

Continuation of Vector/Matrix Review

Note Title

LECTURE 04

★ Range & Nullspace (or Kernel)

Def. $A \in \mathbb{R}^{m \times n}$.

$$\text{range}(A) := \{y \in \mathbb{R}^m \mid y = A\mathbb{x}, \mathbb{x} \in \mathbb{R}^n\}$$

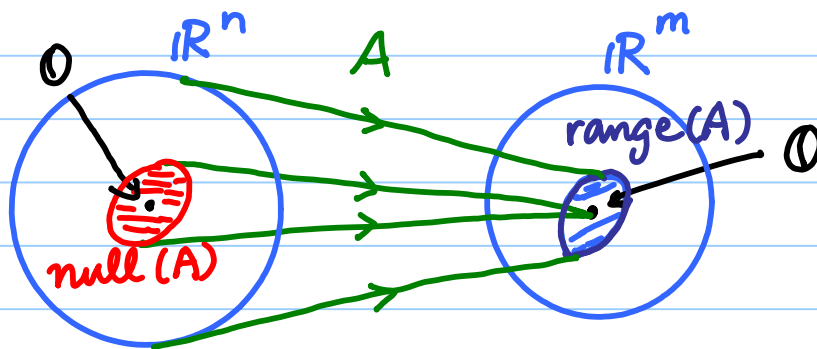
often written as $\text{Ran}(A)$ or $\text{Im}(A)$.

This is also called the image column space of A .

$$\text{null}(A) := \{\mathbb{x} \in \mathbb{R}^n \mid A\mathbb{x} = \mathbf{0}\}$$

is called the nullspace (or kernel) of A

$\text{Ker}(A)$



Thm $\text{range}(A) = \text{span}\{a_1, \dots, a_n\}$
= a set of all possible
linear combi. of $\{a_1, \dots, a_n\}$

(Proof) Need to show two things

(1) $\text{range}(A) \subset \text{span}\{a_1, \dots, a_n\}$

(2) $\text{span}\{a_1, \dots, a_n\} \subset \text{range}(A)$

Now, (1) is easy since any $y \in \text{range}(A)$
by definition, $\exists \mathbb{x} \in \mathbb{R}^n$ s.t. $y = A\mathbb{x}$.

This is a lin. combi. of col vectors of A
So, $y \in \text{span}\{a_1, \dots, a_n\}$.

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(2) Take any $y \in \text{span}\{a_1, \dots, a_n\}$.
By definition, $\exists \{x_1, \dots, x_n\}$ s.t.
 $y = x_1 a_1 + \dots + x_n a_n = A x \in \text{range}(A)$
by setting $x = (x_1, \dots, x_n)^T$ //

★ Linear Independence, Bases

Def. The vectors $\{a_1, \dots, a_n\}$, $a_j \in \mathbb{R}^m$
are called **linearly independent** if
$$\sum_{j=1}^n x_j a_j = 0 \iff x_j = 0, 1 \leq j \leq n$$

A set of m linearly independent
vectors in $\mathbb{R}^m$ is called a **basis**
in $\mathbb{R}^m$. $\Rightarrow$ a matrix representation
of a basis in $\mathbb{R}^m$ is an $m \times m$ matrix.
Note that any vector in $\mathbb{R}^m$ can be
written as a lin. combi. of the m
basis vectors in $\mathbb{R}^m$

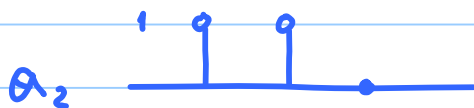
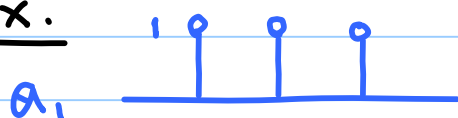
Def. The **dimension** of $\text{span}\{a_1, \dots, a_n\}$
is the maximal number of linearly
independent vectors among $\{a_1, \dots, a_n\}$

i.e., if $\exists j$, $a_j = x_1 a_1 + \dots + x_{j-1} a_{j-1}$
 $+ x_{j+1} a_{j+1} + \dots + x_n a_n$

then such a_j is useless in some sense
(or more precisely, it is redundant).

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Ex.



In $\mathbb{R}^3$, these are linearly dependent.

$$a_1 = a_2 + a_3$$

So $\dim \text{span}\{a_1, a_2, a_3\} = 2$.

We cannot write any vector in $\mathbb{R}^3$ by a lin. combi. of $\{a_2, a_3\}$.

Only a certain subset of vectors in $\mathbb{R}^3$ can be written as a lin. combi of $\{a_2, a_3\}$ (no control on the first and second entries of a vector in $\mathbb{R}^3$.)

* Rank

Def. The **column rank** of A

$$:= \dim(\text{range}(A))$$

$$= \# \text{ of linearly indep. col. vec's of } A.$$

The **row rank** of A

$$:= \dim(\text{range}(A^T))$$

$$= \# \text{ of linearly indep. row vec's of } A.$$

$$\text{rank}(A) := \dim(\text{range}(A))$$

$A \in \mathbb{R}^{m \times n}$ is said to be of **full rank** if $\text{rank}(A) = \min(m, n)$.

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Thm. $A \in \mathbb{R}^{m \times n}$, $m \geq n$ is of full rank
 $\Leftrightarrow \forall x, y \in \mathbb{R}^n$, $x \neq y$,
 $Ax \neq Ay$.

(Proof) $[\Rightarrow]$ If $\text{rank}(A) = n$, i.e., full rank,
then $\{a_1, \dots, a_n\}$ are lin. indep.

So, they form a basis of $\text{range}(A)$.

This means that $\forall b \in \text{range}(A)$,
 $\exists! x \in \mathbb{R}^n$ s.t. $b = \sum_{j=1}^n x_j a_j$.
there exists a unique ...

$[\Leftarrow]$ Suppose A is not of full rank.

Then $\{a_1, \dots, a_n\}$ are linearly dep.

i.e., $\exists c \in \mathbb{R}^n$, $c \neq 0$ s.t.

$$\sum_{j=1}^n c_j a_j = 0, \text{ i.e., } Ac = 0.$$

Then set $y = x + c \neq x$.

$$\begin{aligned} \text{But } Ay &= A(x + c) = Ax + \underbrace{Ac}_{=0} \\ &= Ax \quad \text{contradiction!} \quad \# \end{aligned}$$

★ Inverse

Def. A is said to be **nonsingular**
or **invertible** $\Leftrightarrow A$ is square and
of full rank

Hence $A \in \mathbb{R}^{m \times m}$: nonsingular

$\Rightarrow \{a_1, \dots, a_m\}$ form a basis of $\mathbb{R}^m$

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That means: the canonical basis vector $e_j \in \mathbb{R}^m$ can also be written as a lin. combi. of $\{a_1, \dots, a_m\}$

$$e_j = \sum_{i=1}^m z_{ij} a_i, \quad \exists z_{ij}, \quad 1 \leq i \leq m$$

$$\Rightarrow e_j = A z_j, \quad z_j = (z_{1j}, \dots, z_{mj})^T$$

$$\text{So, } [e_1 | e_2 | \dots | e_m] = A [z_1 | z_2 | \dots | z_m]$$

$$\Leftrightarrow \underbrace{I}_{m \times m \text{ identity matrix}} = A Z$$

Such matrix $Z \in \mathbb{R}^{m \times m}$ is called the **inverse** of A and written as A^{-1} .

Any nonsingular matrix has a unique inverse, and $AA^{-1} = A^{-1}A = I$.

Thm (Equivalences of a nonsingular matrix)

For $A \in \mathbb{R}^{m \times m}$, the following statements are equivalent:

- (a) A has an inverse A^{-1}
- (b) $\text{rank}(A) = m$
- (c) $\text{range}(A) = \mathbb{R}^m$
- (d) $\text{null}(A) = \{0\}$
- (e) 0 is not an eigenvalue of A
- (f) 0 is not a singular value of A
- (g) $\det(A) \neq 0$.

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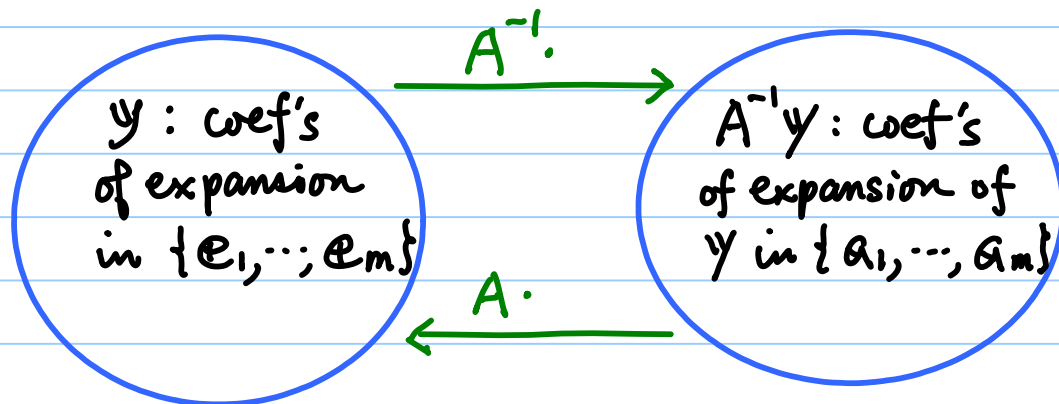
★ Matrix⁻¹ ^{times} vector

$$y = A x, \quad A: \text{nonsingular}$$

$$\Rightarrow x = A^{-1} y.$$

This means that $A^{-1} y$ represents an expansion coefficients of y in the basis of col's of A .

So, Multiplication by A^{-1} is a change of basis operation!



Note: $y = I^{-1} y$.