1. (Problem 2 on page 78 of Do Carmo’s Riemannian geometry) It is possible to introduce a Riemannian metric on the tangent bundle $TM$ of a Riemannian manifold $M$ in the following manner. Let $(p, v) \in TM$ and $V, W$ be tangent vectors in $TM$ at $(p, v)$. Choose curves in $TM$

$$\alpha : t \rightarrow (p(t), v(t)), \beta : t \rightarrow (q(t), w(t)),$$

with $p(0) = q(0) = p, v(0) = w(0) = v$ and $V = \alpha'(0), W = \beta'(0)$. Define an inner product on $TM$ by

$$< V, W >_{(p,v)} = < \pi_* V, \pi_* W >_p + < \frac{DV}{dt}(0), \frac{DW}{dt}(0) >_p,$$

where $\pi_*$ is the tangent map induced by the natural projection $\pi : TM \rightarrow M$.

(a) Show that this inner product is well-defined and introduces a Riemannian metric on $TM$.

(b) A tangent vector $V$ in $TM$ at $(p, v)$ is called vertical if $\pi_* V = 0$, i.e. if $V$ is tangent to the fiber $\pi^{-1}(p) \cong T_p M$. A tangent vector $W$ in $TM$ at $(p, v)$ is called horizontal if it is orthogonal to $T_p M$ under the above metric. Show that

$$< V, V >_{(p,v)} = < V, V >_p, \text{ if } V \text{ is vertical}$$

$$< W, W >_{(p,v)} = < \pi_* W, \pi_* W >_p, \text{ if } W \text{ is horizontal}$$

(c) A curve $t \rightarrow (p(t), v(t))$ in $TM$ is horizontal if its tangent vector is horizontal for all $t$. Show that the curve $t \rightarrow (p(t), v(t))$ is horizontal if and only if the vector field $v(t)$ is parallel along $p(t)$ in $M$.

2. Consider $S^2$ with the standard metric induced from $\mathbb{R}^2$. For any (piecewise smooth) loop $\gamma$ based at $p = (0, 0, 1)$ in $S^2$, let $P_\gamma : T_p S^2 \rightarrow T_p S^2$ be the parallel transport (under Levi-Civita connection) along $\gamma$.

(a) Show that $P_\gamma$ is the identity map if $\gamma$ is a great circle.

(b) Find $\gamma$ such that $P_\gamma = -id$.

3. Show that great circles in $S^n$ are geodesics, where $S^n$ is given the metric induced from the standard metric on $\mathbb{R}^{n+1}$. 