MATH 765-HOMEWORK 1 (DUE ON JAN 30)

(1) Find the integral curves of $-y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}$ and $x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$ on $\mathbb{R}^2$.

(2) Consider $D_n = \{(u_1, u_2, \cdots, u_n)| |u| < 1\} \subset \mathbb{R}^n$ with metric given by $g = \sum_{i=1}^{n} \frac{4(du_i)^2}{(1 - |u|^2)^2}$. Consider $H^n = \{x_0^2 - x_1^2 - x_2^2 - \cdots - x_n^2 = 1|x_0 \geq 1\} \subset \mathbb{R}^{n,1}$ where $\mathbb{R}^{n,1}$ is $\mathbb{R}^{n+1}$ with the pseudo Riemannian metric $\tilde{g} = -(dx_0)^2 + (dx_1)^2 + \cdots + (dx_n)^2$.

Show that $\tilde{g}$ induces a Riemannian metric on $H^n$ and that $D_n$ is isometric to $H^n$.

Hints: 1. The connected component containing the identity of the group of isometries of $\mathbb{R}^{n,1}$ acts on $H^n$ transitively. Thus it suffices to check $\tilde{g}$ is positive-definite on $T_pH^n$ at one point $p = (1,0,\cdots,0)$. 2. Consider the map given by $u_i = \frac{x_i}{1 + x_0}, 1 \leq i \leq n$ from $H^n$ to $D_n$.

(3) Let $\mathbb{H} = \{z = x + iy|x, y \in \mathbb{R}, y > 0\}$, with metric given by $g = \frac{(dx)^2 + (dy)^2}{y^2}$.

• Show that $\mathbb{H}$ is isometric to $D_2$ in problem 1.

• Show that $SL(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right| a, b, c, d \in \mathbb{R}, ad - bc = 1 \}$ acts on $\mathbb{H}$ by isometries, where the action is given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}$. 
