in the limit when $N \to \infty$.

3. Let F be the distribution function

$$F(x) = \begin{cases} 0 & x < 0, \\ \frac{1}{3} + \frac{1}{6}x & 0 \le x < 1, \\ \frac{1}{2} & 1 \le x < 2, \\ 1 - \frac{1}{4}e^{2-x} & x \ge 2. \end{cases}$$

Compute the lower and upper quantile functions of F, defined by

$$\begin{aligned} X_*(p) &= \sup\{x : F(x) < p\}, \\ X^*(p) &= \inf\{x : F(x) > p\}, \end{aligned} \qquad (0 < p < 1).$$

A recommended way is to plot F on paper and then figure out the quantiles by "eyeballing". Of course, the answer should be spelled out in precise formulas.

4. A drunken archer shoots at a target hanging on a wall 1 unit of distance away. Since he is drunk, his arrow ends up going in a random direction at an angle chosen uniformly in $(-\pi/2, \pi/2)$ (an angle of 0 means he will hit the target precisely) until it hits the wall. Ignoring gravity and the third dimension, compute the distribution function (and density function if it exists) of the random distance from the hitting point of the arrow to the target.

5. (Optional question - solution not required, but recommended)
(a) Let $(\Omega_1, \mathcal{F}_1, \mathbf{P}_1)$ be a probability space, let $(\Omega_2, \mathcal{F}_2)$ be a measurable space, and let $f : \Omega_1 \to \Omega_2$ be a measurable function. Verify that the function $\mathbf{P}_2 : \mathcal{F}_2 \to [0, 1]$ defined by

$$\mathbf{P}_2(A) = \mathbf{P}_1(f^{-1}(A))$$

is a probability measure. This probability measure is called the **push-forward measure of $\mathbf{P}_1$ under f**.

(b) For a real number x, denote the **integer part of** x by

$$\lfloor x \rfloor = \sup\{n \in \mathbb{Z} : n \leq x\},$$

and denote the **fractional part of** x by

$$\{x\} = x - \lfloor x \rfloor.$$

Let $((0, 1), \mathcal{B}, \mathbf{P})$ be the unit interval with the σ-algebra of Borel subsets and the Lebesgue probability measure, corresponding to the experiment of choosing a uniform random number in $(0, 1)$. Define a sequence of functions $R_1, R_2, \ldots : (0, 1) \to \mathbb{R}$ by

$$R_n(x) = \begin{cases} 0 & 0 \leq \{2^{n-1}x\} < 1/2, \\ 1 & 1/2 \leq \{2^{n-1}x\} < 1. \end{cases}$$

For any $n \in \mathbb{N}$ and $a_1, a_2, \ldots, a_n \in \{0,1\}$, denote by $B_n(a_1, \ldots, a_n)$ the set

$$B_n(a_1, \ldots, a_n) = \{x \in (0,1) : R_1(x) = a_1, R_2(x) = a_2, \ldots, R_n(x) = a_n\}.$$

Find a good explicit description for this set ("the set of all x's such that ..."), and deduce from it that

$$\mathbf{P}(B_n(a_1, \ldots, a_n)) = \frac{1}{2^n}.$$

(c) Define a function $f : (0,1) \to \{0,1\}^{\mathbb{N}}$ by

$$f(x) = (R_1(x), R_2(x), R_3(x), \ldots).$$

Prove that f is a measurable function when the space $\{0,1\}^{\mathbb{N}}$ is equipped with the σ-algebra generated by the sets

$$A_n(1) = \{(x_1, x_2, \ldots) \in \{0,1\}^{\mathbb{N}} : x_n = 1\}.$$

(d) Prove that the push-forward of Lebesgue measure under f is the probability measure corresponding to the random experiment of an infinite sequence of fair coin tosses.