

More about SVD!

LECTURE 15

★ "A Change of Bases" viewpoint

$$A = U \Sigma V^T \in \mathbb{R}^{m \times n}$$

Pick any $\mathbf{x} \in \mathbb{R}^n$ and consider

$$\tilde{\mathbf{x}} = V^T \mathbf{x}$$

Then $\tilde{\mathbf{x}}$ is the expansion coefficient of $\mathbf{x}$ w.r.t. the ONB $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ why? You should know this by now.

But, just in case,

$$\tilde{\mathbf{x}} = V^T \mathbf{x} \Leftrightarrow \mathbf{x} = V \tilde{\mathbf{x}}$$

$$= \tilde{x}_1 \mathbf{v}_1 + \dots + \tilde{x}_n \mathbf{v}_n$$

linear comb. of

$\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$.

//

Now, let $\mathbf{b} = A \mathbf{x} \in \mathbb{R}^m$

Expand $\mathbf{b}$ w.r.t. the ONB $\{\mathbf{u}_1, \dots, \mathbf{u}_m\}$

$$\hat{\mathbf{b}} = U^T \mathbf{b} = U^T A \mathbf{x} = U^T A V \tilde{\mathbf{x}}$$

$$= \underbrace{U^T U}_{= I_m} \Sigma \underbrace{V^T V}_{= I_n} \tilde{\mathbf{x}} = \Sigma \tilde{\mathbf{x}}$$

Now, we know that Σ is diagonal!

This again shows that

" Σ represents the essence of A in a much clearer manner!"

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★ SVD vs Eigenvalue Decomposition

Let $A \in \mathbb{R}^{m \times m}$ be diagonalizable,
i.e., $\exists$ the eigenvalue decomposition:

$$A = X \Lambda X^{-1}$$

Note: where $X = [x_1 \cdots x_m] \in \mathbb{C}^{m \times m}$
Even if $A \in \mathbb{R}^{m \times m}$, satisfying $A x_j = \lambda_j x_j$, $j=1, \dots, m$
its eigval's & eigvec's may be complex-valued!

$$\Lambda = \text{diag}(\lambda_1, \dots, \lambda_m) \in \mathbb{C}^{m \times m}$$

$$A x_j = \lambda_j x_j, \quad j=1, \dots, m$$

$$\Leftrightarrow$$

$$A X = X \Lambda$$

Note that the eigenvectors $\{x_1, \dots, x_m\}$ form a basis of $\mathbb{C}^m$, but not necessarily orthonormal in general unless $A^* = A$ (unitary)

EX.

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\text{Here } A^* := (\bar{a}_{ji}) = \bar{A}^T$$

conjugate transposition of $A \in \mathbb{C}^{m \times m}$

"unitarity" is a generalization of "symmetry".

With the eigenvalue decomposition,

$$b = A x \quad \text{can be simplified as}$$

$$\tilde{b} = \underbrace{\Lambda}_{\text{diagonal}} \tilde{x}$$

$$\text{via } \begin{cases} \tilde{b} = X^{-1} b \\ \tilde{x} = X^{-1} x \end{cases}$$

change of bases again!

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So, we can summarize as follows:

- SVD: Use two different ONB's U, V and work for any matrix.
- EIG: Use one basis (not ONB in general) and work only for square matrices.

★ Matrix Properties via SVD

Let $A \in \mathbb{R}^{m \times n}$.

$$p := \min(m, n)$$
$$r := \# \text{ nonzero singular values} \\ \leq p.$$

Thm $\text{rank}(A) = r$.

(Proof) Let $A = U \Sigma V^T$.

Since U, V are orthogonal matrices, they are of full rank.

Hence, $\text{rank}(A) = \text{rank}(\Sigma)$
 $= \# \text{ nonzero diagonal entries}$

Recall $\langle u_1, \dots, u_r \rangle = r$ 

$$:= \text{span}\{u_1, \dots, u_r\} \rightarrow$$

Thm $\text{range}(A) = \langle u_1, \dots, u_r \rangle$

$$\text{null}(A) = \langle v_{r+1}, \dots, v_n \rangle$$

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(Proof) Since $\Sigma \in \mathbb{R}^{m \times n}$ is diagonal with only r nonzero entries,

$$\text{range}(\Sigma) = \langle e_1, \dots, e_r \rangle \subset \mathbb{R}^m$$

$$\Leftrightarrow \text{range}(A) = \langle u_1, \dots, u_r \rangle \subset \mathbb{R}^m. \quad \checkmark$$

On the other hand, it is clear that for any vector $x \in \mathbb{R}^n$ s.t.

$$x = \underbrace{[0, 0, \dots, 0]}_r, x_{r+1}, \dots, x_n]^T,$$

$$\Sigma x = \begin{bmatrix} \sigma_1 & & & \\ & \ddots & & \\ & & \sigma_r & \\ & & & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ x_{r+1} \\ \vdots \\ x_n \end{bmatrix} = 0.$$

So, $\text{null}(\Sigma) = \langle e_{r+1}, \dots, e_n \rangle \subset \mathbb{R}^n$

Then, for such x , we have

$$\begin{aligned} A V x &= U \Sigma V^T V x \\ &= U \Sigma x = 0 \end{aligned}$$

i.e., Any member of $\text{null}(A)$ should be of the form $V x$, $x \in \text{null}(\Sigma)$

i.e., $\text{null}(A) = \langle v_{r+1}, \dots, v_n \rangle \subset \mathbb{R}^n$ ///

Thm $\|A\|_2 = \sigma_1, \quad \|A\|_F = \sqrt{\sigma_1^2 + \dots + \sigma_r^2}$

(Proof) Since U, V are orthogonal,
 $\|A\|_2 = \|\Sigma\|_2 = \max_{1 \leq j \leq r} \{|\sigma_j|\} = \sigma_1 \quad \checkmark$

The Frobenius norm is also invariant w.r.t. rotations (ortho. matrix multiplications)

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$$\text{Hence, } \|A\|_F = \|\Sigma\|_F = \sqrt{\sigma_1^2 + \dots + \sigma_r^2} //$$

Thm The nonzero singular values of A are the square roots of the nonzero eigenvalues of $A^T A$ or $A A^T$.

$$\begin{aligned} (\text{Proof}) \quad A^T A &= (U \Sigma V^T)^T (U \Sigma V^T) \\ &= V \Sigma^T \Sigma V^T \end{aligned}$$

$$\Leftrightarrow (A^T A) V = V (\underbrace{\Sigma^T \Sigma})$$

$$\begin{aligned} &\text{" } \text{diag}(\sigma_1^2, \dots, \sigma_r^2, 0, \dots, 0) \\ &\in \mathbb{R}^{n \times n} \end{aligned}$$

So, the col's of V are the eigenvectors of $A^T A$ and their nonzero eigval's are $\sigma_1^2, \dots, \sigma_r^2$

You can show similarly that the col's of U are the eigenvectors of $A A^T$, and their nonzero eigval's are $\sigma_1^2, \dots, \sigma_r^2$. //

$$\text{Thm} \quad A^T = A \Rightarrow \sigma_i(A) = |\lambda_i(A)|$$

(Proof) HW #3 Prob 3 says:

Any symmetric matrix has only real-valued eigenvalues and the eigenvectors form an ONB.

$$\begin{aligned} \text{So, } A &= Q \Lambda Q^T, \quad Q: \text{ortho}, \Lambda: \text{diag} \\ &= Q |\Lambda| \text{sgn}(\Lambda) Q^T \end{aligned}$$

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where $|\Lambda| := \begin{bmatrix} |\lambda_1| & & 0 \\ & \ddots & \\ 0 & & |\lambda_m| \end{bmatrix}$
 $\text{sgn}(\Lambda) := \begin{bmatrix} \text{sgn}(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & \text{sgn}(\lambda_m) \end{bmatrix}$

Now, it's clear that $Q \text{sgn}(\Lambda)$ is orthogonal if Q is orthogonal.
why?

$$\begin{aligned} & (Q \text{sgn}(\Lambda))(Q \text{sgn}(\Lambda))^T \\ &= Q \text{sgn}(\Lambda) \text{sgn}(\Lambda) Q^T \\ &= Q Q^T = I_m \end{aligned}$$

So, $A = \underbrace{Q}_U \underbrace{|\Lambda|}_\Sigma \underbrace{(Q \text{sgn}(\Lambda))^T}_{V^T} \quad \equiv$

Thm For $A \in \mathbb{R}^{m \times m}$,
 $|\det(A)| = \prod_{i=1}^m \sigma_i = \sigma_1 \cdot \sigma_2 \cdot \dots \cdot \sigma_m$

(Proof) We'll use the following facts.

- $\det(AB) = \det(A) \cdot \det(B)$.
- $\det(A^T) = \det(A)$
- $\det(\text{diag}(a_1, \dots, a_m)) = \prod_{i=1}^m a_i$
- For any Q : orthogonal, $|\det(Q)| = 1$.
why? $\det(Q^T Q) = \det(Q^T) \cdot \det(Q) = (\det(Q))^2$
 $= \det(I) = 1$. so, $|\det(Q)| = 1$ ✓

Then, $|\det(A)| = |\det(U \Sigma V^T)| = |\det(\Sigma)|$
 $= \prod \sigma_i \quad \equiv$