

Vectors & Matrices (Review)

Note Title

LECTURE 03

★ Matrix - Vector Multiplication

$$\mathbf{x} \in \mathbb{R}^n$$

$$A = (a_{ij}) \in \mathbb{R}^{m \times n}$$

i.e., m rows $\times$ n cols

$$m \left\{ \begin{bmatrix} \vdots \\ \vdots \\ \vdots \end{bmatrix} \begin{bmatrix} \vdots \\ \vdots \\ \vdots \end{bmatrix} \right\}^n$$

$A \quad \mathbf{x}$

$$\text{Now, } \mathbf{y} = \underline{A\mathbf{x}} \in \mathbb{R}^m$$

Very important to notice that
 $\mathbf{y}$ is a linear combination of
the column vectors of A .

$$\mathbf{y} = A\mathbf{x} = [a_1 | a_2 | \dots | a_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

a_j is the j th col. vector of A .
 $\underbrace{\quad}_{\mathbb{R}^m}$

a_j is often written as $a_{\cdot j}$

$$\mathbf{y} = \underline{x_1 a_1 + x_2 a_2 + \dots + x_n a_n}$$

$$y_i = x_1 a_{i1} + x_2 a_{i2} + \dots + x_n a_{in}$$

$$= \sum_{j=1}^n x_j a_{ij} = \sum_{j=1}^n a_{ij} x_j, \quad 1 \leq i \leq m$$

Thm Let $F_A : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a map
defined as

$$F_A : \underbrace{\mathbf{x}}_{\mathbb{R}^n} \mapsto \underbrace{A\mathbf{x}}_{\mathbb{R}^m}$$

Then, F_A is a linear map.
i.e., $\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n, \forall \alpha \in \mathbb{R}$.

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$$\begin{cases} F_A(x+y) = F_A(x) + F_A(y) \\ F_A(\alpha x) = \alpha F_A(x) \end{cases}$$

Conversely, for any linear map $F: \mathbb{R}^n \rightarrow \mathbb{R}^m$,
there exists a unique matrix $A \in \mathbb{R}^{m \times n}$,
s.t. $F = F_A$.

(Proof) It's easy to prove F_A is a linear map. ($A(x+y) = Ax + Ay$, $A(\alpha x) = \alpha Ax$ using the definition of a matrix-vector product.)

Showing the converse is more challenging.

Let F be a linear map

Let $\{e_1, \dots, e_n\}$ be the canonical basis of $\mathbb{R}^n$, i.e., $e_j = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow j$, $1 \leq j \leq n$

Set $F(e_j) = a_j \in \mathbb{R}^m$

Let $A = [a_1 \dots a_n] \in \mathbb{R}^{m \times n}$

Now pick any $x \in \mathbb{R}^n$, we can always write $x = x_1 e_1 + \dots + x_n e_n$

$$\begin{aligned} \text{Then } F(x) &= F(x_1 e_1 + \dots + x_n e_n) \\ &= F(x_1 e_1) + \dots + F(x_n e_n) \\ &= x_1 F(e_1) + \dots + x_n F(e_n) \\ &= x_1 a_1 + \dots + x_n a_n \\ &= Ax = F_A(x) \end{aligned}$$

About the uniqueness, let $A, B \in \mathbb{R}^{m \times n}$

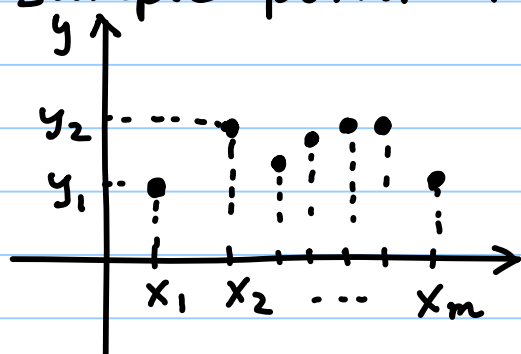
Suppose $F_A = F_B$. Then

$$F_A(e_j) = a_j = F_B(e_j) = b_j, 1 \leq j \leq n \Rightarrow A = B$$

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Example : A Vandermonde Matrix

Let $\{x_1, \dots, x_m\}$ be a set of sample points. For example, y_i may be temperature of this room measured at time x_i , $i=1, \dots, m$



Assume $x_i \neq x_j$
if $i \neq j$

Now consider a space of polynomials

$$P_{n-1}[x] := \{p(x) = c_0 + c_1x + \dots + c_{n-1}x^{n-1},$$

$$c_j \in \mathbb{R}, j=0, \dots, n-1\}$$

It is clear that $P_{n-1}[x]$ is a linear (vector) space since

$\forall p, q \in P_{n-1}[x]$, easy to check

$$p + q \in P_{n-1}[x],$$

$$\alpha p \in P_{n-1}[x], \quad \forall \alpha \in \mathbb{R}$$

Hence a map from a coefficient vector $\mathbf{c} = [c_0, \dots, c_{n-1}]^T \in \mathbb{R}^n$ to the vectors of sampled polynomial values $\mathbf{y} = [p(x_1), \dots, p(x_m)]^T \in \mathbb{R}^m$ is linear! Say $\mathbf{y} = \tilde{F}(\mathbf{c})$
 $\tilde{F}$ linear map

So, according to the theorem in the previous page, $\exists A \in \mathbb{R}^{m \times n}$ for such F .

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What is this matrix A ?

$$A = \begin{bmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_m & x_m^2 & \dots & x_m^{n-1} \end{bmatrix} \in \mathbb{R}^{m \times n}$$

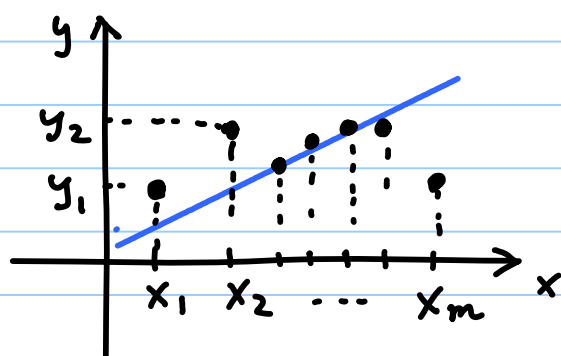
which is called the $m \times n$

Vandermonde matrix.

It's clear that

$$y = A \mathbb{C} \Leftrightarrow \begin{bmatrix} p(x_1) \\ \vdots \\ p(x_m) \end{bmatrix} = \begin{bmatrix} 1 & x_1 & \dots & x_1^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_m & \dots & x_m^{n-1} \end{bmatrix} \begin{bmatrix} c_0 \\ \vdots \\ c_{n-1} \end{bmatrix}$$

This matrix is often used in the least squares polynomial fitting to a set of measurements or noisy data.



In the case of a line fitting, $n = 2$.

But you may have many points, i.e., m : large.

Then, you want to find a line s.t., the size of $y - A \mathbb{C}$ is small.
residual error

Note in the case of a line fitting, $\mathbb{C} = \begin{bmatrix} c_0 \\ c_1 \end{bmatrix}$. We'll discuss this problem later.

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★ Matrix - Matrix Multiplication

$$C = A B$$

$$A \in \mathbb{R}^{m \times k}, B \in \mathbb{R}^{k \times n} \\ \Rightarrow C \in \mathbb{R}^{m \times n}$$

Note that

$$[c_1 \cdots c_n] = [a_1 \cdots a_k] [b_1 \cdots b_n]$$

$$\text{i.e., } c_j = A b_j \quad 1 \leq j \leq n$$

So each c_j is a linear combination of column vectors of A with the coefficient vector b_j .

Example 1. Outer product

$$\text{Let } u \in \mathbb{R}^m = \mathbb{R}^{m \times 1}, \\ v \in \mathbb{R}^n = \mathbb{R}^{n \times 1}.$$

Then, the outer product between u and v is :

$$u v^T = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} [v_1 \cdots v_n] = \begin{bmatrix} u_1 v_1 & \cdots & u_1 v_n \\ \vdots & & \vdots \\ u_m v_1 & \cdots & u_m v_n \end{bmatrix} \in \mathbb{R}^{m \times n}$$

This matrix has rank 1 because

$$u v^T = [v_1 u, \cdots, v_n u]$$

i.e., each column is just a scalar multiple of the same vector u .

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Example 2.

$$B = AR, \quad R: \text{upper triangular}$$

$$\begin{matrix} \mathbb{R}^{m \times n} & \mathbb{R}^{m \times n} & \mathbb{R}^{n \times n} \\ \uparrow & \uparrow & \uparrow \\ \mathbb{R}^{m \times n} & \mathbb{R}^{m \times n} & \mathbb{R}^{n \times n} \end{matrix}$$

$$R = \begin{bmatrix} 1 & \cdots & 1 \\ & \ddots & \vdots \\ & & 1 \\ \bigcirc & & & 1 \end{bmatrix}$$

all entries below the main diagonal $\equiv 0$

$$[b_1 \cdots b_n] = [a_1 \cdots a_n] \begin{bmatrix} 1 & \cdots & 1 \\ & \ddots & \vdots \\ & & 1 \\ \bigcirc & & & 1 \end{bmatrix}$$

$$\begin{aligned} \text{So, } b_j &= A r_j \\ &= [a_1 \cdots a_n] \begin{bmatrix} 1 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_j \\ &= a_1 + \cdots + a_j = \sum_{k=1}^j a_k \end{aligned}$$